Translations of

MATHEMATICAL MONOGRAPHS

Volume 184

Financial Markets

Stochastic Analysis and the Pricing of Derivative Securities

A. V. Mel'nikov

American Mathematical Society

Selected Titles in This Series

184 **A. V. Mel′nikov,** Financial markets, 1999
183 **Hajime Sato,** Algebraic topology: an intuitive approach, 1999
182 **I. S. Krasil′shchik and A. M. Vinogradov, Editors,** Symmetries and conservation laws for differential equations of mathematical physics, 1999
181 **Ya. G. Berkovich and E. M. Zhmud′,** Characters of finite groups. Part 2, 1999
180 **A. A. Milyutin and N. P. Osmolovskii,** Calculus of variations and optimal control, 1998
179 **V. E. Voskresenskiĭ,** Algebraic groups and their birational invariants, 1998
178 **Mitsuo Morimoto,** Analytic functionals on the sphere, 1998
177 **Satoru Igari,** Real analysis—with an introduction to wavelet theory, 1998
176 **L. M. Lerman and Ya. L. Umanskiy,** Four-dimensional integrable Hamiltonian systems with simple singular points (topological aspects), 1998
175 **S. K. Godunov,** Modern aspects of linear algebra, 1998
174 **Ya-Zhe Chen and Lan-Cheng Wu,** Second order elliptic equations and elliptic systems, 1998
173 **Yu. A. Davydov, M. A. Lifshits, and N. V. Smorodina,** Local properties of distributions of stochastic functionals, 1998
172 **Ya. G. Berkovich and E. M. Zhmud′,** Characters of finite groups. Part 1, 1998
171 **E. M. Landis,** Second order equations of elliptic and parabolic type, 1998
170 **Viktor Prasolov and Yuri Solovyev,** Elliptic functions and elliptic integrals, 1997
169 **S. K. Godunov,** Ordinary differential equations with constant coefficient, 1997
168 **Junjiro Noguchi,** Introduction to complex analysis, 1998
167 **Masaya Yamaguti, Masayoshi Hata, and Jun Kigami,** Mathematics of fractals, 1997
166 **Kenji Ueno,** An introduction to algebraic geometry, 1997
165 **V. V. Ishkhanov, B. B. Lur′e, and D. K. Faddeev,** The embedding problem in Galois theory, 1997
164 **E. I. Gordon,** Nonstandard methods in commutative harmonic analysis, 1997
163 **A. Ya. Dorogovtsev, D. S. Silvestrov, A. V. Skorokhod, and M. I. Yadrenko,** Probability theory: Collection of problems, 1997
162 **M. V. Boldin, G. I. Simonova, and Yu. N. Tyurin,** Sign-based methods in linear statistical models, 1997
161 **Michael Blank,** Discreteness and continuity in problems of chaotic dynamics, 1997
160 **V. G. Osmolovskiĭ,** Linear and nonlinear perturbations of the operator div, 1997
159 **S. Ya. Khavinson,** Best approximation by linear superpositions (approximate nomography), 1997
158 **Hideki Omori,** Infinite-dimensional Lie groups, 1997
157 **V. B. Kolmanovskiĭ and L. E. Shaĭkhet,** Control of systems with aftereffect, 1996
156 **V. N. Shevchenko,** Qualitative topics in integer linear programming, 1997
155 **Yu. Safarov and D. Vassiliev,** The asymptotic distribution of eigenvalues of partial differential operators, 1997
154 **V. V. Prasolov and A. B. Sossinsky,** Knots, links, braids and 3-manifolds. An introduction to the new invariants in low-dimensional topology, 1997
153 **S. Kh. Aranson, G. R. Belitsky, and E. V. Zhuzhoma,** Introduction to the qualitative theory of dynamical systems on surfaces, 1996
152 **R. S. Ismagilov,** Representations of infinite-dimensional groups, 1996
151 **S. Yu. Slavyanov,** Asymptotic solutions of the one-dimensional Schrödinger equation, 1996
150 **B. Ya. Levin,** Lectures on entire functions, 1996

(*Continued in the back of this publication*)

This page intentionally left blank

Financial Markets

Stochastic Analysis and the Pricing of Derivative Securities

This page intentionally left blank

10.1090/mmono/184

Translations of
MATHEMATICAL MONOGRAPHS

Volume 184

Financial Markets

Stochastic Analysis and the Pricing of Derivative Securities

A. V. Mel'nikov

American Mathematical Society
Providence, Rhode Island

А. В. Мельников

ФИНАНСОВЫЕ РЫНКИ

«Научное издательство ТВП», Москва, 1997

Translated from the Russian by H. H. McFaden

1991 *Mathematics Subject Classification.* Primary 90–02, 90A12, 90A09;
Secondary 90A60, 60H30, 60G42, 60G40.

ABSTRACT. This is one of the first textbooks in the new subject of "actuarial and financial mathematics". The abandonment of a fixed gold price at the beginning of the 1970s and the opening of the Chicago market in the trading of option contracts, combined with the appearance of the famous papers of Black and Scholes and of Merton on pricing options, led to that rare situation when the opportunity for rapid practical utilization creates a favorable environment for purely theoretical developments (in the area of pricing derivative instruments on a financial market). Due to the powerful effect of contemporary mathematical methods and stochastic analysis, there was a transition from arithmetic to mathematics as a basis for finance. Financial mathematics not only has acquired the outlines of an independent science, but also has found effective and promising applications in financial and insurance markets. This is the direction of the current book, which is a self-contained and sufficiently broad introduction to the general mathematical theory of pricing for derivative securities.

Library of Congress Cataloging-in-Publication Data

Mel'nikov, A. V., 1953-
[Finansovye rynki. English]
Financial markets : stochastic analysis and the pricing of derivative securities / A. V. Mel'nikov ; [translated from the Russian by H. H. McFaden].
p. cm. — (Translations of mathematical monographs; v. 184)
Includes bibliographical references and index.
ISBN 0-8218-1082-0 (alk. paper)
1. Derivative securities–Prices–Mathematical models. 2. Options (Finance)–Prices–Mathematical models. I. Title. II. Series.
HG6024.A3M43813 1999 99-17735
332.64'5–dc21 CIP

---

10 9 8 7 6 5 4 3 2 1 04 03 02 01 00 99

Contents

This page intentionally left blank

Foreword

Financial mathematics is at present going through a period of intensive development, especially in the area connected with contemporary *stochastic analysis*. It is the methods of the general theory of random processes that have turned out to be most suitable for an adequate description of the evolution of *basic* (bonds and stocks) and *derivative* (forwards, futures, options, and so on) securities.

Historically, the first work (1900) in this direction was the dissertation of Bachelier [13], a student of Poincaré who, several years before Einstein and 23 years before Wiener, gave a mathematical definition of the concept of 'Brownian motion', used it to model the dynamics of stock prices, and gave a formula for the investment cost of an option. The main deficiency of Bachelier's model, which was the possible negativity of the stock prices, was removed in 1965 by the well-known economist Samuelson, who proposed a *geometric Brownian motion* for describing these prices. This model now bears the names of Black and Scholes, who in 1973 [15] obtained precise formulas for computing the fair price and hedging strategies for European options in the framework of the model.

