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Algebraic Topology: An Intuitive Approach

Hajime Sato

American Mathematical Society

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Volume 183

Algebraic Topology: An Intuitive Approach

Hajime Sato

Translated by
Kiki Hudson

American Mathematical Society
Providence, Rhode Ilsand

Editorial Board

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ISŌ KIKA
(ALGEBRAIC TOPOLOGY)

by Hajime Sato

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ABSTRACT. This book develops an introduction to algebraic topology mainly through simple examples built on cell complexes. The topics covered include homeomorphisms, homotopy equivalences, the torus, the Möbius strip, closed surfaces, the Klein bottle, cell complexes, fundamental groups, homotopy groups, homology groups, cohomology groups, fiber bundles, vector bundles, spectral sequences, and characteristic classes.

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Preface

Topology forms a branch of geometry emphasizing connectedness as the most fundamental aspect of a geometrical object. In topology, therefore, one ignores virtually all geometrical traits other than connectedness, such as any form of change in a geometrical object that stretching or shrinking might cause. Classification in topology is a crude tool, but one that never fails to determine if a geometrical object is connected or not. If a geometrical object is connected then we investigate to what degree it is connected. Just as the state of connectedness characterizes the essence of many phenomena we encounter in our daily lives, it is often necessary to describe to what extent a certain object is connected or separated. Thus the terms one employs in topology are increasingly becoming important and useful in other branches of mathematics as well as in various fields in the natural sciences.

There are numerous algebraic topology books and many of them are excellent; yet we have dared to add another book on this subject. The single most difficult thing one faces when one begins to learn a new branch of mathematics is to get a feel for the mathematical sense of this subject. To somebody who has mastered the subject this essential common sense should be as familiar as the air around him. It takes a long time for a beginner to get to this point. The purpose of this book is to help an aspiring first–time reader acquire this topological atmosphere in a short period of time.

I believe that the most efficient way to fulfill this purpose is to investigate simple but meaningful examples in some concrete terms. It is important that the reader grasp a mathematical object with his or her own hands. By touching it one can feel its physical quality and then keep this as one's own. This book is a simple manual that the reader can follow, and in fact the reader who follows our instructions step by step will end up with a real working model of algebraic topology.

In order to pursue this objective we have therefore sacrificed generality and limited the objects of our discussion to the simplest but most essential cases. We did not try to expand the theory to its fullest extent to make our book an encyclopedic reference; instead, we use the easiest possible examples to help the reader see the backbone of our discussion.

We will be greatly pleased if the reader enjoys reading our book while acquiring several essential methods or approaches to discuss algebraic topology. We must await the reaction of the reader to see if our plan will succeed. We will appreciate it if the reader gives us any feedback (criticisms and comments)[1].

The basic framework of the book comes from the seminar notes "Practical Topology for Physicists" given by Akihiro Tsuchiya and compiled by Yasuhiko Yamada at the University of Nagoya in 1986. I am deeply indebted to Mr. Tsuchiya for permitting me to use his seminar notes as well as for giving me much useful advice throughout every stage of the writing. My thanks also go to Tadayoshi Mizutani, Tetsuya Ozawa, Yoshinori Machida, and Shigeo Ichiraku, who not only read the entire manuscript carefully, finding many mistakes, but also suggested various ways to improve the final product. Last but not least, I would like to thank the editors at Iwanami Shoten.

Hajime Sato
July 1996

[1] See Preface to the English Translation

Preface to the English Translation

It is a great pleasure to me that the American Mathematical Society chose to publish my book "Algebraic Topology: An Intuitive Approach" in their translation series.

Since the publication of the original version of this book in 1996, several of my friends (including the translator) have complained that the gap between my claim that *no previous knowledge of mathematics is required...* and the actual contents of the book is too big. So I have provided the reader who has no knowledge of sets, topology, groups, *etc.* with a basic minimal list of definitions and results that may prove useful, together with readable references. This is in the Appendix at the end of the book. This does not really change my original view that the book is readable for anybody who wishes to find out about algebraic topology. I think that technical terms help both the reader and the author organize their thoughts, but they will not do much good unless both the reader and the author have "good vibes" about the subject. I have also used the book for my topology seminar (for seniors) and came to see that the reading got a little rough toward the end of the book. This is all right too, since it simply shows that good vibes alone cannot conquer everything; however, I have modified some of those troublesome spots, filling in missing links and so on.

I am grateful to the translator, Kiki Hudson, for conveying my writing style and philosophy as faithfully as possible in her translation. We discussed all the changes verbally, and consequently she had to do more writing than translating. This is especially so with the Appendix. I would also like to thank Martin Guest for valuable suggestions, Yoshinori Machida for spotting numerous typos, and the AMS editors for presenting the book in splendid style.

