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Algebraic Groups and Their Birational Invariants

V. E. Voskresenskiĭ

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Algebraic Groups and Their Birational Invariants

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АЛГЕБРАИЧЕСКИЕ ГРУППЫ И ИХ БИРАЦИОНАЛЬНЫЕ ИНВАРИАНТЫ

Translated from the original Russian manuscript
by Boris Kunyavski

2000 *Mathematics Subject Classification.* Primary 20G15, 20G30;
Secondary 14G05, 14G25.

ABSTRACT. This book, which can be viewed as a significantly revised version of the author's book "Algebraic Tori" (Nauka, Moscow, 1977), studies birational properties of linear algebraic groups focusing on arithmetical applications. The main topics are forms and Galois cohomology, Picard group and Brauer group, birational geometry of algebraic tori, arithmetic of algebraic groups, Tamagawa numbers, R-equivalence, projective toric varieties, invariants of finite transformation groups, index-formulas. Many of the results and applications are recent. An extensive bibliography with additional comments can serve as a guide for further reading.

The book will be useful for researchers and graduate students working in Algebraic geometry, algebraic groups, and related fields.

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Contents

Preface

In his well-known 1962 ICM address, Borel gave a panoramic picture of a relatively fresh (for that period) domain, the arithmetic of linear algebraic groups, presenting its achievements along with arising problems [**Borel2**]. In particular, he pointed out that investigation of important characteristics of a connected algebraic k-group G such as the Tamagawa number $\tau(G)$ or the set $Ш(G)$ of principal homogeneous spaces of G trivial in all completions of a number field k led to a really dramatic situation. Conjectures concerning the numbers $\tau(G)$ and $|Ш(G)|$ were disproved almost immediately after being stated. Retrospectively, we see that the arithmetic of algebraic groups lacked deeper information on birational nature of the variety of the group G. Hironaka's results on desingularization [**Hi**] became a very important source of additional information. Even earlier investigation of surfaces by Manin and Shafarevich showed that the group $H^1(k, \operatorname{Pic} \overline{X})$, where $\overline{X} = X \otimes_k k_s$, is a birational invariant in the class of smooth projective surfaces X defined over k, and this fact was successfully employed [**Manin1**]. The very first application of birational technique to the study of linear algebraic groups led to nontrivial results [**Vo2**]. Birational geometry provided new insight into causal relations in the vast world of linear algebraic groups and their homogeneous spaces. By now, the birational geometry of algebraic groups is a well-developed area. It records many fundamental results, including, for example, the solution of E. Noether's problem on the field of invariants of a permutation group and Zariski's problem on stably rational varieties; it also establishes the birational nature of many arithmetical invariants.

In this book we present the main results in the area, which have been obtained in the past 25 years. Sometimes we do not give detailed proofs restricting ourselves by sketches and comments. First we give a general remark. Various questions necessarily lead to consideration of algebraic varieties defined over a nonclosed field k or over a ring. Let X be such a variety, k_s a separable closure of k, $\mathcal{G} = \operatorname{Gal}(k_s/k)$ the Galois group of k_s/k. Then we have an object comprising very rich information, the variety $\overline{X} = X \otimes_k k_s$ defined over the field k_s and viewed together with the action of $\mathcal{G}$ on $\overline{X}$ via the second factor. Thus we obtain an impressive collection of associated $\mathcal{G}$-sets and Galois $\mathcal{G}$-modules: points $X(k_s)$, differential forms, $\operatorname{Pic} \overline{X}$, $\operatorname{Br} \overline{X}$, etc. Thus the study of the variety X can be divided into two stages: the geometric part dealing with the investigation of $\overline{X}$ over k_s, and the algebraic part focusing on Galois modules associated to the scheme $\overline{X}$. Any attempt to give a comprehensive unified exposition of these two stages would make the book voluminous and unreadable. In the present work, we give a detailed overview of the second algebraic aspect of the study of varieties assuming the geometric background known. We are mainly interested in algebraic groups and their homogeneous spaces. This topic is reflected in several monographs [**Borel3**], [**Hum**], [**Pl/Ra**], [**SGA3**], [**Sp**], [**Vi/Oni**]. Our style of exposition is close to Springer's [**Sp**], and we refer the reader to that review.