Employing the heuristic argument that stock prices are either rising or falling at any moment of time, Cox, Ross, and Rubinstein [19] proposed regarding these changes as *discrete* and introduced a *binomial model* of a financial market. They showed that the binomial model has a geometric Brownian motion 'as a limit', and the formula obtained for a fair price converges to the Black–Scholes formula.

These now-classical papers have become a direct basis for the application and development of methods from contemporary stochastic analysis in the mathematical theory of finance. It is in this direction, with the use of elements of functional analysis and convex analysis, that deep results have been obtained about the structure of prices and about the properties of arbitrage and completeness of a financial market.

The goal of this book is to present, in a sufficiently self-contained form, the methods and results of the contemporary theory of financial computations for a discrete market. It gives a representation of basic techniques in stochastic analysis: martingales, semimartingales, stochastic exponents, Itô's formula, Girsanov's theorem, and so on. The discreteness of the models considered above leads to a whole series of technical simplifications, and often to greater clarity of the results obtained. Yet at the same time, this discrete theory contains in itself many elements of the very complex techniques and problems in the general theory. Therefore, the book can be regarded as a sufficiently broad introduction to the contemporary mathematics of financial computations with derivative securities.

In large part this book is based on the material and approaches expounded in [12], and it represents the content of the course of lectures "Stochastic analysis in finance" given by the author in 1994–1997 in the Mechanics and Mathematics

Department of Moscow State University. This explains its theoretical character and direction.

The author sincerely thanks A. N. Shiryaev, Yu. M. Kabanov, and D. O. Kramkov for many useful discussions and for their help, criticism, and constant support. The book was published at the proposal of the scientific publishing house "TVP". The author is very grateful to V. I. Khokhlov, both for making this proposal and for the amount of work he put into editing the book.

November 22, 1998 *A. V. Mel'nikov*

Notation

$(\Omega, \mathcal{F}, \mathbf{P})$	a probability space
$X, Y, \dots$	random variables (RVs)
$\tau, \sigma, \dots$	stopping times
$(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$	a stochastic base
$\mathcal{O}$	the optional σ-algebra
$\mathcal{P}$	the predictable σ-algebra
$\sigma\{\varepsilon_1, \dots, \varepsilon_n\}$	the σ-algebra generated by the random variables $\varepsilon_1, \dots, \varepsilon_n$
$\mathbb{P}^*$	the set of martingale measures
$[\![\tau, \sigma[\![$	a stochastic interval
$\mathbf{P}\{A\}$	the probability of an event A
$\mathbf{E}\,X$	the mathematical expectation of X
$\mathbf{D}\,X$	the variance of X
$\mathbf{E}\,(X \mid \mathcal{A})$	the conditional mathematical expectation of X with respect to the σ-algebra $\mathcal{A}$
$\mathcal{E}_n(U)$	the stochastic exponential with respect to a sequence U
$\langle M \rangle$	the quadratic characteristic (compensator) of a square-integrable martingale M
$\Phi(x)$	the standard normal distribution
$L_2(\Omega, \mathcal{F}, \mathbf{P})$	the set of random variables X such that $\mathbf{E}\,\lvert X \rvert^2 < \infty$
$\mathbf{P} \sim \mathbf{P}^*$	equivalence of two probability measures
$\mathbf{R}^d$	d-dimensional Euclidean space
$\mathbb{Z}_+$	the set $\{0, 1, \dots\}$ of nonnegative numbers
$\mathbf{I}_A$	the indicator function of a set A
$\varnothing$	the empty set

ΔX_n	$= X_n - X_{n-1}$
$a \wedge b$	$= \min\{a, b\}$
$a \vee b$	$= \max\{a, b\}$
a^+	$= a \vee 0$
$[a]$	the integer part of a number a
$f \sim g$	equivalence of functions f and g as $x \to a$, which means that $\lim_{x\to a} f(x)/g(x) = 1$
$o(x)$	a function such that $\lim_{x\to 0} o(x)/x = 0$
$\binom{N}{n}$	the number of combination of N things, taken n at a time
X^π	the value of a strategy (portfolio) π
$\mathbb{C}(N)$	the fair price of an option with expiration time N
$\mathbb{C}_T(\Delta)$	the fair price of an option wth expiration time T and discreteness step Δ
(f, N)	a contingent claim
(B, S)-market	a market made up of the assets B and S
SF	the set of self-financing strategies
$\mathrm{SF}_{\mathrm{arb}}$	the set of arbitrage strategies
$\Pi(x, f, N)$	the set of hedging strategies for (f, N) with initial value x
GF	the set of G-financing strategies

This page intentionally left blank

Hints for solving the problems

2.1. Prove the more general fact that $\mathbf{E}(M_{\tau_2} \mid \mathcal{F}_{\tau_1}) = M_{\tau_1}$ for any stopping times $0 \le \tau_1 \le \tau_2 \le N$, and hence $\mathbf{E}M_{\tau_2} = \mathbf{E}M_{\tau_1} = \mathbf{E}M_0$. Consider a set $A \in \mathcal{F}_{\tau_1}$ and the set $B = A \cap \{\omega : \tau_1 = n\}$, and show that

$$\int_A M_{\tau_2} d\mathbf{P} = \int_A M_{\tau_1} d\mathbf{P}, \quad \text{or} \quad \int_{B\cap\{\tau_2 \ge n\}} M_{\tau_2} d\mathbf{P} = \int_{B\cap\{\tau_2\ge n\}} M_n d\mathbf{P}.$$

These equalities are consequences of the following chain of relations, which are valid in view of the martingale property of M:

$$\begin{aligned}
\int_{B\cap\{\tau_2\ge n\}} M_n d\mathbf{P} &= \int_{B\cap\{\tau_2=n\}} M_n d\mathbf{P} + \int_{B\cap\{\tau_2>n\}} M_n d\mathbf{P} \\
&= \int_{B\cap\{\tau_2=n\}} M_n d\mathbf{P} + \int_{B\cap\{\tau_2>n\}} \mathbf{E}(M_{n+1} \mid \mathcal{F}_n)\, d\mathbf{P} \\
&= \int_{B\cap\{\tau_2=n\}} M_n d\mathbf{P} + \int_{B\cap\{\tau_2>n\}} M_{n+1} d\mathbf{P} \\
&= \int_{B\cap\{n\le\tau_2\le n+1\}} M_{\tau_2} d\mathbf{P} + \int_{B\cap\{\tau_2\ge n+2\}} M_{n+2} d\mathbf{P},
\end{aligned}$$

and so on.

