

Translations of
**MATHEMATICAL
MONOGRAPHS**

Volume 180

**Calculus of Variations
and Optimal Control**

A. A. Milyutin

N. P. Osmolovskii



American Mathematical Society

Selected Titles in This Series

- 180 **A. A. Milyutin and N. P. Osmolovskii**, Calculus of variations and optimal control, 1998
- 179 **V. E. Voskresenskii**, Algebraic groups and their birational invariants, 1998
- 178 **Mitsuo Morimoto**, Analytic functionals on the sphere, 1998
- 177 **Satoru Igari**, Real analysis—with an introduction to wavelet theory, 1998
- 176 **L. M. Lerman and Ya. L. Umanskiy**, Four-dimensional integrable Hamiltonian systems with simple singular points (topological aspects), 1998
- 175 **S. K. Godunov**, Modern aspects of linear algebra, 1998
- 174 **Ya-Zhe Chen and Lan-Cheng Wu**, Second order elliptic equations and elliptic systems, 1998
- 173 **Yu. A. Davydov, M. A. Lifshits, and N. V. Smorodina**, Local properties of distributions of stochastic functionals, 1998
- 172 **Ya. G. Berkovich and E. M. Zhmud'**, Characters of finite groups. Part 1, 1998
- 171 **E. M. Landis**, Second order equations of elliptic and parabolic type, 1998
- 170 **Viktor Prasolov and Yuri Solovyev**, Elliptic functions and elliptic integrals, 1997
- 169 **S. K. Godunov**, Ordinary differential equations with constant coefficient, 1997
- 168 **Junjiro Noguchi**, Introduction to complex analysis, 1998
- 167 **Masaya Yamaguti, Masayoshi Hata, and Jun Kigami**, Mathematics of fractals, 1997
- 166 **Kenji Ueno**, An introduction to algebraic geometry, 1997
- 165 **V. V. Ishkhanov, B. B. Lur'e, and D. K. Faddeev**, The embedding problem in Galois theory, 1997
- 164 **E. I. Gordon**, Nonstandard methods in commutative harmonic analysis, 1997
- 163 **A. Ya. Dorogovtsev, D. S. Silvestrov, A. V. Skorokhod, and M. I. Yadrenko**, Probability theory: Collection of problems, 1997
- 162 **M. V. Boldin, G. I. Simonova, and Yu. N. Tyurin**, Sign-based methods in linear statistical models, 1997
- 161 **Michael Blank**, Discreteness and continuity in problems of chaotic dynamics, 1997
- 160 **V. G. Osmolovskii**, Linear and nonlinear perturbations of the operator div, 1997
- 159 **S. Ya. Khavinson**, Best approximation by linear superpositions (approximate nomography), 1997
- 158 **Hideki Omori**, Infinite-dimensional Lie groups, 1997
- 157 **V. B. Kolmanovskii and L. E. Shaikhet**, Control of systems with aftereffect, 1996
- 156 **V. N. Shevchenko**, Qualitative topics in integer linear programming, 1997
- 155 **Yu. Safarov and D. Vassiliev**, The asymptotic distribution of eigenvalues of partial differential operators, 1997
- 154 **V. V. Prasolov and A. B. Sossinsky**, Knots, links, braids and 3-manifolds. An introduction to the new invariants in low-dimensional topology, 1997
- 153 **S. Kh. Aranson, G. R. Belitsky, and E. V. Zhuzhoma**, Introduction to the qualitative theory of dynamical systems on surfaces, 1996
- 152 **R. S. Ismagilov**, Representations of infinite-dimensional groups, 1996
- 151 **S. Yu. Slavyanov**, Asymptotic solutions of the one-dimensional Schrödinger equation, 1996
- 150 **B. Ya. Levin**, Lectures on entire functions, 1996
- 149 **Takashi Sakai**, Riemannian geometry, 1996
- 148 **Vladimir I. Piterbarg**, Asymptotic methods in the theory of Gaussian processes and fields, 1996
- 147 **S. G. Gindikin and L. R. Volevich**, Mixed problem for partial differential equations with quasihomogeneous principal part, 1996