Certainly, the above mentioned subdivision into two stages is too rough. Even from the formal point of view, the variety $\overline{X}$ is only one fiber of a k-scheme X. In the number field case, one can consider, for example, the p-adic completions k_p of k and the corresponding p-adic varieties $X \otimes_k k_p$ with their own collections of Galois modules. Furthermore, arithmetic problems require introducing integer structures in k-varieties, and one is immediately led to a problem of classifying such structures, computing their characteristics, etc. To treat such problems, one has to use subtle devices such as Tamagawa measures and Siegel formulas. We consider this technique in detail.

The major part of the text is related to the theory of algebraic tori which is presented from the point of view of birational geometry. In the general theory of complex linear algebraic groups, algebraic tori always played an auxiliary, though important, role. This is understandable because over an algebraically closed field a torus has a very simple structure: it is a product of several copies of the multiplicative group $\mathbb{G}_m$. The situation changes radically when one passes to a nonclosed field k. In this case, the category of algebraic tori over a field k is dual to the category of integral representations of the Galois group $\mathcal{G}$, and hence the study of algebraic tori provides an amount of work sufficient for all future generations of mathematicians. The birational geometry of algebraic tori turned out to be a good pattern for stating (and subsequently proving) conjectures in the general case of linear algebraic groups. Conversely, several general conjectures (like Zariski's conjecture) are being tried using the proving ground of algebraic tori, and the information obtained leads to nontrivial results. During the past twenty years a new domain was created, the theory of toric varieties. It allows one to reduce many nonlinear problems of algebraic geometry and topology to problems about integral points in certain polyhedra. One can point out many important problems such as resolution of singularities, estimating the number of solutions to a system of equations, birational geometry of linear algebraic groups. Successful development of the theory of toric varieties can be explained by its natural origin. Let M be a free abelian group of rank n. We may view M with the group structure written in either additive, or multiplicative form. The isomorphism $M \cong \mathbb{Z}^n$ allows one to regard M as a lattice in the vector space $M \otimes k$. On the other hand, by embedding M as a multiplicative subgroup in the field of rational functions $k(M) = k(x_1, \dots, x_n)$, where $x_1, \dots, x_n$ is a basis of the multiplicative group M, we are led to birational geometry of integral representations, i.e., the theory of toric varieties. To present the theory of tori, we badly need the technique of group schemes. We give a digest of group schemes in the very beginning. We freely use various cohomology theories, referring the reader to [**Ca/Ei**], [**Gro2**], [**Serre2**].

The theory of invariants of finite transformation groups was developed in a natural framework of birational geometry of group varieties. This topic is a significant subject of our attention.

The notion of R-equivalence introduced by Manin [**Manin2**] is an analogue of the notion of a fundamental group of a continuous variety. It turned out to be a natural one in the category of algebraic groups G over a field k. The classes of R-equivalence in $G(k)$ form a group with respect to multiplication in $G(k)$, and this group is a birational invariant. The Manin group is trivial over $\mathbb{C}$ and $\mathbb{R}$, probably this being the reason for its relatively recent discovery [**CT/San1**]. The study of this invariant enabled one, in particular, to establish a close relationship between algebraic geometry and central simple algebras [**Vo10**].

We give a detailed exposition of the arithmetic of linear algebraic groups using the discovered direct connection of arithmetical characteristics of a group G with its birational properties. We present in detail the construction of the famous Tamagawa measure on adele groups and discuss a motivation for introducing this notion in treating diophantine problems. We compute Tamagawa numbers in many important cases. During the course of the computation, we elucidate appearance of Eisenstein series and Gindikin–Karpelevich integrals, so that this part can serve as a good introduction to harmonic analysis on semisimple groups. We prove Siegel's formula in Tamagawa's form and illustrate its application in a series of classical examples. We present the detailed computation of local p-adic volumes which are necessary for getting precise formulas. A similar approach to algebraic tori provides index formulas for number fields which generalize classical formulas of Dirichlet and Hasse.