2.2. For any $A \in \mathcal{F}_{N-1}$ the properties of mathematical expectations give us that, on the one hand,

$$\int_A Y\, d\widetilde{\mathbf{P}} = \int_A \widetilde{\mathbf{E}}(Y \mid \mathcal{F}_{N-1})\, d\widetilde{\mathbf{P}} = \int_A \widetilde{\mathbf{E}}(Y \mid \mathcal{F}_{N-1}) Z_{N-1} d\mathbf{P}$$

and on the other hand,

$$\int_A Y\, d\widetilde{\mathbf{P}} = \int_A Y Z_N d\mathbf{P} = \int_A \mathbf{E}(Y Z_N \mid \mathcal{F}_{N-1})\, d\mathbf{P}.$$

2.3. Letting $\Delta\widehat{U}_n = \exp(\Delta U_n) - 1$, show that

$$\exp(U_n) = \exp\Bigl(\sum_{k=1}^{n} \Delta U_n\Bigr) = \prod_{k=1}^{n} (1 + \Delta\widehat{U}_k) = \mathcal{E}_N(\widehat{U}).$$

2.4. Since

$$\mathcal{E}_n^{-1}(A) = \bigl(\prod \mathbf{E}(\exp(\alpha_k \Delta V_k) \mid \mathcal{F}_{k-1})\bigr)^{-1},$$

it follows that

$$Z_n = \exp\Big(\sum_{k=1}^{n} \alpha_k \Delta V_k\Big)\big(\prod \mathbf{E}(\exp(\alpha_k \Delta V_k) \mid \mathcal{F}_{k-1})\big)^{-1}$$
$$= \prod_{k=1}^{n} \exp(\alpha_k \Delta V_k)(\mathbf{E}(\exp(\alpha_k \Delta V_k) \mid \mathcal{F}_{k-1}))^{-1},$$

and hence

$$\mathbf{E}(Z_n \mid \mathcal{F}_{n-1}) = Z_{n-1}\mathbf{E}(\exp(\alpha_n \Delta V_n)(\mathbf{E}(\exp(\alpha_n \Delta V_n) \mid \mathcal{F}_{n-1}))^{-1} \mid \mathcal{F}_{n-1}) = Z_{n-1}.$$

3.1. We represent the stochastic sequence V_n in (3.2) according to the Doob decomposition in the form $V_n = M_n + A_n$, where M is a martingale and A is a predictable sequence. By Girsanov's theorem, the stochastic sequence

$$M_n^* = (V - A)_n^* = (V - A)_n - \sum_{k=1}^{n} \mathbf{E}(Z_{k-1}^{-1} Z_k \Delta(V - A)_k \mid \mathcal{F}_{k-1})$$

is a martingale with respect to the measure $\mathbf{P}^*$ with local density Z_n with respect to $\mathbf{P}$. In view of Theorem 2.1 the density Z must be chosen so that

$$\begin{aligned} r_n = \Delta U_n &= \Delta A_n + \mathbf{E}(\mathbf{E}(Z_{n-1}^{-1} Z_n \Delta(V - A)_n \mid \mathcal{F}_{n-1}) \\ &= \Delta A_n + \mathbf{E}(Z_{n-1}^{-1} Z_n \Delta V_n \mid \mathcal{F}_{n-1}) - Z_{n-1}^{-1} \Delta A_n \mathbf{E}(Z_n \mid \mathcal{F}_{n-1}) \\ &= \mathbf{E}(Z_{n-1}^{-1} Z_n \Delta V_n \mid \mathcal{F}_{n-1}) = \mathbf{E}(Z_{n-1}^{-1} Z_n \rho_n \mid \mathcal{F}_{n-1}), \end{aligned}$$

that is, the formula connecting r, ρ, and Z is preserved for the "martingale" case.

3.2. To construct a spatially infinite market ($d = \infty$) we consider the space $\mathbf{R}^{\mathbb{Z}_+}$ of numerical sequences. Let $\Omega = \mathbb{Z}_+$, $\mathcal{F}_0 = \{\varnothing, \Omega\}$, $\mathcal{F}_1 = \mathcal{F} = 2^\Omega$, and $\mathbf{P} = \sum_{n=1}^{\infty} 2^{-n}\delta_n$, where $\delta_n\{k\} = 1$ for $k = n$ and 0 for $k \neq n$. We define an $\mathbf{R}^{\mathbb{Z}_+}$-valued stochastic sequence $(S_n)_{n=0,1}$ by

$$S_0 \equiv 0, \quad S_1^j(\omega) = \begin{cases} 1 & \text{for } \omega = j, \\ -1 & \text{for } \omega = j+1, \\ 0 & \text{otherwise,} \end{cases}$$

where S^j is the jth coordinate. Then the relation $\mathbf{P}^*\{j\} = \mathbf{P}^*\{j+1\}$ must hold for a martingale measure $\mathbf{P}^*$, which cannot be true for arbitrary $j \in \mathbb{Z}_+$. Consequently, such a measure does not exist.

For the other "infinite" case ($N = \infty$) we follow [**20**] and define a "risky" asset $S = (S_k)_{k \in \mathbb{Z}_+}$ to be the sequence of partial sums of independent random variables

$$\xi_k = \begin{cases} 1 & \text{with probability } p \neq 1/2, \\ -1 & \text{with probability } 1 - p, \end{cases}$$

and we let $\mathcal{F}_k = \sigma(\xi_1, \dots, \xi_k)$. Now if $\mathbf{P}^*$ is a martingale measure, then $\mathbf{P}^*(\xi_k = 1 \mid \mathcal{F}_{k-1}) = 1/2$ (a.s.). Further, by the law of large numbers for the measures $\mathbf{P}$ and $\mathbf{P}^*$, $S_n/n \to 2p - 1 \neq 0$ $\mathbf{P}$-a.s. and $S_n/n \to 0$ $\mathbf{P}^*$-a.s. as $n \to \infty$, and this means that $\mathbf{P}$ and $\mathbf{P}^*$ are not equivalent.

3.3. The assertions a)–c) are simple exercises with conditional expectations. From them,

$$X_0^N = \mathbf{E}X_N^N Z_N \geq \mathbf{E}\{X_N^N Z_N \mathbf{I}_{\{-C_N \leq X_N^N < 0\}}\} + \mathbf{E}\{X_N^N Z_N \mathbf{I}_{\{X_N^N \geq \varepsilon\}} \mathbf{I}_{\{Z_N \geq \varepsilon\}}\}$$
$$\geq -C_N + \varepsilon^2 \mathbf{P}\{X_N^N \geq \varepsilon,\ Z_N \geq \varepsilon\}.$$

Further,

$$X_0^N + C_N \geq \varepsilon \mathbf{P}\{X_N^N \geq \varepsilon,\ Z_N \geq \varepsilon\} \geq \varepsilon^2(\mathbf{P}\{X_N^N \geq \varepsilon\} - \mathbf{P}\{Z_N < 0\}).$$

Passing to the limit, we get that $\mathbf{P}\{Z_\infty < \varepsilon\} \geq \limsup_{N\to\infty} \mathbf{P}\{X_N^N \geq \varepsilon\}$. If the portfolio π^{*N} is asymptotically arbitrage-free, then there exists an $\varepsilon' > 0$ such that $\limsup_{N\to\infty} \mathbf{P}\{X_N^{*N} \geq \varepsilon'\} = \delta$ for some $\delta > 0$. Consequently, $\mathbf{P}\{Z_\infty < \varepsilon\} \geq \delta$ for $\varepsilon > \varepsilon'$, which contradicts a condition of the problem. We remark that the condition $\mathbf{P}\{Z_\infty < 0\} = 1$ means the continuity of $\{\mathbf{P}^N\}$ with respect to $\{\mathbf{P}^{*N}\}$ in the sense that if $A^N \in \mathcal{F}_N$ and if $\mathbf{P}^{*N}(A^N) \to 0$ as $N \to \infty$, then $\mathbf{P}^N(A^N) \to 0$ as $N \to \infty$.