Hajime Sato

September 1998

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Objectives

As I stated in the Preface, in topology we investigate one aspect of geometrical objects almost exclusively of the others: that is, whether a given geometrical object is connected or not connected. We classify objects according to the nature of their connectedness. One focuses on the connectivity, ignoring changes caused by stretching or shrinking.

One can measure the length of a geometrical object in meters and the weight in kilograms. How do we measure the extent to which a geometrical object is connected? Can we develop a system with suitable units and numbered scales?

For example, we can use the number of holes in a geometrical object. But then what is a hole and how do we count the number of holes? In this book, you will find a mathematical interpretation of these concepts, termed "homotopy groups","homology groups", and "cohomology groups". These are some of the major concerns in algebraic topology. We actually go beyond counting the number of holes and develop "characteristic classes" to describe how a geometrical object bends globally. Intuitively the "i-th homotopy group" describes the "i-dimensional *round holes*" and "i-th homology group" reveals the number of "i-dimensional *rooms*" in a geometrical object.

In the problem described above, which may appear to be too slippery to grasp, it would be nice if the reader would come to understand and appreciate how contemporary mathematics has constructed the theory of algebraic topology, translating geometrical concepts into algebraic terms. It has managed to express these problems cleanly and algebraically in group-theoretical terms (involving almost only the additive group of integers or cyclic groups of integers modulo prime numbers). I want the reader to spend a few minutes before beginning the book imagining the problem of classifying geometrical objects only with a yardstick that measures their connectedness. Then after finishing the book the reader should compare its contents with this original concept. If the concept and reality are far apart you will have opened a door to a brave new world, and if they are rather close your mathematical intuition will have proved to be excellent (and you will continue to go on the right track with conviction).

If you already have any familiarity with algebraic topology, you might rightly guess from the table of contents that the following are the key words in the book:

homeomorphisms, homotopy equivalences, torus, Möbius strip, closed surfaces, Klein bottle, cell complexes, fundamental groups, homotopy groups, homology groups, cohomology groups, fiber bundles, vector bundles, spectral sequences, characteristic classes, etc.

If you have seen some (or all) of these words somewhere before and they have vaguely interested you, then you will find upon finishing the book that they are not difficult at all but that they form some of the basic concepts in contemporary mathematics. If you have had nothing to do with them so far, I hope that the strange sound they make intrigues you enough to start the book.

Topology has developed (perhaps unintentionally) on the strength of several attractive geometrical figures which serve as characteristic examples for the theory. This pattern may not be unique in topology; we may see it repeated in other branches of mathematics and possibly in every other academic discipline.

I emphasize again that the purpose of this book is to familiarize the reader with the way to think about algebraic topology. I use the axiomatic approach to introduce homology and cohomology theories, and will later construct concrete examples such as simplicial homology groups, as I feel that this order might work better to sharpen the reader's intuitive understanding.

Needless to say, algebraic topology evolved from general topology (the theory of topological spaces). If you have already studied general topology (especially its geometrical aspects), for instance if you have read Chapters from I to XI in *Topology* of James Dugundji[2], you will be ideally prepared; however, I have tried to keep my explanation basically intuitive so that even readers with no previous knowledge of general topology will be able to follow the book.

The reader might feel a need for the theory of groups, but essentially all you need in order to read this book is to understand the following two concepts:

---

[2] *Topology* by James Dugundji, William C. Brown, 1989

(1) The addition or subtraction of two integers gives another integer (we say that the set $\mathbb{Z}$ of the integers is an additive group).

(2) In certain situations, we regard two integers which differ by a fixed prime number p to be equal (we say that we consider integers $\bmod p$). We write $\mathbb{Z}_p$ for the set of the integers $\bmod p$. The addition and subtraction of integers carry over to those operations $\bmod p$ (we say that $\bmod p$ is a cyclic group of order p).

The only talent this book demands of the reader is a flexible and resilient mind.

