IWANAMI SERIES IN MODERN MATHEMATICS

Translations of
MATHEMATICAL MONOGRAPHS

Volume 185

Algebraic Geometry 1
From Algebraic Varieties to Schemes

Kenji Ueno

American Mathematical Society

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10.1090/mmono/185

Volume 185

Algebraic Geometry 1
From Algebraic Varieties to Schemes

Kenji Ueno

Translated by
Goro Kato

American Mathematical Society
Providence, Rhode Island

代数幾何 1

代数多様体からスキームへ

DAISŪ KIKA (ALGEBRAIC GEOMETRY 1)

by Kenji Ueno
with financial support
from the Japan Association for Mathematical Sciences

Originally published in Japanese
by Iwanami Shoten, Publishers, Tokyo, 1997

Translated from the Japanese by Goro Kato

2000 *Mathematics Subject Classification.* Primary 14-01.

ABSTRACT. This is the first in a series of three books by the author, aimed at introducing the reader to Grothendieck's scheme theory as a method for studying algebraic geometry. This first book contains the definition and main properties of schemes, together with necessary material from the theory of algebraic varieties and category theory. The author also includes many examples.

The book is aimed at graduate and upper level undergraduate students who want to learn modern algebraic geometry.

Library of Congress Cataloging-in-Publication Data

Ueno, Kenji, 1945–
[Daisū kika. English]
Algebraic geometry / Kenji Ueno ; translated by Goro Kato.
p. cm. — (Translations of mathematical monographs, ISSN 0065-9282 ; v. 185) (Iwanami series in modern mathematics)
Includes index.
Contents: 1. From algebraic varieties to schemes
ISBN 0-8218-0862-1 (v. 1 : pbk. : acid-free)
1. Geometry, Algebraic. I. Title. II. Series. III. Series: Iwanami series in modern mathematics.
QA564.U3513 1999
516.3′5—dc21 99-22304
CIP

Printed in the United States of America.

∞ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.

10 9 8 7 6 5 4 3 2 06 05 04 03 02 01

To the memory of Hisao Miyauchi

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Preface

It has often been said that algebraic geometry is a difficult field in mathematics. There certainly was a time when algebraic geometry was a difficult geometry. In particular, the theory of algebraic curves of the Italian school from the late nineteenth century through the first half of the twentieth century was indeed difficult. Intuitive arguments proceeded without rigorous proofs. Legend has it that one of the leaders of the Italian school, Enriques, once said, "It is a nobleman's work to find theorems, and it is a slave's work to prove them. Mathematicians are noblemen." Their sharpness of intuition might well convince us of that legend, but it was nearly impossible for common mathematicians to follow the arguments.

The plans to provide mathematically solid foundations for such intuitionistic algebraic geometry were carried out by van der Waerden, Zariski, Weil, Chevalley, and others, using abstract algebra as it developed in the 1930's. Zariski and Weil provided a foundation for algebraic geometry for their time period. Based on their foundation, Weil was able to prove the Riemann hypothesis for an algebraic curve defined over a finite field, establishing the closely related theory of abelian varieties over a field of positive characteristic, and Zariski established birational geometry over a field of arbitrary characteristic. The theorems of Weil and Zariski were among the main results of their era.

An important aspect of Grothendieck's later foundation was to adapt the categorical approach to algebraic geometry by rewriting the very foundations. This view is an ultimate example of Bourbaki's "structurism." Initially there was resistance to accepting this approach. However, more algebraic geometors began to appreciate solving problems by reaching the essence of the matter thorough generalization. Nowadays, Grothendieck's scheme theory is considered as the most natural and flexible theory available in algebraic geometry. Grothendieck's claim that not only an object in an absolute situation, but also an object in a relative situation must be studied, has been considered to be most natural, thanks to the usefulness of the representable functor theory.