4.1. For a complete no-arbitrage $(1, S)$-market (3.1) Theorem 4.1 and Jensen's inequality give us that

$$\begin{aligned}
\mathbb{C}_{(N+1)} &= \mathbf{E}^*((S_{N+1} - K)^+) = \mathbf{E}^*((S_N(1 + \rho_{N+1}) - K)^+) \\
&= \mathbf{E}^*(\mathbf{E}^*((S_{N+1} - K)^+) \mid \mathcal{F}_N) \\
&\geq \mathbf{E}^*((S_N \mathbf{E}^*((1 + \rho_{N+1}) \mid \mathcal{F}_N) - K)^+) \\
&= \mathbf{E}^*((S_N - K)^+) = \mathbb{C}_{(N)}.
\end{aligned}$$

4.2. The cases a) and c) are simple exercises, so we dwell on the more complicated case c) for a complete $(1, S)$-market (3.2) with a unique martingale measure $\mathbf{P}^*$. Since the function $(x - K)^+$ is convex with respect to K, we have for $p \in [0, 1]$ that

$$\begin{aligned}
&p(S_0\mathcal{E}_N(V) - K_1)^+ + (1-p)(S_0\mathcal{E}_N(V) - K_2)^+ \\
&\quad = \mathcal{E}_N(V)(p(S_0 - K_1\mathcal{E}_N^{-1}(V))^+ + (1-p)(S_0 - K\mathcal{E}_N^{-1}(V))^+) \\
&\quad \geq \mathcal{E}_N(V)(S_0 - pK_1\mathcal{E}_N^{-1}(V) - (1-p)K_2\mathcal{E}_N^{-1}(V))^+ \\
&\quad = (S_0\mathcal{E}_N(V) - pK_1 - (1-p)K_2)^+.
\end{aligned}$$

Taking the mathematical expectation with respect to the measure $\mathbf{P}^*$ now gives the necessary inequality:

$$p\mathbb{C}(S_0, N, K_1) + (1-p)\mathbb{C}(S_0, N, K_2) \geq \mathbb{C}(S_0, N, pK_1 + (1-p)K_2).$$

Convexity with respect to S_0 is proved similarly.

4.3. Let π_1 and π_2 be two minimal hedges on a complete (B, S)-market (3.1) with martingale measure $\mathbf{P}^*$. Then with regard to this measure the sequences

$$M_n^i = B_n^{-1} X_n^{\pi^i} = \mathbb{C}(N) B_0^{-1} + \sum_{k=1}^n B_k^{-1} \gamma_k^i S_{k-1}(\rho_k - r_k), \qquad i = 1, 2,$$

are martingales, and $M_N^1 = B_N^{-1} f = M_N^2$. Their difference

$$M_n = M_n^1 - M_n^2 = \sum_{k=1}^n B_k^{-1}(\gamma_k^1 - \gamma_k^2) S_{k-1}(\rho_k - r_k)$$

is also a martingale such that $M_n = \mathbf{E}^*(M_N \mid \mathcal{F}_n)$ and $M_0 = M_N = 0$. Consequently, $M_n \equiv 0$, and hence $\gamma_k^1 = \gamma_k^2$, $k \le N$.

4.4. It is clear that the required exchange rate is equal to the ratio S_n^1/S_n^2, and thus is a solution of the difference equation $\Delta(S^1/S^2)_n = (S_0^1/S_0^2)(\rho_n^1 - \rho_n^2)/(1+\rho_n^2)$.

5.1. Using the procedure described in the proof of Theorem 5.1 for constructing $\tau^* = \tau_1^* = \min\{0 \le k \le 1 : Y_k = f_k/B_k\}$ and $\mathbb{C}(1) = \sup_{0\le\tau\le 1} \mathbf{E}^*(1 + r)^{-1}\beta^*(S_\tau - 1)^+$, we get

a) $\frac{f_0}{B_0} = \frac{(S_0-1)^+}{1} = 0 < \mathbf{E}^* Y_1 = \alpha\beta p^*(\lambda - 1) \implies \tau^* \equiv 1$, and $\mathbb{C}(1) = \mathbf{E}^* Y_1 = \alpha\beta p^*(\lambda - 1)$;

b) $\frac{f_0}{B_0} = \lambda - 1$, $\mathbf{E}^* Y_1 = \alpha\beta p^*(\lambda^2 - 1) = (1 + r)^{-1}\beta(\lambda^2 - 1)\frac{r-(\lambda^{-1}-1)}{\lambda-1-(\lambda^{-1}-1)} = \beta(\lambda - \alpha)$, and if $\beta > \frac{\lambda-1}{\lambda-\alpha}$, then $\mathbf{E}^* Y_1 > \lambda - 1 \implies \tau^* \equiv 1$ and $\mathbb{C}(1) = \mathbf{E}^* Y_1 = \beta(\lambda - \alpha)$.

5.2. As in the solution of Problem 5.1, to find the optimal stopping time τ^* we must compare the quantities $f_0/B_0 = \lambda^k - 1$ and $\mathbf{E}^* Y_1 = \mathbf{E}^*(S_1 - 1)^+\alpha\beta = \alpha\beta p^*(\lambda^{k+1}-1)+\alpha\beta(1-p^*)(\lambda^{k-1}-1) = \alpha\beta(p^*(\lambda^{k+1}-\lambda^{k-1})+\lambda^{k-1}-1) = \beta(\lambda^k-\alpha)$. Thus, $\tau_1^* = 1$ for $\lambda^k - 1 < \beta(\lambda^k - \alpha)$, and otherwise $\tau_1^* = 0$. Correspondingly,

$$\mathbb{C}(1) = \sup_{0\le\tau\le 1} \mathbf{E}^*(1 + r)^{-\tau^*}\beta^{\tau^*}(S_{\tau^*} - 1)^+ = \max(\lambda^k - 1, \beta(\lambda^k - \alpha)).$$

These arguments lead to the solution.

6.1. The solution here is analogous to 4.3.

6.2. From (6.6) and (6.7),

$$\mathbf{E}^*\mathcal{E}_N^{-1} X_N^{\Pi(G)} = X_0^{\Pi(G)} - \sum_{k=1}^N \mathcal{E}_{k-1}^{-1}\mathbf{E}^*\Delta G_k \le X_0^{\Pi(G)} - \sum_{k=1}^N \mathcal{E}_{k-1}^{-1}\mathbf{E}^*\Delta D_k.$$

Hence, $X_0^{\Pi(G)} \ge \mathbf{E}^*\mathcal{E}_N^{-1} X_N^{\Pi(G)} + \sum_{k=1}^N \mathcal{E}_{k-1}^{-1}\mathbf{E}^*\Delta D_k$. In particular, for $\Delta G_n = \mathbb{C}_G S_{n-1}$ and $\Delta D_n = \mathbb{C}_D S_{n-1}$ the resulting inequality is valid when $\mathbb{C}_G \ge \mathbb{C}_D$.