List of Symbols

Symbol	Meaning	Page
$f_0 \simeq f_1$	homotopic	3
$[X, Y]$	homotopy set	4
$X \simeq Y$	X and Y have the same homotopy type	4
D^n	n-dimensional ball	9
S^{n-1}	$(n-1)$-dimensional sphere	9
I	closed unit interval $[0, 1]$	10
$P^n(\mathbb{R})$	n-dimensional real projective plane	11
e^i	(open) i-cell	13
$\bar{e}^i$	closed i-cell	13
$\pi_n(X, x_0)$	n-th homotopy group of X	25
$\pi_n(X)$	n-th homotopy group of X	25
$h_p(X)$	p-th homology group of X	31
$h_*(X)$	direct sum $\sum_{p=0}^{\infty} h_p(X)$ of $h_p(X)$	31
pt	singleton set	32
$H_p(X; G)$	$h_p(X)$ for $h_0(X) \cong G$	32
$H_*(X; G)$	direct sum $\sum_{p=0}^{\infty}$ of $H_p(X; G)$	32
CA	cone over A	36
$\tilde{h}_*(X)$	reduced homology of group X	38
C	chain complex	45
$Z_p(C)$	group of p-cycles	45
$B_p(C)$	group of p-boundaries	45
$\sigma^j \prec \sigma^n$	simplex σ^j belongs to the boundary of σ^n (σ^j is a face of σ^n that is different from σ^n)	48
$C_q(\mathcal{S}; \mathbb{Z})$	q-th chain group of $\mathcal{S}$ over $\mathbb{Z}$	52
$H_q(\mathcal{S}; \mathbb{Z})$	q-th homology group of $\mathcal{S}$ over $\mathbb{Z}$	53
$P^n(\mathbb{C})$	n-dimensional complex projective space	56
$h^p(X)$	p-th cohomology group of X	59
$\mathcal{S}$	simplicial complex	49

Symbol	Meaning	Page
$h^*(X)$	direct sum $\sum_{p=0}^{\infty} h^p(X)$ of $h^p(X)$	59
δ^p, δ	coboundary homomorphism	60
$C^q(\mathcal{S}; G)$	q-th cohomology chain of $\mathcal{S}$ over G	60
$C^*(\mathcal{S}; G)$	cochain complex of $\mathcal{S}$ over G	61
$H^q(\mathcal{S}; G)$	q-th cohomology group of $\mathcal{S}$ over G	61
$Z^q(\mathcal{S}; G)$	group of q-cochains of $\mathcal{S}$ over G	61
$B^q(\mathcal{S}; G)$	group of q-coboundaries of $\mathcal{S}$ over G	61
$G_1 \otimes G_2$	tensor product	
$\mathrm{Hom}(G_1, G_2)$	abelian group of homomorphisms from G_1 to G_2	66
$\mathrm{Tor}(G_1, G_2)$	torsion	66
$\mathrm{Ext}(G_1, G_2)$	abelian group of the extensions of G_2 by G_1	67
$\times$	cross product	68
$\triangle$	diagonal map	69
$\cup$	cup product	69
(E, π, B, F)	fiber bundle	74
$F \to E \xrightarrow{\pi} B$	fiber bundle	74
E	total space	74
B	base space	74
F	fiber	74
π	projection	74
$G^{\mathbb{R}}(m, n)$	real Grassmannian manifold	80
$G^{\mathbb{C}}(m, n)$	complex Grassmannian manifold	80
$BO(n)$	classifying space of real n-vector bundles	83
$BU(n)$	classifying space of complex n-vector bundles	83
$\mathrm{Lk}(\sigma, \mathcal{S})$	link complex of σ in $\mathcal{S}$	106

Appendix

This chapter offers a short list of basic definitions and results at an introductory level, which will help the reader start our book.

Sets

We will not give a rigorous definition of sets. A *set* is a collection of items. The items in a set are its *elements* or members. We often denote a set by a capital letter and list its elements by lower case letters within braces. For instance, $A = \{a, b, c\}$ represents a set consisting of the elements a, b and c. Suppose $P(x)$ is some statement about x. Then we write $\{x|\, P(x)\}$ to represent the set of all x for which $P(x)$ is valid. For instance $\mathbb{Z}_e = \{x|\, x \text{ is an even integer}\}$ says that $\mathbb{Z}_e$ is the set of the even integers; we can also write $\mathbb{Z}_e = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$. If x belongs to a set A, we write $x \in A$. If x is not in A, we write $x \notin A$.

DEFINITION 1. Let X be a set. A set Y is a subset of X (we write $Y \subset X$ or $X \supset Y$) if every $x \in Y$ satisfies $x \in X$. If $X \subset Y$ and $Y \subset X$ we write $X = Y$ and say that they are the same.

If $Q(x)$ is some statement about x, then $S = \{x \in X \,|\, Q(x)\}$ reads "S is the subset of X for which $Q(x)$ is true". If a set is empty we call it the empty set (all empty sets are equal) and denote it by $\emptyset$. We say that X is finite if it contains only finitely many elements.