(See the AMS catalog for earlier titles)

This page intentionally left blank



Calculus of Variations and Optimal Control

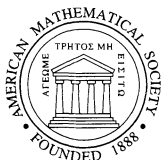
This page intentionally left blank

Translations of
**MATHEMATICAL
MONOGRAPHS**

Volume 180

**Calculus of Variations
and Optimal Control**

A. A. Milyutin
N. P. Osmolovskii



American Mathematical Society
Providence, Rhode Island

EDITORIAL COMMITTEE

AMS Subcommittee

Robert D. MacPherson

Grigorii A. Margulis

James D. Stasheff (Chair)

ASL Subcommittee Steffen Lempert (Chair)

IMS Subcommittee Mark I. Freidlin (Chair)

А. А. Милютин, Н. П. Осмоловский

ВАРИАЦИОННОЕ ИСЧИСЛЕНИЕ И ОПТИМАЛЬНОЕ УПРАВЛЕНИЕ

Translated from the original Russian manuscript
by Dimitrii Chibisov

1991 *Mathematics Subject Classification*. Primary 49–02, 49K15.

ABSTRACT. The theory of Pontryagin minimum is developed for problems in the calculus of variations. The application of the notion of Pontryagin minimum to the calculus of variations is a distinctive feature of the book. A new theory of quadratic conditions for a Pontryagin minimum, which covers broken extremals, is developed, and corresponding sufficient conditions for a strong minimum are obtained. Some classical theorems of the calculus of variations are generalized.

The book can be used by researchers and graduate students in mathematics working in the calculus of variations and control theory, and in applications to mechanics, physics, and engineering.

Library of Congress Cataloging-in-Publication Data

Milyutin, A. A.

[Variatsionnoe ischislenie i optimal'noe upravlenie. English]

Calculus of variations and optimal control / A.A. Milyutin, N.P. Osmolovskii ; [translated by Dimitrii Chibisov].

p. cm. — (Translations of mathematical monographs ; v. 180)

Includes bibliographical references.

ISBN 0-8218-0753-6 (alk. paper)

I. Calculus of variations. 2. Control theory. 3. Mathematical optimization. I. Osmolovskii, N. P. (Nikolai Pavlovich), 1948–. II. Title. III. Series.

QA315.M49813 1998

515'.64—dc21

98-29674

CIP

Copying and reprinting. Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy a chapter for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication (including abstracts) is permitted only under license from the American Mathematical Society. Requests for such permission should be addressed to the Assistant to the Publisher, American Mathematical Society, P. O. Box 6248, Providence, Rhode Island 02940-6248. Requests can also be made by e-mail to reprint-permission@ams.org.

© 1998 by the American Mathematical Society. All rights reserved.

The American Mathematical Society retains all rights

except those granted to the United States Government.

Printed in the United States of America.

∞ The paper used in this book is acid-free and falls within the guidelines established to ensure permanence and durability.

Visit the AMS home page at URL: <http://www.ams.org/>

10 9 8 7 6 5 4 3 2 1 03 02 01 00 99 98

Contents

Preface	xi
Introduction	1
Part 1. First Order Conditions	5
Chapter 1. Theory of a Weak Minimum for the Problem on a Fixed Time Interval	7
1. Problems of the calculus of variations	7
2. The problem on a fixed time interval. Necessary conditions for a weak extremum	8
3. Two examples	14
4. Weak extremals	18
Chapter 2. Theory of the Maximum Principle	23
5. Formulation of the maximum principle for the problem of § 1	23
6. Invariance of Pontryagin's convergence under the change of the independent variable	26
7. Proof of the maximum principle	32
8. Expansion formulas	40
Chapter 3. Extremals and the Hamiltonian of a Control System	43
9. Extremals	43
10. Solutions of a Hamiltonian system and extremals	47
11. Examples	51
11.1. Time-optimal control problems	51
11.2. Two optimal control problems on a fixed time interval	55
11.3. Isoperimetric problem	63
11.4. Singular extremals	66
12. Convexification of the right-hand side of a control system (sliding modes)	69
Chapter 4. Hamilton–Jacobi Equation and Field Theory	87
13. The Hamilton–Jacobi equation and sufficient conditions for a strong minimum	87
14. Solutions of the Hamilton–Jacobi equation and extremals	92
15. The field of extremals	105
15.1. The general field theory	105
15.2. A linear control system	112