7.1. Suppose that on a $(1, S)$-market the original measure $\mathbf{P} = \mathbf{P}^*$ is a martingale measure. Minimizing the remaining risk

$$R_n^\Pi = \mathbf{E}\Bigl(\Bigl(f - X_n^\Pi - \sum_{k=1}^N \gamma_k S_{k-1}\rho_k\Bigr)^2 \mid \mathcal{F}_n\Bigr)$$

with respect to X and γ, we get the equations

$$\frac{\partial R_{N-1}^\Pi}{\partial X_{N-1}} = 2\mathbf{E}(f - X_{N-1}^\Pi - \gamma_N S_{N-1}\rho_N \mid \mathcal{F}_{N-1}) = 0,$$
$$\frac{\partial R_{N-1}^\Pi}{\partial \gamma_{N-1}} = 2\mathbf{E}((f - X_{N-1}^\Pi - \gamma_N S_{N-1}\rho_N)S_{N-1}\rho_N \mid \mathcal{F}_{N-1}) = 0.$$

Therefore, the risk-minimizing strategy is determined by the relations $\widetilde{X}_{N-1} = \mathbf{E}(f \mid \mathcal{F}_{N-1})$ and $\widetilde{\gamma}_N = \mathbf{E}(f\rho_N \mid \mathcal{F}_{N-1})/(S_{N-1}\mathbf{E}(\rho_N^2 \mid \mathcal{F}_{N-1}))$ (the procedure is extended in the natural way to $k < N$: $\widetilde{\gamma}_k = \mathbf{E}(f\rho_k \mid \mathcal{F}_{k-1})/(S_{k-1}\mathbf{E}(\rho_k^2 \mid \mathcal{F}_{k-1}))$, $\widetilde{X}_k = \mathbf{E}(f \mid \mathcal{F}_{k-1})$). For an optimal strategy $\widehat{\pi}$ it follows from the martingale property of $G_n = \sum_{k=1}^n \Delta G_k$ and $M_n = \sum_{k=1}^n \rho_k$ that $\widehat{X}_n = \mathbf{E}(f \mid \mathcal{F}_n)$. Since M

and G are orthogonal, it follows that $0 = \mathbf{E}(\rho_k \Delta G_k \mid \mathcal{F}_{k-1}) = -\mathbf{E}(\rho_k \Delta \widehat{X}_k \mid \mathcal{F}_{k-1}) + \widehat{\gamma}_k \mathbf{E}(\rho_k \Delta S_k \mid \mathcal{F}_{k-1})$ or $\widehat{\gamma}_k = \mathbf{E}(f\rho_k \mid \mathcal{F}_{k-1})/(S_{k-1}\mathbf{E}(\rho_k^2 \mid \mathcal{F}_{k-1}))$. Comparison of the formulas obtained leads to the necessary assertion of the problem.

7.2. For a self-financing portfolio $\pi_n = (\beta_n, \gamma_n)$ the value X_n^π is determined by the formula (3.5′). Using (3.5′) and the notation $X_n^\gamma = \mathcal{E}_n^{-1} X_n^\pi$, $f_N = \mathcal{E}_n^{-1} f$, $X_0^\pi = x$, and $\Delta S_n^1 = \mathcal{E}_n^{-1} S_{n-1}(\rho_n - r_n)$, we get that $X_n^\gamma = X_n^\gamma(x) = x + \sum_{k=1}^n \gamma_k \Delta S_k^1$. Hence, the original problem reduces to finding an $x^* > 0$ and a predictable sequence γ^* such that

$$R_N^{\gamma^*}(x^*) \equiv \mathbf{E}(X_N^{\gamma^*}(x^*) - f_N)^2 = \inf_{x,\gamma} R_N^\gamma(x).$$

Here the first component β^* of the optimal self-financing portfolio π^* can be uniquely recovered from γ^*.

Note also that (S_n^1) and $(X_n^\gamma(x))$ are martingales, and $x = \mathbf{E}X_n^\gamma(x)$. Further, rewriting $R_N^\gamma(x) = (\mathbf{E}(f_N - x))^2 + \mathbf{E}((X_N^\gamma(x) - x) - (f_N - \mathbf{E}f_N))^2$ and differentiating with respect to x, we have that $(R_N^\gamma(x))' = -2\mathbf{E}(f_N - x) = 0$. Consequently, $x^* = \mathbf{E}f_N$. Now take hypothetical optimal variables to be $\gamma_n^* = \mathbf{E}(f_N \Delta S_n^1 \mid \mathcal{F}_{n-1})/\mathbf{E}((\Delta S_n^1)^2 \mid \mathcal{F}_{n-1})$, and form the martingale

$$L_n^* = \mathbf{E}(f_N \mid \mathcal{F}_n) - \mathbf{E}\Big(\sum \gamma_k^* \Delta S_k^1 \,\Big|\, \mathcal{F}_n\Big) - x^*.$$

Setting $n = N$, we get, for example, the representation

$$f_N = x^* + \sum_{k=1}^N \gamma_k^* \Delta S_k^1 + L_N^*.$$

Since the martingales $\sum_{k=1}^N \gamma_k^* \Delta S_k^1$ and L_N^* are orthogonal,

$$\mathbf{E}(\Delta L_n^* \gamma_n^* \Delta S_n^1 \mid \mathcal{F}_{n-1}) = 0.$$

Next,

$$\begin{aligned} R_N^\gamma(x^*) &= \mathbf{E}\bigg(f_N - \Big(x^* + \sum_{k=1}^N \gamma_k^* \Delta S_k^1\Big)\bigg)^2 = \mathbf{E}\bigg(\sum_{k=1}^N (\gamma_k^* - \gamma_k)\Delta S_k^1 + L_N^*\bigg)^2 \\ &= \mathbf{E}\bigg(\sum_{k=1}^N (\gamma_k^* - \gamma_k)\Delta S_k^1\bigg)^2 + \mathbf{E}(L_N^*)^2 \geq \mathbf{E}(L_N^*)^2. \end{aligned}$$

It is clear that equality here is possible only when $\gamma_k = \gamma_k^*$.