When A is an infinite set we often use an *index set* Λ to label its elements. For instance we can write $\{a_i\}_{i \in \mathbb{N}}$ instead of $\{a_1, a_2, a_3, \ldots, \}$. Here $\mathbb{N}$ is the set of the natural numbers. An indexing set is most likely to appear when one defines a set whose elements are sets (in this case we use the expression *a family of sets*).

DEFINITION 2. Let A and B be subsets of a set X (We often ignore the set X and pretend A and B are just sets). We define the union $A \cup B$, the intersection $A \cap B$ and the cartesian product $A \times B$ of A and B by

$$A \cup B = \{ x \in X | x \in A \text{ or } x \in B \},$$
$$A \cap B = \{ x \in X | x \in A \text{ and } x \in B \},$$
$$A \times B = \{ (x, y) \,|\, x \in X, y \in B \}.$$

Two sets A and B are *disjoint* when $A \cap B = \emptyset$.

Evidently the operations $\cup$ and $\cap$ satisfy the following properties for any sets A, B and C.
(1) $A \cup A = A \cap A = A$.
(2) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
(3) $(A \cup B) \cup C = A \cup (B \cup C)$; $(A \cap B) \cap B = A \cap (B \cap C)$ (associativity).

Because of the associativity property we can denote by $A_1 \cup A_2 \cup \cdots \cup A_n$ the union of n sets A_i. We write this more briefly as $\bigcup_i A_i$, $i = 1, 2, \ldots, n$. Similarly, $\bigcap_i A_i$ denotes the intersection of n sets A_i.

The cartesian product $\times$ satisfies the associativity property: $A \times (B \times C) = (A \times B) \times C$. It is also distributive over the union and the intersection.

We can generalize these operations over a family of sets indexed by a set Λ.

If $B \subset A$ then the complement $A - B$ of B in A is defined by $A - B = \{ x \in A \,|\, x \notin B \}$. Sometimes we just write $-B$.

DEFINITION 3. A map f from A to B, written as $f : A \to B$, is a subset f of $A \times B$ with the properties: (1) for each $x \in X$ there is $y \in Y$ such that $(x, y) \in f$; (2) if (x, y) and (x, y') are both in f, then $y = y'$.

A map $f : A \to B$ is an *injection* (we also say that f is one-to-one) if $f(x) = f(y)$ implies $x = y$. A map $f : A \to B$ is a surjection (f is onto) if for every $y \in B$ there is $x \in A$ with $f(x) = y$. If $f : A \to B$ is both injective and surjective we say that it is a *bijection*, or a bijective map.

We have the identity map $I_A : A \to A$ of A defined by $I_A(x) = x$, $x \in A$. In particular, if A is a subset of B then we have the inclusion (map) $i : A \to B$ defined by $i(x) = x$, $x \in A$. The inclusion (map) i is the identity map I_A of A if we choose to ignore the set B.

By the inverse image of a subset $B' \subset A$ by f we mean the subset $\{x \in A \,|\, f(x) \in B' \}$. With an abuse of notation we indicate this set by $f^{-1}(B)$.

The *composite* of maps $f : A \to B$ and $g : B \to C$ is a map $g \circ f : A \to C$ defined by $g \circ f(x) = g(f(x))$ (check that this makes sense).

If A' is a subset of A, then the composite $f \circ i : A' \to B$ of $i : A' \to A$ and $f : A \to B$ is the *restriction* of f to A'. We denote this restriction by $f|_{A'}$.

DEFINITION 4. We say $\sim$ is a binary relation on a set A if for any two elements a and b (in this order) of A, a is related to b by $\sim$ (we write this as $a \sim b$) or a is not related to b by $\sim$ ($a \nsim b$). A binary relation $\sim$ in A is an equivalence relation if it satisfies the following properties:

(1) $a \sim a$ for every $a \in A$ (reflexivity).
(2) $a \sim b$ implies $b \sim a$ for $a, b \in A$ (symmetry).
(3) $a \sim b$ and $b \sim c$ implies $a \sim c$ for $a, b, c \in A$ (transitivity).

Let $\sim$ be an equivalence relation on A. For an element $a \in A$ we denote by $[a]$ the subset $\{x \in A \,|\, x \sim a\}$. Then we say that $[a]$ is the *equivalence class* of a with respect to $\sim$ and that a (or any element in $[a]$) is a *representative* of $[a]$. We denote by $A/\sim$ (read "A modulo tilde") the set of all equivalence classes of A with respect to A and say that it is the *quotient set* of A (with respect to $\sim$). Each element of $A/\sim$ is some subset of A, and if $[a] \neq [b]$ then $[a] \cap [b] = \emptyset$. Moreover the union of the equivalence classes of A is equal to A. Note that the number of equivalence classes may not be finite and in that case we must consider an infinite union. But we do not go into a theoretical discussion of this nature. Let's say that we only consider the circumstance where this type of union exists. We define the projection $\pi : A \to A/\sim$ by $\pi(x) = [x]$ (verify that this is well–defined).