Terminology and notation are more or less standard. In particular, $\mathbb{Z}$ is the ring of integers, $\mathbb{Q}$ is the field of rational numbers, $\mathbb{R}$ is the field of real numbers, $\mathbb{C}$ is the field of complex numbers, $\mathbb{F}_p$ is the prime field of p elements, $\mathbb{Q}_p$ is the field of p-adic numbers. All rings are assumed to have unity, A^* denotes the group of invertible elements of a ring A. If $a, b \in \mathbb{Z}$, we denote by (a, b) the greatest common divisor. The symbol $|X|$ stands for the cardinality of a set X. We denote by $(L : k)$ the degree of an extension L/k, and by $(G : H)$ the index of a subgroup H in a group G. By an algebraic variety over a field k we usually mean a geometrically irreducible, reduced, separated scheme of finite type over k. We denote by $k[X]$ the ring of regular functions and by $k(X)$ the field of rational functions of X.

It is my pleasure to express deep gratitude to Boris Kunyavskiĭ who kindly agreed to translate the manuscript into English. He is one of the active developers of geometry and arithmetic of algebraic groups, and his advice and critical comments were always useful.

V. E. Voskresenskiĭ

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Bibliographical Remarks

As noted in the preface, when studying algebraic groups defined over a non-closed field, one has to regard them as group objects in the category of schemes. The theory of group schemes is presented in detail in [**SGA3**], but the present book is mainly aimed at the reader with some background in algebraic groups over a closed field. Such a reader is used to identify a given group with the set of its geometric points which might only give a superficial understanding of problems typical for the non-closed case. I believe that while reading this book, an interested reader could be stimulated for serious study of the general theory of schemes. In Chapter 1 we gathered some basics on functors, group schemes, and cohomology which are necessary for the subsequent exposition. Chapter 1 also contains many important examples which do not just illustrate the formulated results but also serve as a motivation for introducing a particular notion and indicate directions for subsequent constructions.

A special role in the present theory is played by diagonal groups and their forms. In Chapter 1, they are studied in detail. We follow the approach suggested in Grothendieck's talk on the seminar on group schemes. Our needs being modest enough, we restrict our attention to groups over affine schemes. Most results concernings the Brauer group of an algebraic variety are also due to Grothendieck [**Gro2**].

The theory of forms is assumed to be known, although one can hardly give a good reference for its detailed exposition. In [**Weil1**] and [**Lang/Tate**], the theory of forms was first distinguished as a special topic. In Serre's lectures [**Serre2**] one can find a detailed exposition of Galois cohomology. The operation of restriction of the ground field was introduced by Weil [**Weil3**], see also [**Gro1**] for the descent theory in a more general setting.

The fibration into maximal tori in a reductive group was studied in a pioneering paper by Chevalley [**Chev**], and our exposition follows the approach suggested in [**SGA3**]. The module of rational characters of a generic torus in a semisimple group has been computed by the author [**Vo13**]. The Picard group of a connected linear algebraic k-group was studied in [**Vo1**] and [**Po**]. The Brauer group of a linear group and its compactification is described in detail in [**CT/San1**] and [**San**].

In Manin's and Shafarevich's papers on the theory of surfaces, one first encounters a new birational invariant $[\operatorname{Pic}\overline{X}]$, where X is a smooth projective surface over a non-closed field k [**Manin1**]. The fact that the structure of the Galois module $\operatorname{Pic}\overline{X}$ is important for the arithmetic of a surface X was mentioned as early as in [**Segre2**]. Segre considered the birational invariant $[\operatorname{Pic}\overline{X}]$ in the case where X is a smooth projective model of a connected linear algebraic group G and studied it in a series of subsequent papers. It turned out that the category of algebraic tori is the most natural domain for using the invariant $[\operatorname{Pic}\overline{X}(T)] = p_k(T)$.

