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Volume 180

Calculus of Variations and Optimal Control

A. A. Milyutin
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American Mathematical Society

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10.1090/mmono/180

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Calculus of Variations and Optimal Control

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American Mathematical Society
Providence, Rhode Island

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ВАРИАЦИОННОЕ ИСЧИСЛЕНИЕ
И ОПТИМАЛЬНОЕ УПРАВЛЕНИЕ

Translated from the original Russian manuscript
by Dimitrii Chibisov

1991 *Mathematics Subject Classification.* Primary 49–02, 49K15.

ABSTRACT. The theory of Pontryagin minimum is developed for problems in the calculus of variations. The application of the notion of Pontryagin minimum to the calculus of variations is a distinctive feature of the book. A new theory of quadratic conditions for a Pontryagin minimum, which covers broken extremals, is developed, and corresponding sufficient conditions for a strong minimum are obtained. Some classical theorems of the calculus of variations are generalized.

The book can be used by researchers and graduate students in mathematics working in the calculus of variations and control theory, and in applications to mechanics, physics, and engineering.

Library of Congress Cataloging-in-Publication Data

Miliutin, A. A.
[Variatsionnoe ischislenie i optimal'noe upravlenie. English]
Calculus of variations and optimal control / A.A. Milyutin, N.P. Osmolovskii ; [translated by Dimitrii Chibisov].
p. cm. — (Translations of mathematical monographs ; v. 180)
Includes bibliographical references.
ISBN 0-8218-0753-6 (alk. paper)
1. Calculus of variations. 2. Control theory. 3. Mathematical optimization. I. Osmolovskii, N. P. (Nikolai Pavlovich), 1948– . II. Title. III. Series.
QA315.M49813 1998
515'.64—dc21 98-29674
CIP

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10 9 8 7 6 5 4 3 2 1 03 02 01 00 99 98

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Preface

The almost 300 year history of the calculus of variations involves the names of many outstanding mathematicians of the past. The subject has been treated in a number of excellent monographs. The foundations of this science seemed to be firmly established and subject to no revision. However, this proved to be not quite so.

About 40 years ago there emerged a new science, optimal control theory, which deals with more compicated problems than those in the calculus of variations. The new ideas and methods of optimal control provided us with a new outlook on its predecessor, the calculus of variations. This was one of the stimulating reasons for the authors to write this book.

Like the majority of treatises on the calculus of variations and optimal control, our book begins with formulating the problem and determining the class of variations to be considered in the subsequent theory. The key concept in our setup is that of a control system. The principal type of control variations are spiky variations, i.e., variations that take nonnegligible values on a set of small measure.

These variations lead to the notion of minimum that is stronger than a weak minimum usually considered in the calculus of variations; we refer to it as a Pontryagin minimum. Both concepts, the control system and Pontryagin minimum, originate from optimal control. Their formulation is fairly simple and requires no preliminary knowledge.

We specify a class of problems, related to an arbitrary control system, whose properties are similar to those of problems in the calculus of variations. We refer to these problems as problems of the calculus of variations, although formally they are more general and allow us to illustrate theoretical developments by examples from optimal control.

We explore the interaction between the ideas of the calculus of variations and optimal control within the framework of this class of problems. The fundamental concepts of the calculus of variations such as an extremal, a field of extremals, the second variation of a functional, quadratic conditions, are related to a weak minimum. We study how these concepts transform when we apply them to a Pontryagin minimum and develop the theory of Pontryagin minimum for problems of the calculus of variations. It turns out that a Pontryagin minimum allows us to construct a deeper and more natural theory than a weak minimum. In particular, it requires weaker smoothness assumptions. Moreover, this theory is self-consistent, and we dare to recommend our book as an introduction to the calculus of variations. On the other hand, the book provides a good introduction to optimal control.

The book is accessible to graduate students in mathematics and can be recommended to all of those who use extremum theory in their research or applied studies. We hope that it will be of interest to mathematicians working in extremum theory due to the novelty of material and the possibility to extend the results to a wider variety of problems. Moreover, the book can be used for teaching courses in the calculus of variations and optimal control. In fact, its first version emerged from such courses taught by the authors at Moscow State University in the 1980s.

We are thankful to V. M. Tikhomirov and S. I. Gelfand who suggested that we write the book and assisted in accomplishing this task. We are grateful to the American Mathematical Society for the publication of our book. We express our gratitude to D. M. Chibisov who translated the book into English. The authors thank the G. Soros Cultural Initiative Foundation and the Russian Foundation for Fundamental Research for substantial financial support.

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ISBN 0-8218-0753-6

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