7.3. Suppose that an option has cost $x > \mathbb{C}^*$, and it was bought at this price. Then the seller has the following arbitrage opportunity. For this it is necessary to invest capital $y \in (\mathbb{C}^*, x)$ on a (B, S)-market in such a way that $X_0^{\pi^*}(y) = y$ and $X_N^{\pi^*}(y) \geq f$. This is possible in view of the definition of $\mathbb{C}^*$. Here the nonrisky gain of the seller is strictly positive:

$$(x - f) + (X_N^{\pi^*}(y) - y) = (x - y) + (X_N^{\pi^*}(y) - f) \geq x - y > 0.$$

But if the buyer acquired the contract at a price $x < \mathbb{C}^*$, then an arbitrage opportunity is realized by the buyer. According to the definition of $\mathbb{C}^*$, there exists for $y \in (\mathbb{C}_*, x)$ a portfolio π^* such that

$$X_0^{\pi^*}(y) = y, \qquad X_N^{\pi^*}(y) \leq f.$$

Here the buyer proceeds as follows: he borrows y and invests it according to the portfolio π^* and receives the nonrisky profit

$$(f - X_N^{\pi^*}(y)) + (y - x) \geq y - x > 0.$$

7.4. There is a hint in the formulation of the problem.

8.1. Since the total expenditures of the parties in a forward contract are zero, for the contract to be represented as a composition of a call option and a put option it is necessary that these options have equal prices: $\mathbb{C}_N = \mathbb{P}_N$. Further, it follows from the call-put parity $\mathbb{C}_N = \mathbb{P}_N + S_0 - K(1+r)^{-N}$ that $K = S_0(1+r)^N$. The latter quantity coincides precisely with the strike price for the forward contract.

8.2. A forward contract (on an asset $(S_n)_{n\leq N}$) with strike price F_n is equivalent to accepting the contingent claim

$$f_k = \begin{cases} 0, & k = n, \dots, N-1, \\ S_N - F_n, & n = N. \end{cases}$$

It follows from the general theory of pricing contingent claims that a no-arbitrage price $\mathbb{C}_n^F$ must be found within the limits

$$\mathbb{C}_n = \inf_{\widetilde{\mathbf{P}} \in \mathbb{P}^*} \widetilde{\mathbf{E}}(S_N - F_n \mid \mathcal{F}_n) \leq \mathbb{C}_n^F \leq \sup_{\widetilde{\mathbf{P}} \in \mathbb{P}^*} \widetilde{\mathbf{E}}(S_N - F_n \mid \mathcal{F}_n) = \mathbb{C}^n.$$

On the other hand, the cost $\mathbb{C}_n^F$ for the forward contract is equal to zero. Consequently, $\mathbb{C}_n \leq 0$ and $\mathbb{C}^n \geq 0$, which implies the required assertion.

8.3. To determine the futures prices F_n^* on the complete (B,S)-market with martingale measure $\mathbf{P}^*$ we use "backward induction". In concluding the futures contract at time $N-1$ for the purchase of a unit of the asset at the price F_{N-1}^* it is necessary to deposit in the margin account $\Delta M_n = m_n M_{n-1}$, $-1 < m_n \leq r_n$, the quantity

$$\alpha_N F_{N-1}^* = \mu_N M_{N-1},$$

where μ_N is the number of units in this account. At time N the investor gets

$$S_N - F_{N-1}^* + \mu_N M_N.$$

Consequently, the contingent claim

$$f_N = S_N - F_N^* + \mu_N M_N$$

is connected with this futures contract, and its cost is

$$\mathbb{C}_{N-1} = \mathbf{E}^*\Big(\frac{B_{N-1}}{B_N}(S_N - F_{N-1}^* + \mu_N M_N) \Bigm| \mathcal{F}_{N-1}\Big).$$

Since there are no arbitrage opportunities,

$$\alpha_N F_{N-1}^* \geq \mathbf{E}^*\Big(\frac{B_{N-1}}{B_N}(S_N - F_{N-1}^* + \mu_N M_N) \Bigm| \mathcal{F}_{N-1}\Big),$$

and hence

$$F_{N-1}^* \geq \frac{F_{N-1}}{1 + \alpha_N(r_N - m_N)}.$$

Conclusion of the futures contract to sell is equivalent to the contingent claim

$$f'_N = F_{N-1}^* - S_N + \mu_N M_N, \qquad \alpha_N F_{N-1}^* = \mu_N M_{N-1}.$$

In a way analogous to that above we get that

$$F_{N-1}^* \le \frac{F_{N-1}}{1-\alpha_N(r_N-m_N)}\,.$$

Continuing these arguments leads to the establishment of bounds for the futures prices F_n^* for all $n \le N$.

8.4. This can be established directly.

9.1. Let $B_0 = B_1$. Then the yield of the portfolio π takes the form $R = R^\pi = X_1^\pi/X_0 - 1$, and maximization of the mean yield $\mathbf{E}R^\pi$ reduces to maximization of the expectation

$$\mathbf{E}X_1^\pi = X_0 + \gamma_1 S_0(bp + a(1-p)) - \delta S_0|\Delta\gamma_1|$$

with respect to γ_1.

The function obtained is piecewise linear in γ_1 and attains its maximum at the break point $\gamma_1 = \gamma_0$ (the portfolio remains as before) or on the limits $\gamma_1 = 0$ (all the stocks are sold at time zero) and $\gamma_1 = (X_0 - \gamma_0 S_0)/(S_0(1+\delta)) + \gamma_0$ of the possible values. The rest is a direct computation of $\mathbf{E}X_1$ and $\sup_\pi \mathbf{E}R^\pi$.

9.2. For the single-period model of a $(1,S)$-market it follows since γ_1 is deterministic (γ_1 is measurable with respect to $\mathcal{F}_0 = \{\varnothing,\Omega\}$) that the problem reduces to the standard problem of minimizing the quadratic (with respect to γ_1) function $\mathbf{E}(X_1^\pi - f)^2$ in the two regions $\gamma_1 \ge \gamma_0$ and $\gamma_1 < \gamma_0$.

9.3. It is clear that for the function $u(x) = x^\alpha/\alpha$ the corresponding function v is $v(x) = x^{1/(\alpha-1)}$, and the form of the optimal terminal value is $X_N^* = (yZ_N^*)^{1/(\alpha-1)}$. Further,

$$\begin{aligned}\psi(y) &= \mathbf{E}\sup_{X_N>0}(X_N^\alpha/\alpha - yZ_N^*X_N)\\ &= \mathbf{E}((X_N^*)^\alpha/\alpha - (yZ_N^*)X_N^*) = \Big(\frac{1}{\alpha}-1\Big)y^{\alpha/(\alpha-1)}\mathbf{E}(Z_N^*)^{\alpha/(\alpha-1)},\end{aligned}$$

and, with use of the notation $c = \mathbf{E}(Z_N^*)^{\alpha/(\alpha-1)}$,

$$\varphi(x) = \inf_{y>0}(\psi(y) + yx) = \inf_{y>0}(c(1/\alpha - 1)y^{\alpha/(\alpha-1)} + yx).$$

The quantity $\widehat{y} = (x/c)^{\alpha-1}$ minimizes the function $c(1/\alpha-1)y^{\alpha/(\alpha-1)} + yx$, and hence

$$\varphi(x) = \Big(\frac{1}{\alpha}-1\Big)\frac{x^\alpha}{c^{\alpha-1}} + \frac{x^\alpha}{c^{\alpha-1}} = \frac{x^\alpha}{\alpha c^{\alpha-1}}\,,\qquad X_N^* = \frac{x}{c}(Z_N^*)^{1/(\alpha-1)}.$$