For example, the class $p_k(T)$ determines the variety of T up to stable equivalence [**Vo8**], [**Vo9**]. Furthermore, the class $p_k(T)$ can be defined in a purely algebraic way. It turned out that the Picard module $\operatorname{Pic}\overline{X}(T)$ has an interesting property: $H^{-1}(\pi, \operatorname{Pic}\overline{X}(T)) = 0$ for any subgroup π of the splitting group Π of T [**Vo8**], [**Vo9**]. This provides a canonical resolution for any finite dimensional torsion-free Galois module $\hat{T}$. Such a resolution uniquely determines the class $p_k(T)$. Colliot-Thélène and Sansuc [**CT/San1**] suggested calling such a resolution flasque since it induces a flasque fibration. The theory of flasque resolutions led to substantial progress in the study of the semigroup of stable equivalence $Z(L/k)$ and its maximal subgroup $Z^0(L/k)$ [**Vo8**], [**Vo9**], [**Endo/Miy1**]–[**Endo/Miy6**], [**Ku1**], [**Chi1**], [**Chi2**]. The group $Z^0(L/k)$ turned out to be isomorphic to a group studied earlier by Dress [**Dress**]. It is of finite type but not always finite. In [**Chi2**] it is shown that in most cases the semigroup $Z(L/k)$ is not finitely generated. If T is a torus with a cyclic splitting field, the class $p_k(T)$ is invertible in the semigroup of similarity classes of modules. This fundamental fact has been first established by Endo and Miyata [**Endo/Miy3**]. Using the above result, Chistov [**Chi1**] proved Theorem 1 of 5.3 giving birational classification of tori with a cyclic splitting field (up to stable rationality). Many interesting results in this domain are due to Lenstra [**Le1**] who studied the fields of invariants of finite abelian transformation groups.

Introducing birational characteristics in the framework of linear algebraic groups allowed one to tackle an old problem of rationality of the field of invariants of a finite transformation group acting linearly on a finite dimensional space. The first example $(\mathbb{Q}, 47)$ of the field of invariants which is not rational over $\mathbb{Q}$ is due to Swan [**Swan2**]. At the same time, by another occasion, the same example was given by the author [**Vo3**], [**Vo4**] who noticed that the field of invariant of a finite abelian group acting linearly on a vector space V can be naturally described as the function field of some torus isogenous to a maximal torus of $\mathrm{GL}(V)$. This allowed one to use the theory of birational invariants of algebraic tori and translate the problem into the language of finite dimensional modules. Just that language was used in [**Le1**] and [**Endo/Miy3**] where the classification of fields of invariants of finite abelian linear groups was completed.

The first example of a non-rational field of invariants of a finite group G acting on a linear space over a closed field k has been constructed by Saltman [**Sal1**]. He managed to compute the unramified Brauer group of $k(V)^G$ which in many cases turned out to be non-trivial. Bogomolov [**Bog2**] suggested an elegant construction for such a computation. See also an expository paper [**CT/San4**] where the authors present their own interpretation of the problem, and [**Barge**] where there are interesting results concerning quotients of finite groups acting on toric varieties. In [**For1**], [**For2**], one can find a modern exposition of the classical problem on reducing a pair of matrices to the simplest form. Namely, it is shown that the field of rational functions of the quotient $(\mathrm{M}(n) \oplus \mathrm{M}(n))/\mathrm{GL}(n)$ can be realized as the field of rational functions of some algebraic torus which can be described explicitly. In [**Be/LB**], the computations were made for n dividing 420, for all such n the above quotient turned out to be stably rational.

Yet another direction in the theory of algebraic groups appeared as a result of successful study of R-equivalence on algebraic tori [**CT/San1**]. Continuing their research, the author noticed [**Vo10**], [**Vo11**] that R-equivalence on the unimodular group of a simple algebra is intimately tied with Platonov's results [**Pl2**], [**Pl3**] on the Tannaka–Artin problem. Recently Merkurjev [**Me2**] invented a device for computing R-equivalence on semisimple groups of adjoint type and showed that among them there are non-rational groups, thus disproving a well-known conjecture.

Problems concerning the arithmetic of algebraic groups are presented in detail with all necessary comments. We only note that one can refer to [**ANT**] or [**Borev/Sha**] for basics on number fields.

Finally, we should mention some related research areas which are left aside. First, this is analytic arithmetic of algebraic tori and toric varieties which arise naturally from classical problems of analytic number theory [**Draxl1**], [**Moroz1**]; see [**Moroz2**] and references therein for further development. For another approach fitting into a general theory of asymptotic distribution of rational points on algebraic varieties, we refer to [**Bat/Tsch1**]–[**Bat/Tsch3**]. This topic is being extensively developed and certainly deserves a separate publication.

Yet another analytic aspect, related to class numbers of algebraic tori and L-functions, was developed in [**Shyr2**], [**Moril**]. We can also point out some miscellaneous areas of applications such as rational surfaces (via their Néron–Severi tori) [**Ku/Ts**], [**K/S/Ts**], arithmetic of bilinear forms [**Gras/Cor**], and dense sphere packings [**Ku5**].

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