15.3. The field in the problem with a general control system and a primitive endpoint part	116
15.4. Isoperimetric problem	121
Chapter 5. Transformations of Problems and Invariance of Extremals	125
16. Invariance of extremals	125
16.1. Change of variables	125
16.2. Change of the independent variable	134
16.3. Passage to a parametric form	140
17. Calculus of variations problems with pointwise equality constraints	140
18. Problems with pointwise mixed state–control equality and inequality constraints	148
Part 2. Quadratic Conditions	153
Chapter 1. Quadratic Conditions and Conjugate Points for Broken Extremals	155
1. Quadratic conditions in the simplest problem of the calculus of variations	155
1.1. The setup and assumptions	155
1.2. Minimum on a set of sequences	157
1.3. First order conditions	158
1.4. An additional condition of the Weierstrass–Erdmann type	160
1.5. The Legendre condition	162
1.6. Quadratic conditions for a weak minimum	162
1.7. Quadratic conditions for a Pontryagin minimum	163
1.8. Sufficient conditions for a bounded-strong minimum	166
1.9. θ -weak minimum	167
2. Conjugate points and conditions for positive definiteness of the quadratic form	168
2.1. Passage of the quadratic form through zero: the classical Jacobi condition	168
2.2. Positive definiteness of Ω on G_2	177
3. Conditions for nonnegativeness of the quadratic form	186
3.1. Nonnegativeness of ω on E_2	186
3.2. Nonnegativeness of Ω on G_2	187
3.3. Abstract model	189
3.4. Nonnegativeness of Ω on G_2 (continued)	190
4. Investigation of a broken extremal by means of conjugate points theory: an example	196
Chapter 2. Quadratic Conditions for a Pontryagin Minimum and Sufficient Conditions for a Strong Minimum: Proofs	211
5. Higher orders, γ -conditions, and constant C_γ	211
5.1. Order γ	211
5.2. γ -conditions	214
5.3. Constant C_γ	215
6. Expansion of the integral functional on local sequences of variations	217
6.1. The structure of local sequences. The main lemma	217
6.2. Representation for the increment δF on local sequences	222

6.3. Proof of Lemma 6.1, continued	223
6.4. Proof of Lemma 6.1, concluded	225
6.5. Proof of Theorem 1.1	228
6.6. Estimation of $\ \delta x\ _C$ on local sequences	228
7. Upper bound for C_γ	229
7.1. The constant C_γ^{loc}	229
7.2. Extension of the set $\Pi^{\text{loc}}(E)$	230
7.3. Canonical representation for sequences in $\Pi_{O(\gamma)}^{\text{loc}}$	233
7.4. Passage to sequences with $\delta v = 0$	234
7.5. Passage to sequences with discontinuous state components	235
7.6. The set of sequences S^3	241
7.7. The sets of sequences S^4 and S^5	241
7.8. The space $Z(\theta)$ and subspace G	243
7.9. Passage to G_2	244
7.10. The CT-strict maximum principle	245
8. Lower bound for C_γ	249
8.1. Extension of the set $\Pi(E)$	249
8.2. Passage to local sequences	249
8.3. Simplifications in the definition of $C_\gamma(\Phi, \Pi_{o(\sqrt{\gamma})}^{\text{loc}})$	256
8.4. Application of the Legendre condition	258
8.5. Passage to sequences with discontinuous state components	260
8.6. Condition $D^k(H) \geq 2C_\Gamma(H)$	261
8.7. Passage to the equality in the differential constraint	265
8.8. The final lower bound for C_γ . The result of deciphering	266
8.9. Proof of Theorem 5.1	267
8.10. Proof of Theorem 5.2	267
9. Sufficient conditions for bounded-strong and strong minima in the simplest problem of the calculus of variations	271
9.1. Sufficient conditions for a bounded-strong minimum. Proof of Theorem 1.5	271
9.2. γ -sufficiency on $\bar{\Pi}^S$	275
9.3. Sufficient conditions for a strong minimum	275
Chapter 3. Quadratic Conditions in the General Problem of the Calculus of Variations and Related Optimal Control Problems	283
10. Formulation of quadratic conditions in the general problem of the calculus of variations	283
10.1. The problem setting and assumptions	283
10.2. First order conditions; the set M_0	284
10.3. Critical cone	286
10.4. The quadratic form	287
10.5. Quadratic necessary condition	289
10.6. Strong and bounded-strong minima	290
10.7. The strict maximum principle and strictly Legendrian elements	291
11. Quadratic conditions in the general problem of the calculus of variations on a fixed time interval	293
11.1. Formulation of quadratic conditions	293