To find an optimal strategy $\pi_n^* = (\beta_n^*, \gamma_n^*)$ we introduce the proportion α_n^* of the risky part of the portfolio. Equating the two expressions for the terminal value gives us that

$$\mathcal{E}_N\Big(\sum \alpha_k^*\rho_k\Big) = c^{-1}\mathcal{E}_N^{1/(\alpha-1)}\Big(\sum\Big(-\frac{m}{d}(\rho_k - m)\Big)\Big),$$

and, as a consequence,

$$\alpha_k^* = \frac{1}{b}\Big\{\Big(\frac{a}{a-b} + \Big(\frac{-bp}{a(1-p)}\Big)^{1/(\alpha-1)}\frac{b}{b-a}\Big)^{-1} - 1\Big\}.$$

As a result, π_k^* is found from the formulas

$$\beta_k^* = X_{k-1}^*(1-\alpha_k^*), \quad \gamma_k^* = \alpha_k^* X_{k-1}^*/S_{k-1}.$$

10.1. Use the Black–Scholes formula directly.

10.2. Considering a (B, S, Δ)-market with parameters satisfying (10.6), we find that a martingale measure satisfies

$$p^* = \frac{\left(e^{r\Delta}-1\right)-\left(1-e^{-\sigma\sqrt{\Delta}}\right)}{e^{\sigma\sqrt{\Delta}}-e^{-\sigma\sqrt{\Delta}}} = \frac{1}{2}\left(1+\frac{r}{\sigma}\sqrt{\Delta}\right)+o(\Delta).$$

Further, since this is a "binomial" model,

$$e^{r\Delta}\mathbb{C}(s,t) = p^*\mathbb{C}(Se^{\sigma\sqrt{\Delta}}, t+\Delta) + (1-p^*)\mathbb{C}(Se^{-\sigma\sqrt{\Delta}}, t+\Delta).$$

From Taylor's formula we get that as $\Delta \to 0$

$$e^{r\Delta}\mathbb{C}(s,t) = (1+r\Delta)\mathbb{C}(s,t) + o(\Delta);$$

$$\mathbb{C}(Se^{\sigma\sqrt{\Delta}}, t+\Delta) = \mathbb{C}(s,t) + \frac{\partial}{\partial t}\mathbb{C}(s,t) + \frac{\partial\mathbb{C}}{\partial s}s\sigma\sqrt{\Delta} + \frac{1}{2}\frac{\partial^2\mathbb{C}}{\partial s^2}s^2\sigma^2\Delta + o(\Delta);$$

$$\mathbb{C}(Se^{-\sigma\sqrt{\Delta}}, t+\Delta) = \mathbb{C}(s,t) + \frac{\partial}{\partial t}\mathbb{C}(s,t) - \frac{\partial\mathbb{C}}{\partial s}s\sigma\sqrt{\Delta} + \frac{1}{2}\frac{\partial^2\mathbb{C}}{\partial s^2}s^2\sigma^2\Delta + o(\Delta).$$

Consequently,

$$(1+r\Delta)\mathbb{C}(s,t) = \mathbb{C}(s,t) + \frac{\partial\mathbb{C}}{\partial t}\Delta + \frac{\partial\mathbb{C}}{\partial s}rs\Delta + \frac{1}{2}\frac{\partial^2\mathbb{C}}{\partial s^2}s^2\sigma^2\Delta + o(\Delta),$$

and passage to the limit as $\Delta \to 0$ leads to the Black–Scholes equation.

Bibliography

1. A. N. Burenin, *Futures, forward, and option contracts*, Trivola, Moscow, 1995. (Russian)
2. J. Jacod and A. N. Shiryaev, *Limit theorems for stochastic processes*, Springer–Verlag, Berlin, 1987; Russian transl., vols. 1, 2, Fizmatlit, 1994.
3. D. O. Kramkov and A. N. Shiryaev, *On rational pricing of a "Russian option" in the symmetric binomial model of a (B, S)-market*, Teor. Veroyatnost. i Primenen. **39** (1994), 191–200; English transl. in Theory Probab. Appl. **39** (1994).
4. A. V. Mel′nikov and A. N. Shiryaev, *Criteria for the absence of arbitrage in the financial market*, Progress in the Theory of Probability and its Applications II (A. N. Shiryaev, et al., eds.), TVP, Moscow, 1996, pp. 121–134.
5. A. V. Mel′nikov, M. L. Nechaev, and V. M. Stepanov, *On a discrete model for a financial market and methods for pricing securities*, preprint no. 3, Actuarial and Financial Center for Scientific Investigation, Moscow, 1996. (Russian)
6. A. A. Novikov, *Pricing options of American type: a minimax statistical approach*, Teor. Veroyatnost. i Primenen. (to appear); English transl. in Theory Probab. Appl..
7. A. A. Pervozvanskiĭ and T. N. Pervozvanskaya, *Financial markets: pricing and risk*, Infra-M, Moscow, 1994. (Russian)
8. S. T. Rachev and L. Rüschendorf, *Models for option prices*, Teor. Veroyatnost. i Primenen. **39** (1994), 150–190; English transl. in Theory Probab. Appl. **39** (1994).
9. H. Robbins, D. Siegmund, and Y. S. Chow, *Great expectations: The theory of optimal stopping*, Houghton–Mifflin, Boston, 1971; Russian transl., "Nauka", Moscow, 1977.
10. A. N. Shiryaev, *Probability*, 2nd ed., "Nauka", Moscow, 1989; English transl., Springer–Verlag, Berlin–New York, 1996.
11. A. N. Shiryaev, *On some basic concepts and stochastic models in financial mathematics*, Teor. Veroyatnost. i Primenen. **39** (1994), 5–22; English transl. in Theory Probab. Appl. **39** (1994).
12. A. N. Shiryaev, Yu. M. Kabanov, D. O. Kramkov, and A. V. Mel′nikov, *Toward a theory of pricing options of European and American types.* I. *Discrete time*, Teor. Veroyatnost. i Primenen. **39** (1994), 23–79; English transl. in Theory Probab. Appl. **39** (1994).
13. L. Bachelier, *Théorie de la spéculation*, Ann. École Norm. Sup. **17** (1900), 21–86; reprint, The Random Character of Stock Market Prices (P. H. Coothner, ed.), MIT Press, Cambridge, MA, 1967, pp. 17–78.
14. J. Bardhan and X. Chao, *Pricing options on securities with discontinuous returns*, Stochastic Process. Appl. **48** (1993), no. 1, 123–137.
15. F. Black and M. Scholes, *The pricing of options and corporate liabilities*, J. Polit. Economy **3** (1973), 637–659.
16. D. B. Colwell and R. J. Elliot, *Discontinuous asset prices and non-attainable contingent claims and corporate policy*, Math. Finance **3** (1993), no. 3, 295–368.
17. T. Copeland and J. Westen, *Financial Theory and Corporate Policy*, Addison–Wesley, Reading, MA, 1983.
18. J. C. Cox and M. Rubinstein, *Option Markets*, Prentice–Hall, Englewood Cliffs, NJ, 1985.
19. J. C. Cox, R. A. Ross, and M. Rubinstein, *Option pricing: a simplified approach*, J. Financial Econom. **3** (1979), no. 7, 229–263.
20. R. C. Dalang, A. Morton, and W. Willinger, *Equivalent martingale measures and no-arbitrage in stochastic securities market models*, Stochastics Stochastics Rep. **29** (1990), no. 2, 185–209.
21. R. A. Dana and M. Jeanblanc-Picqué, *Marchés financiers en temps continu (valorisation et équilibre)*, Economica, Paris, 1994.
22. F. Delbaen and W. Schachermayer, *A general version of the fundamental theorem of asset pricing*, Math. Ann. **300** (1994), 463–520.