The set of integers mod p, denoted by $\mathbb{Z}_p$, p a prime number, is the quotient set of $\mathbb{Z}$ by the equivalence relation $\sim$: $a \sim b \Leftrightarrow a - b$ is divisible by p.

Topological spaces and continuous maps

DEFINITION 5. Let X be a set. A topology in X is a family $\mathcal{U}$ of subsets of X, which we call *open sets*, satisfying the following properties:

(1) A union of elements of $\mathcal{U}$ is again an element of $\mathcal{U}$.
(2) A finite intersection of elements of $\mathcal{U}$ is again an element of $\mathcal{U}$.
(3) The empty set $\emptyset$ and X both belong to $\mathcal{U}$.

In terms of open sets we can rephrase these properties:
(1) A union of open sets is an open set.
(2) A finite intersection of open sets is an open set.
(3) $\emptyset$ and X are both open sets.

We say that X is a *topological space* whenever X has some topology $\mathcal{U}$ defined in it. We often write $(X, \mathcal{U})$ to indicate that X is a topological space with topology $\mathcal{U}$.

DEFINITION 6. In a topological space $(X, \mathcal{U})$ a subset $Y \subset X$ is *closed* in X if $-Y$ is open; that is, if $-Y \in \mathcal{U}$.

The closed sets of a topological space $(X, \mathcal{U})$ satisfy the following properties:
(1′) The intersection of closed sets is a closed set.
(2′) The union of finitely many closed set is a closed set.
(3′) X and $\emptyset$ are both closed sets.

DEFINITION 7. Let $(X, \mathcal{U})$ be a topological space and let $x \in X$. We say that $U \in \mathcal{U}$ is a *neighborhood* of x if $x \in U$.

Let $(X, \mathcal{U})$ be a topological space. Let Y be a subset of X. Set

$$\mathcal{U}_Y = \{ U \cap Y \mid U \in \mathcal{U} \}.$$

It is easy to check that $\mathcal{U}_Y$ is a topology in Y, making Y a topological space. We say that $\mathcal{U}_Y$ is the *relative topology* on Y with respect to $\mathcal{U}$ and that $(Y, \mathcal{U}_Y)$ is a *topological subspace* (subspace, for short) of X.

DEFINITION 8. Let X and Y be topological spaces. We say that a map $f : X \to Y$ is *continuous* if the inverse image of an arbitrary open set in Y is open in X.

Suppose X is a topological space and $f : X \to Y$ is a surjection (here Y is a set). Then

$$\mathcal{V} = \{ V \subset Y \mid f^{-1}(V) \text{ is an open subset of } X \}$$

defines the *identification topology* or *quotient topology* on Y.

In particular, if $\sim$ is an equivalence relation on a topological space X, then the projection of X to $X/\sim$ is an onto map. So $X/\sim$ becomes a topological space with the quotient topology with respect to $\sim$. We then say that $X/\sim$ is the *quotient space*

THEOREM 9. *In the above setting $f : X \to Y$ is continuous. Moreover, if we give Y another topology $\mathcal{V}'$ for which f is continuous, then $\mathcal{V}' \subset \mathcal{V}$.*

In other words, the quotient topology is the strongest (largest) topology of Y such that $f : X \to Y$ is continuous.

Let $(Y, \mathcal{V})$ be a topological space. Let $f : X \to Y$ be a map from a set X to Y. Then the family $\{ f^{-1}(V) \mid V \in \mathcal{V} \}$ of subsets of X defines a topology on X, called the *topology induced by* $f : X \to Y$. Evidently $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is continuous. Moreover, if $f : (X, \mathcal{U}') \to (Y, \mathcal{V})$ is continuous with respect to any other topology $\mathcal{U}'$ on X then $\mathcal{U} \subset \mathcal{U}'$. In other words, the topology on X induced by $f : X \to Y$ is the *weakest* (smallest) topology on X making f continuous.

Let X_1 and X_2 be topological spaces and consider their set product $X_1 \times X_2$. Let $\pi_i : X_1 \times X_2 \to X_i$, $i = 1, 2$, be the projections. Then there is a smallest topology on $X_1 \times X_2$ making both π_1 and π_2 continuous. This topology is called the product topology (also called the weak topology) on the *product space* $X_1 \times X_2$ (the same name as before, but notice that this time it has become a topological space). This generalizes to the product of spaces indexed by an index set Λ.