Let me elaborate on my earlier usage of "relative." Consider a simple example of a polynomial with coefficients in integers

$$f(x_1, \ldots, x_n) = 0. \tag{1}$$

We can consider the common zeros of this equation not only as rational numbers, real numbers, or complex numbers, but also, through reduction of equation (1) at a prime number p (i.e., with coefficients in the finite field $\mathbb{Z}/p\mathbb{Z}$), we can consider the common zeros of equation (1) as p-adic numbers. Furthermore, for a homomorphism from the ring of integers to a commutative ring R, we can regard equation (1) as having coefficients in R. Through such relative consideration as above, the nature of the geometry of the common zeros of equation (1) becomes clear. Even though such a relative consideration had been made earlier for individual problems, it was Grothendieck who systematically introduced such a vision to algebraic geometry, to solve Weil's conjectures on congruence zeta functions. He obtained fruitful results, and expanded his theory in "*Éléments de Géométrie Algébrique*" (often abbreviated as EGA). However, much of his incomplete theory has been published as seminar notes. One has no difficulties studying Grothendieck's theory.

This book develops Grothendieck's scheme theory as a method for studying algebraic geometry. Our goal is to develop and apply scheme cohomology to the theories of algebraic curves and algebraic surfaces. In the preface of EGA, Grothendieck even claimed that a knowledge of classical algebraic geometry may hinder the reader from studying scheme theory. It might have been a necessary thing for him to say at the time, as he wanted his radical theory to be understood. However, things are reversed nowadays. One cannot understand and apply scheme theory without knowing classical algebraic geometry. Therefore we will not begin with schemes, but rather we will first describe the classical notion of algebraic varieties, which was introduced in the mid twentieth century.

The major part of this book will be devoted to preparing for the definition of a scheme. We will describe sheaf theory from an elemental viewpoint, with as few prerequisites as possible. From this scheme-theoretic foundation, we will urge the reader onward towards a unified understanding in "*Algebraic Geometry* 2" and "*Algebraic Geometry* 3", in which we study not only algebraic geometry but also the theory of complex analytic spaces. Unfortunately, much preparation is required to reach this higher view. I believe, though, that a careful reader will have little difficulty in understanding this book.

I am thankful to Yuji Shimizu for reading the manuscript, and for his corrections and advice.

October 1996
Kenji Ueno

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Preface to the English Translation

Algebraic geometry plays an important role in several branches of science and technology. The present book is the first of three volumes on scheme theory, the most natural form of algebraic geometry. These three volumes are written for non-specialists, to explain the main ideas and techniques of scheme theory. The original Japanese volumes have been widely accepted as introductory books for scheme theory. The author hopes the present English edition will serve the same role.

My special thanks are due to Professor Goro Kato, who undertook the difficult job of translating the Japanese edition into English. I also express my sincere thanks to the late Mr. H. Miyauchi, editor of Iwanami Shoten Publisher. Without his constant encouragement, the Japanese edition would never have appeared.

May 1999

Kenji Ueno

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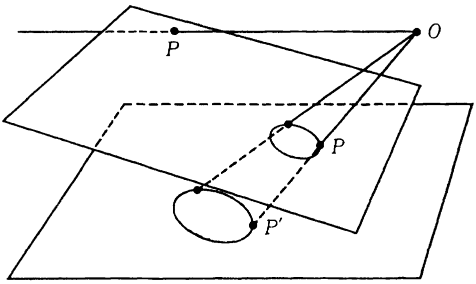
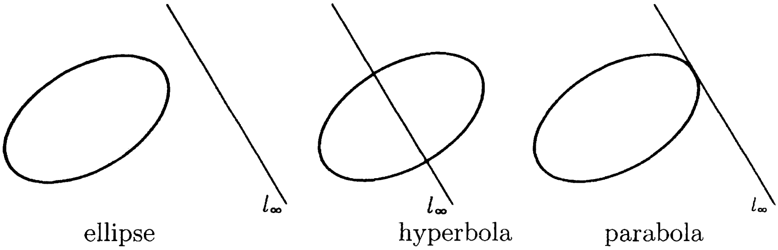
Summary and Goals

Algebraic geometry is a geometry of figures which are defined by equations. Projective geometry encouraged the gradual development of algebraic geometry. To put it simply, projective geometry is the geometry used to study the properties of figures that are invariant under projection from a point.