11.2.	γ -sufficiency	295
11.3.	Discussion of the proofs of quadratic conditions	296
12.	Quadratic conditions in problems that are linear in control	298
12.1.	Linear in control problem on a fixed time interval	298
12.2.	Quadratic conditions in the linear in control problem on a non-fixed time interval	302
12.3.	Quadratic conditions for a piecewise constant control	303
12.4.	Quadratic conditions in the minimum time problem for a system linear in control	308
13.	Quadratic conditions in time-optimal control problems for linear systems with constant coefficients	310
13.1.	Problem setting, maximum principle, and simple sufficient conditions	310
13.2.	Quadratic necessary condition	314
13.3.	Quadratic sufficient condition	318
13.4.	Nonemptiness of the set Ξ	322
13.5.	The case where M_0 is a singleton	323
Chapter 4.	Investigation of Extremals by Quadratic Conditions: Examples	325
14.	Time-optimal control problems for linear systems with constant coefficients	325
14.1.	Two-dimensional chain	325
14.2.	Oscillating system	328
14.3.	Three-dimensional chain	331
14.4.	Oscillating system with an additional integral constraint on the control	336
15.	Investigation of extremals in nonlinear systems	341
15.1.	Saw-shaped extremals	341
15.2.	Isoperimetric problem	344
15.3.	M_0 consisting of many elements	349
16.	Appendix	361
16.1.	Integrals of convex combinations	361
16.2.	A property of systems that are linear in control	366
16.3.	Lyusternik's theorem	368
16.4.	Condition for inconsistency of a system of linear inequalities	368
Bibliography		371

Preface

The almost 300 year history of the calculus of variations involves the names of many outstanding mathematicians of the past. The subject has been treated in a number of excellent monographs. The foundations of this science seemed to be firmly established and subject to no revision. However, this proved to be not quite so.

About 40 years ago there emerged a new science, optimal control theory, which deals with more complicated problems than those in the calculus of variations. The new ideas and methods of optimal control provided us with a new outlook on its predecessor, the calculus of variations. This was one of the stimulating reasons for the authors to write this book.

Like the majority of treatises on the calculus of variations and optimal control, our book begins with formulating the problem and determining the class of variations to be considered in the subsequent theory. The key concept in our setup is that of a control system. The principal type of control variations are spiky variations, i.e., variations that take nonnegligible values on a set of small measure.

These variations lead to the notion of minimum that is stronger than a weak minimum usually considered in the calculus of variations; we refer to it as a Pontryagin minimum. Both concepts, the control system and Pontryagin minimum, originate from optimal control. Their formulation is fairly simple and requires no preliminary knowledge.