DEFINITION 10. A topological space X is connected if it is not a union of two open sets.

A subset of a topological space is connected if it is connected as a subspace (with respect to the relative topology).

THEOREM 11. *A space X is connected if and only if the only subspaces of X that are both open and closed are the empty set $\emptyset$ and X.*

Groups

Suppose that a set G satisfies the following properties: To every pair a, b of elements of G there corresponds a third element $a \cdot b$, in such a way that

(1) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (associativity),

(2) there exists an element ι in G such that $a \cdot \iota = \iota \cdot a$ for every $a \in G$ (ι is the identity element of G),

(3) to every element $a \in G$, there corresponds a unique element $\tilde{a}$ such that $a \cdot \tilde{a} = \tilde{a} \cdot a = \iota$ (every a in G has its inverse element $\tilde{a}$).

Then we say that G is a *group*.

If G satisfies $a \cdot b = b \cdot a$ as well, we say that the G is *abelian* or *commutative*.

Question: what is purple and commutes? Answer: an abelian grape.

EXAMPLE 12. Let $\mathbb{Z}$ be the set of the integers with the usual addition, $+$. Then $(\mathbb{Z}, +)$ is an abelian group (we usually say an additive group). The identity element is 0 and the inverse of a is $-a$, of course. From now on $\mathbb{Z}$ implies this group structure.

On the other hand, $\mathbb{Z}$ with the usual multiplication fails to satisfy property (3), and so it is not a group.

The quotient set $G/\sim$ of a group G with respect to an equivalence relation $\sim$ inherits the group structure of G: $[a] \cdot [b] = [a \cdot b]$. One must show that this operation does not depend on the choice of representatives (easy). We say that $G/\sim$ is the quotient group of G with respect to $\sim$.

A subset H of a group G is a *subgroup* of G if H is *closed* under the group operation of G ($ab \in H$ for any a, $b \in H$).

Let $(G_1, +)$ and $(G_2, +)$ be abelian groups. A map $\phi : G_1 \to G_2$ is a (group) *homomorphism* if $\phi(a+b) = \phi(a) + \phi(b)$ for all a, b in G_1 (the operation $+$ on the left–hand side is for G_1, and that on the right–hand side is for G_2).

For a homomorphism $\phi : G_1 \to G_2$ we use the following notation:

$$\ker\phi = \{\, x \in G_1 \mid \phi(x) = 0 \,\},$$
$$\operatorname{im}\phi = \{\, y \in G_2 \mid \phi(x) = y \text{ for some } x \in G_1 \,\}.$$

Then $\ker\phi$ (read the *kernel* of ϕ) is a subgroup of G_1, and $\operatorname{im}\phi$ (read the *image* of G_2 under ϕ, or simply the image ϕ) is a subgroup of G_2.

The homomorphism $\phi : G_1 \to G_2$ is a monomorphism if $\ker\phi = \emptyset$ or equivalently if ϕ is one–to–one, and ϕ is an epimorphism if $\operatorname{im}\phi = G_2$ or equivalently if ϕ is onto. If ϕ is a monomorphism and an epimorphism then it is an *isomorphism*.

Answers to Exercises

CHAPTER ONE

1.1. Show that both are homeomorphic to the letter I.

1.2. Let $f : \mathsf{P} \to \mathsf{R}$ be the identity map of P as the subspace of R. Let $g : \mathsf{R} \to \mathsf{P}$ be the map which send the subspace P of R onto P (as the identity map of P) and the leg of R to the joint of P. Then $g \circ f = id : \mathsf{P} \to \mathsf{P}$ and so it is enough to show that $f \circ g : \mathsf{R} \to \mathsf{P} \subset \mathsf{P}$ is homotopic to the identity map. We can construct a homotopy by pulling a leg continuously out of the joint.

1.3. Extend the homeomorphism between $\triangle$ and D to a homeomorphism from A onto R.

CHAPTER TWO

2.1. This quotient space is homeomorphic to the quotient space of $\triangle = \{(x, y) \in I^2 | \, y \le x\}$ in which the points on each of the three boundary segments are identified with respect to its center. The latter space is homeomorphic to S^2.

2.2. If one opens up the Möbius band along its latitudinal center line one gets a band that is homeomorphic to $I \times S^1$. By resewing the cut we get the suggested attaching map.

2.3. We get a space homeomorphic to a square by opening up the double torus along a suitable set of four loops joined at a single point. Therefore, the double torus has a cell division consisting of one 0-cell, four 1-cells and one 2-cell. More generally, the n-ple torus has a cell division of one 0-cell, $2n$ 1-cells and one 2-cell.