23. M. V. Dothan, *Prices in Financial Markets*, Oxford Univ. Press, Oxford, 1990.
24. D. Duffie, *Dynamic Asset Pricing Theory*, Princeton Univ. Press, Princeton, NJ, 1992.
25. H. Föllmer and D. Sonderman, *Hedging of non-redundant contingent claims*, Contributions to Mathematical Economics (W. Hildebrand and A. Mas-Colell, eds.), 1986, pp. 205-223.
26. H. Föllmer, *Probabilistic aspects of options*, Discussion paper B-202, Universität Bonn, Bonn, 1991.
27. H. Föllmer and M. Schweizer, *Hedging of contingent claims under incomplete information*, Applied Stochastic Analysis (M. N. A. Davis and R. J. Elliott, eds.), Gordon & Breach, London, 1991, pp. 389–408.
28. J. M. Harrison and D. M. Kreps, *Martingales and arbitrage in multiperiod securities markets*, J. Econom. Theory **20** (1979), 381–408.
29. J. M. Harrison and S. R. Pliska, *Martingales and stochastic integrals in the theory of continuous trading*, Stochastic Process. Appl. **11** (1981), no. 3, 215–260.
30. T. S. Y. Ho and Sang-Bin Lee, *Term structure movements and pricing interest rate contingent claims*, J. Finance **41** (1986), 1011–1029.
31. J. Hull, *Options, futures and other derivative securities*, Prentice-Hall, Englewood Cliffs, NJ, 1992.
32. I. Karatzas, J. P. Lechoczky, S. E. Shreve, and G. L. Xu, *Martingale and duality methods for utility maximization in an incomplete market*, SIAM J. Control Optim. **29** (1991), 702–730.
33. D. O. Kramkov, *Optional decomposition of supermartingales and hedging continent claims in incomplete security markets*, Probab. Theory Related Fields **105** (1996), 459–479.
34. H. Markowitz, *Mean-variance analysis in portfolio choice and capital markets*, Blackwell, Cambridge, MA, 1990.
35. R. Merton, *Option pricing when underlying stock returns are discontinuous*, J. Finan. Econom. **3** (1976), 125–144.
36. R. Merton, *Continuous-time finance*, Blackwell, Cambridge, MA, 1993.
37. W. Schachermayer, *A Hilbert space proof of the fundamental theorem of asset pricing in finite discrete time*, Insurance: Math. Econom. **11** (1992), 249–257.
38. W. Schachermayer, *A counterexample to several problems in the theory of asset pricing*, Math. Finance **3** (1993), no. 2, 217–229.
39. M. Schweizer, *Approximation pricing and the variance-optimal martingale measure*, Ann. Probab. **24** (1996), no. 1, 206–236.
40. M. S. Taqqu and W. Willinger, *The analysis of finite security markets using martingales*, Adv. Appl. Probab. **19** (1987), 1–25.

Subject Index

This page intentionally left blank

Selected Titles in This Series

(*Continued from the front of this publication*)

149 **Takashi Sakai,** Riemannian geometry, 1996
148 **Vladimir I. Piterbarg,** Asymptotic methods in the theory of Gaussian processes and fields, 1996
147 **S. G. Gindikin and L. R. Volevich,** Mixed problem for partial differential equations with quasihomogeneous principal part, 1996
146 **L. Ya. Adrianova,** Introduction to linear systems of differential equations, 1995
145 **A. N. Andrianov and V. G. Zhuravlev,** Modular forms and Hecke operators, 1995
144 **O. V. Troshkin,** Nontraditional methods in mathematical hydrodynamics, 1995
143 **V. A. Malyshev and R. A. Minlos,** Linear infinite-particle operators, 1995
142 **N. V. Krylov,** Introduction to the theory of diffusion processes, 1995
141 **A. A. Davydov,** Qualitative theory of control systems, 1994
140 **Aizik I. Volpert, Vitaly A. Volpert, and Vladimir A. Volpert,** Traveling wave solutions of parabolic systems, 1994
139 **I. V. Skrypnik,** Methods for analysis of nonlinear elliptic boundary value problems, 1994
138 **Yu. P. Razmyslov,** Identities of algebras and their representations, 1994
137 **F. I. Karpelevich and A. Ya. Kreinin,** Heavy traffic limits for multiphase queues, 1994
136 **Masayoshi Miyanishi,** Algebraic geometry, 1994
135 **Masaru Takeuchi,** Modern spherical functions, 1994
134 **V. V. Prasolov,** Problems and theorems in linear algebra, 1994
133 **P. I. Naumkin and I. A. Shishmarev,** Nonlinear nonlocal equations in the theory of waves, 1994
132 **Hajime Urakawa,** Calculus of variations and harmonic maps, 1993
131 **V. V. Sharko,** Functions on manifolds: Algebraic and topological aspects, 1993
130 **V. V. Vershinin,** Cobordisms and spectral sequences, 1993
129 **Mitsuo Morimoto,** An introduction to Sato's hyperfunctions, 1993
128 **V. P. Orevkov,** Complexity of proofs and their transformations in axiomatic theories, 1993
127 **F. L. Zak,** Tangents and secants of algebraic varieties, 1993
126 **M. L. Agranovskiĭ,** Invariant function spaces on homogeneous manifolds of Lie groups and applications, 1993
125 **Masayoshi Nagata,** Theory of commutative fields, 1993
124 **Masahisa Adachi,** Embeddings and immersions, 1993
123 **M. A. Akivis and B. A. Rosenfeld,** Élie Cartan (1869–1951), 1993
122 **Zhang Guan-Hou,** Theory of entire and meromorphic functions: deficient and asymptotic values and singular directions, 1993
121 **I. B. Fesenko and S. V. Vostokov,** Local fields and their extensions: A constructive approach, 1993
120 **Takeyuki Hida and Masuyuki Hitsuda,** Gaussian processes, 1993
119 **M. V. Karasev and V. P. Maslov,** Nonlinear Poisson brackets. Geometry and quantization, 1993
118 **Kenkichi Iwasawa,** Algebraic functions, 1993
117 **Boris Zilber,** Uncountably categorical theories, 1993
116 **G. M. Fel′dman,** Arithmetic of probability distributions, and characterization problems on abelian groups, 1993
115 **Nikolai V. Ivanov,** Subgroups of Teichmüller modular groups, 1992
114 **Seizô Itô,** Diffusion equations, 1992
113 **Michail Zhitomirskiĭ,** Typical singularities of differential 1-forms and Pfaffian equations, 1992

(See the AMS catalog for earlier titles)

ISBN 0-8218-1082-0

9 780821 810828

MMONO/184