We specify a class of problems, related to an arbitrary control system, whose properties are similar to those of problems in the calculus of variations. We refer to these problems as problems of the calculus of variations, although formally they are more general and allow us to illustrate theoretical developments by examples from optimal control.

We explore the interaction between the ideas of the calculus of variations and optimal control within the framework of this class of problems. The fundamental concepts of the calculus of variations such as an extremal, a field of extremals, the second variation of a functional, quadratic conditions, are related to a weak minimum. We study how these concepts transform when we apply them to a Pontryagin minimum and develop the theory of Pontryagin minimum for problems of the calculus of variations. It turns out that a Pontryagin minimum allows us to construct a deeper and more natural theory than a weak minimum. In particular, it requires weaker smoothness assumptions. Moreover, this theory is self-consistent, and we dare to recommend our book as an introduction to the calculus of variations. On the other hand, the book provides a good introduction to optimal control.

The book is accessible to graduate students in mathematics and can be recommended to all of those who use extremum theory in their research or applied studies. We hope that it will be of interest to mathematicians working in extremum

theory due to the novelty of material and the possibility to extend the results to a wider variety of problems. Moreover, the book can be used for teaching courses in the calculus of variations and optimal control. In fact, its first version emerged from such courses taught by the authors at Moscow State University in the 1980s.

We are thankful to V. M. Tikhomirov and S. I. Gelfand who suggested that we write the book and assisted in accomplishing this task. We are grateful to the American Mathematical Society for the publication of our book. We express our gratitude to D. M. Chibisov who translated the book into English. The authors thank the G. Soros Cultural Initiative Foundation and the Russian Foundation for Fundamental Research for substantial financial support.

A. A. Milyutin
N. P. Osmolovskii

This page intentionally left blank

Bibliography

1. A. P. Afanas'ev, V. V. Dikusar, A. A. Milyutin, and S. V. Chukanov, *A necessary condition in optimal control*, "Nauka", Moscow, 1990. (Russian)
2. N. I. Akhiezer, *Lectures on the calculus of variations*, Gostekhizdat, Moscow, 1955; English transl., Blaisdell, New York-London, 1962.
3. ———, *The calculus of variations*, "Vishcha Shkola", Kharkov, 1981; English transl., Harwood, Chur, 1988.
4. V. M. Alekseev, V. M. Tikhomirov, and S. V. Fomin, *Optimal control*, "Nauka", Moscow, 1979; English transl., Consultants Bureau, New York, 1987.
5. A. V. Balakrishnan, *Control theory and the calculus of variations*, Academic Press, New York, 1969.
6. A. V. Balakrishnan and L. Neustadt, eds., *Mathematical theory of control*, Academic Press, New York-London, 1967.
7. L. D. Berkovitz, *A Hamilton-Jacobi theory for a class of control problems*, Colloque sur la théorie mathématique du contrôle optimal, held Bruxelles, April 1969, Centre Belge de Recherches Mathématiques, Van der Louvain, 1970.
8. ———, *Optimal control theory*, Springer-Verlag, Heidelberg, 1974.
9. G. A. Bliss, *Lectures on the calculus of variations*, Univ. Chicago Press, Chicago, 1946.
10. V. G. Boltyanskii, *Mathematical methods of optimal control*, "Nauka", Moscow, 1969; English transl., Holt, Rinehart, and Winston, New York, 1971.
11. A. Bryson and Y.-C. Ho, *Applied optimal control*, Blaisdell, New York-London, 1969.
12. C. Carathéodory, *Calculus of variations and partial differential equations of the first order*, Holden Day, San Francisco, 1965.
13. V. V. Dikusar and A. A. Milyutin, *Qualitative and numerical methods in the maximum principle*, "Nauka", Moscow, 1989. (Russian)
14. A. V. Dmitruk, A. A. Milyutin, and N. P. Osmolovskii, *Lyusternik's theorem and the theory of extrema*, Uspekhi Mat. Nauk **35** (1980), no. 6, 11–46; English transl., Russian Math. Surveys **35** (1980), no. 6, 11–51.
15. A. V. Dmitruk and N. P. Osmolovskii, *Investigation of extremals in the problem of fastest displacement of a pendulum controlled by the suspension point*, Izv. Ross. Akad. Nauk, Ser. Techn. Kibernet. **1992**, no. 6, 101–109; English transl., J. Comput. Systems Sci. Internat. **32** (1994), no. 4, 116–125.
16. A. Ya. Dubovitskii and A. A. Milyutin, *Problems for extremum under constraints*, Zh. Vychislit. Mat. i Mat. Fiz. **5** (1965), no. 3, 395–453; English transl. in U.S.S.R. Comput. Math. and Math. Phys. **5** (1965).
17. I. M. Gelfand and S. V. Fomin, *Calculus of variations*, Fizmatgiz, Moscow, 1961; English transl., Prentice-Hall, Englewood Cliffs, NJ, 1963.
18. M. R. Hestenes, *Calculus of variations and optimal control theory*, Wiley, New York, 1966.
19. J. C. Hsu and A. U. Meyer, *Modern control principles and applications*, McGraw-Hill, New York, 1968.
20. A. D. Ioffe and V. M. Tikhomirov, *Theory of extremal problems*, "Nauka", Moscow, 1974; English transl., North-Holland, Amsterdam, 1979.
21. R. E. Kalman, *The theory of optimal control and the calculus of variations*, Mathematical Optimization Techniques (R. Bellman, ed.), University of California Press, Berkeley, CA, 1963, pp. 309–331.
22. L. D. Landau and E. M. Lifshits, *Field theory*, Gostekhizdat, Moscow-Leningrad, 1948; English transl., *The classical theory of fields*, Addison-Wesley, Cambridge, MA, 1951.