CHAPTER THREE

3.1. Given a map of $(I^n, \partial I^n)$ into (S^k, x_0), we can choose a point $x_1 \in S^k$ and change the map by a small homotopy so that its image misses $x_1 \, (\neq x_0) \in S^k$. Now $S^k - x_1$ is homeomorphic to the interior of D^k. Follow the proof of $\pi_n(D^k) = 0$.

3.2. $\mathbb{Z}_2$.

3.3. The group generated by $\alpha_1, \beta_1, \alpha_2, \beta_2$ with the relation

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1}.$$

CHAPTER FOUR

4.1. For $p \neq q$, $h^0(S^p \vee S^q) \cong G$, $h^p(S^p \vee S^q) \cong G$, $h^q(S^p \vee S^q) \cong G$; all others are 0. For $p = q$, $h^p(S^p \vee S^q) \cong G$, $h^p(S^p \vee S^q) \cong G \oplus G$; all others are 0.

CHAPTER FIVE

5.1.

$$H_j(S^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & j = 0, n, \\ 0, & \text{otherwise}; \end{cases} \quad H_j(D^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & j = 0, \\ 0, & \text{otherwise}. \end{cases}$$

5.2. From the Mayer–Vietoris sequence for $T^2 = T_0^2 \cup_{S^1} D^2$ we get the homology groups of T_0^2 (the robot's glove):

$$H_0(T_0^2;\mathbb{Z}) \cong \mathbb{Z}, \quad H_1(T_0^2;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}, \quad H_i(T_0^2;\mathbb{Z}) = 0, \quad i \geq 2.$$

Further, we use the Mayer–Vietoris sequence for

$$M_2 = T_0^2 \cup_{S^1} T_0^2$$

to obtain

$$H_0(M_2;\mathbb{Z}) \cong \mathbb{Z},\ H_1(M_2;\mathbb{Z}) \cong \mathbb{Z}^4 = \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z},\ H_2(M_2;\mathbb{Z}) \cong \mathbb{Z}.$$

5.3. Divide M_n into one 0-cell, $2n$ 1-cells and one 2-cell. Then the boundary operators are all zero maps. Therefore, we get

$$H_0(M_n;\mathbb{Z}) \cong \mathbb{Z},\ H_1(M_n;\mathbb{Z}) \cong \mathbb{Z}^{2n} = \overbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}^{2n \text{ copies}},\ H_2(M_n;\mathbb{Z}) \cong \mathbb{Z}.$$

CHAPTER SIX

6.1. $H_0(T^2;\mathbb{Z}) \cong \mathbb{Z}$, $H^1(T^2;\mathbb{Z}) \cong \mathbb{Z}^2 = \mathbb{Z} \oplus \mathbb{Z}$, $H^2(T^2;\mathbb{Z}) \cong \mathbb{Z}$.

6.2. $H^0(P^2(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}$, $H^1(P^2(\mathbb{R});\mathbb{Z}) = 0$, $H^2(P^2(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}_2$.

CHAPTER SEVEN

7.1. $H_0(P^2(\mathbb{R}) \times P^2(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}$, $H_1(P^2(\mathbb{R}) \times P^2(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $H_2(P^2(\mathbb{R}) \times P^2(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}_2$, $H_3(P^2(\mathbb{R}) \times P^2(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}_2$, $H_4(P^2(\mathbb{R}) \times P^2(\mathbb{R});\mathbb{Z}) = 0$.

7.2. $H^0(P^2(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}$, $H^1(P^2(\mathbb{R});\mathbb{Z}) = 0$, $H_2(P^2(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}_2$.

7.3. $H^0(P^2(\mathbb{R}) \times P^2(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}$, $H^1(P^2(\mathbb{R}) \times P^2(\mathbb{R});\mathbb{Z}) = 0$, $H^2(P^2(\mathbb{R}) \times P^2(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $H^3(P^2(\mathbb{R}) \times P^2(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}_2$, $H^4(P^2(\mathbb{R}) \times P^2(\mathbb{R});\mathbb{Z}) \cong \mathbb{Z}_2$.

7.4. $f^* : H^2(S^2;\mathbb{Z}) \cong \mathbb{Z} \to H^2(S^2 \vee S^4;\mathbb{Z}) \cong \mathbb{Z}$ is an isomorphism. Hence, an arbitrary element a_i of $H^2(S^2 \vee S^4;\mathbb{Z})$ is of the form $a_i = f^*(\hat{a}_i)$, $\hat{a}_i \in H^2(S^2;\mathbb{Z})$. Since $\hat{a}_1 \cup \hat{a}_2 \in H^4(S^2;\mathbb{Z}) = 0$, it follows that $a_1 \cup a_2 = f^*(\hat{a}_1) \cup f^*(\hat{a}_2) = f^*(\hat{a}_1 \cup \hat{a}_2) = 0$.