23. M. A. Lavrent'yev and L. A. Lyusternik, *A course in the calculus of variations*, Gostekhizdat, Moscow-Leningrad, 1950. (Russian)
24. E. B. Lee and L. Markus, *Foundations of optimal control*, Wiley, New York, 1967.
25. E. S. Levitin, A. A. Milyutin, and N. P. Osmolovskii, *Higher order conditions for a local minimum in the problems with constraints*, *Uspekhi Mat. Nauk* **33** (1978), no. 6, 85-148; English transl., *Russian Math. Surveys* **33** (1978), no. 6, 97-168.
26. A. A. Milyutin, A. E. Ilyutovich, N. P. Osmolovskii, and S. V. Chukanov, *Optimal control in linear systems*, "Nauka", Moscow, 1993. (Russian)
27. L. W. Neustadt, *Optimization: A theory of necessary conditions*, Princeton University Press, Princeton, NJ, 1975.
28. N. P. Osmolovskii, *Theory of higher order conditions in optimal control*, Doctor of Sci. Thesis, Moscow, 1988. (Russian)
29. ———, *Higher order necessary and sufficient conditions for Pontryagin and bounded-strong minima in an optimal control problem*, *Dokl. Akad. Nauk SSSR, Ser. Cybernetics and Control theory* **303** (1988), no. 5, 1052-1056; English transl., *Sov. Phys. Dokl.* **33** (1988), no. 12, 883-885.
30. ———, *Quadratic conditions for nonsingular extremals in optimal control (a theoretical treatment)*, *Russian J. Math. Physics* **2** (1995), no. 4, 487-516.
31. L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, *The mathematical theory of optimal processes*, Fizmatgiz, Moscow, 1961; English transl., Pergamon Press, New York, 1964.
32. F. L. Chernous'ko, L. D. Akulenko, and B. N. Sokolov, *Control of oscillations*, "Nauka", Moscow, 1980. (Russian)
33. J. Varga, *Optimal control of differential and functional equations*, Academic Press, New York, 1972.
34. L. C. Young, *Lectures on the calculus of variations and optimal control theory*, Saunders, Philadelphia, PA, 1969.

ISBN 0-8218-0753-6



9 780821 807538

AMS *on the Web*
www.ams.org