7.5. $H_0(P^2(\mathbb{R});\mathbb{Z}_2) \cong \mathbb{Z}_2$, $H_1(P^2(\mathbb{R});\mathbb{Z}_2) \cong \mathbb{Z}_2$, $H_2(P^2(\mathbb{R});\mathbb{Z}_2) \cong \mathbb{Z}_2$.

7.6. $H^0(P^2(\mathbb{R});\mathbb{Z}_2) \cong \mathbb{Z}_2$, $H^1(P^2(\mathbb{R});\mathbb{Z}_2) \cong \mathbb{Z}_2$, $H^2(P^2(\mathbb{R});\mathbb{Z}_2) \cong \mathbb{Z}_2$, $H^2(P^2;\mathbb{Z}_2) \cong \mathbb{Z}_2$.

CHAPTER EIGHT

8.1. The natural projection $\pi : I \times S^1 \to I$ becomes the projection of the Klein bottle onto S^1. The local triviality is obvious, and we have the fiber bundle over S^1 with the fiber S^1 whose total space is the Klein bottle.

8.2. There is a one-to-one correspondence between the real Grassmannian manifold $G^{\mathbb{R}}$ and S^1 (the real numbers $\mathbb{R}$ plus the point at infinity). Similarly, each complex line through the origin in the complex plane has the slope that corresponds to a complex number (including the point infinity). The space $\mathbb{C}$ of complex numbers plus the point ∞ is S^2.

CHAPTER NINE

9.1. The spectral sequence collapses and we have

$$H_0(E;\mathbb{Z}) \cong \mathbb{Z},\ H_2(E;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z},\ H_4(E;\mathbb{Z}) \cong \mathbb{Z}.$$

All others are zero.

9.2. See the discussion in §9.3.

9.3. As in §9.3, we show that the sequence

$$\cdots \to H^{p+j}(E;\mathbb{R}) \to E_2^{p,j} \xrightarrow{d_{j+1}} E_2^{p+j+1,0} \xrightarrow{\pi^*} H^{p+j+1}(E;\mathbb{R}) \to \cdots$$

is exact. We can do this by setting $\Omega \equiv d_{n+1}(1)$, $1 \in H^0(B;\mathbb{R}) \cong \mathbb{R}$, to show that $\Psi(u) = (-1)^p d_{j+1}(u)$ just as in §9.5, and then by sliding the sign factor one term over.

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Recommended Reading

SET THEORY AND ALGEBRA

Essentially any undergraduate text will do. Some of the classics are the following:

[1] Birkhoff, G. and Mac Lane, S., *A Survey of Modern Algebra*, A. K. Peters, 1997.

[2] Halmos, P., *Naive Set Theory*, Undergraduate texts in mathematics, Springer–Verlag New York, 1987.

[3] Lang, S., *Algebra*, Advanced Book Program, Addison–Wesley Publishing Company, 1984.

POINT SET TOPOLOGY

[4] Dugundji, J, *Topology*, William C. Brown, 1989.

[5] Kelley, J., *General Topology*, Graduate texts in mathematics; 27, Springer–Verlag, New York and Tokyo, 1975.

ALGEBRAIC TOPOLOGY

[6] Fulton, W., *Algebraic Topology, A First Course*, Graduate texts in mathematics, Springer–Verlag New York, 1995.

[7] Greenberg, F. and Harper, R., *Algebraic Topology, A First Course*, Mathematics lecture notes series, vol. 58, Benjamin/Cummings Pub. Co., 1981.

[8] Massey, W., *A Basic Course in Algebraic Topology*, Graduate texts in mathematics; 127, Springer–Verlag New York, 1991.

[9] Spanier, E., *Algebraic Topology*, McGraw–Hill, New York, 1966.

[10] Whitehead, G., *Elements of Homotopy Theory*, Graduate texts in mathematics; 61, Springer–Verlag, 1978.

MANIFOLDS, LIE GROUPS, CHARACTERISTIC CLASSES

[11] Guillemin, V. and Pollack A., *Differential Topology*, Prentice Hall, 1974.

[12] Milnor, J. and Stasheff, J., *Characteristic Classes*, Annals of mathematics studies; 76, Princeton University Press, 1974.

[13] Miura, M. and Toda, H., *Topology of Lie Groups*, Translation of mathematical monographs; 91, American Mathematical Society, 1991.
[14] Steenrod, N., *The Topology of Fibre Bundles*, Princeton mathematical series; 14, 1951.

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