In the above figure, when the line from point O, the origin of projection, to a point P on a plane is parallel to the lower plane, the point P is not projected onto the lower plane. Thus, a point at infinity was introduced in projective geometry. For example, in two-dimensional projective geometry the totality of points at infinity, i.e., a line at infinity, needs to be added. With this adjustment, there is more geometric harmony than in Euclidean geometry. For instance, a parabola intersects with (more precisely, is tangent to) a unique point on the line at infinity. A hyperbola intersects with two distinct points on the line at infinity. An ellipse, a parabola, and a hyperbola are essentially the same geometric object on the projective plane, namely, an irreducible quadratic curve.

In coordinate geometry and projective geometry it is important to study the intersection points of two distinct curves. For example, let us consider a unit circle and a line on a plane. The points of



According to the location of the line at infinity,
the quadratic curve becomes an ellipse,
a hyperbola or a parabola.

intersection can be obtained by solving

$$x^2 + y^2 = 1,$$
$$ax + by + c = 0.$$

They intersect if there are real solutions to the system of equations, and they do not intersect if the system has complex solutions. There is no real reason for distinguishing complex solutions from real solutions. Thus, it is more natural to consider geometry over complex numbers.

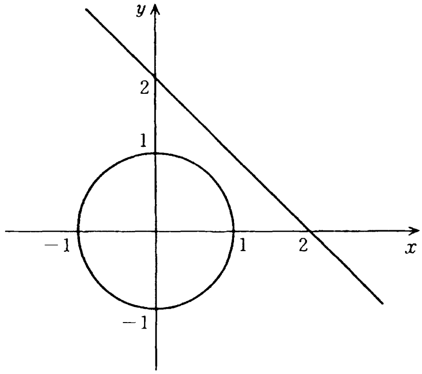
By considering projective geometry over the complex numbers, one obtains complex projective geometry. In complex projective geometry, an irreducible quadratic curve, like an ellipse, parabola, or hyperbola, always intersects with a line at two points (a tangent point is counted as two points).

In plane coordinate geometry, one can parameterize the unit circle as

$$\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right).$$

The point $(-1, 0)$ cannot be expressed in this parametric presentation. However, in the projective plane, the point $(-1, 0)$ corresponds to a point at infinity on a projective line, i.e., one-dimensional projective space. This correspondence is given over the complex numbers.

Thus an irreducible quadratic curve may be considered as a one-dimensional projective line. Figures which have a one-to-one correspondence through algebraic equations can be identified in the sense



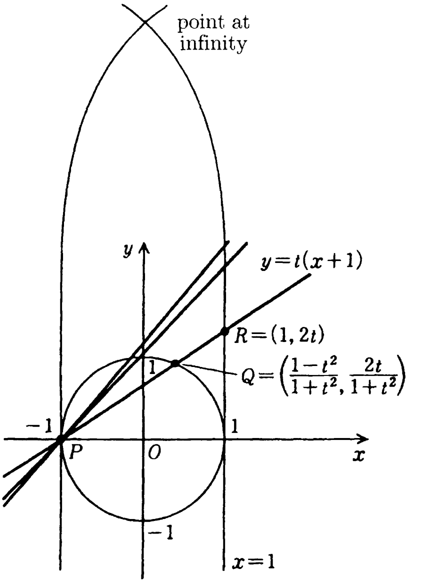
A unit circle $x^2 + y^2 = 1$ intersects the line $x + y = 2$ at the complex points $\left(1 + \frac{i}{\sqrt{2}}, 1 - \frac{i}{\sqrt{2}}\right)$ and $\left(1 - \frac{i}{\sqrt{2}}, 1 + \frac{i}{\sqrt{2}}\right)$.

of algebraic geometry rather than projective geometry. In this situation, flexibility in studying various figures is increased. This is a simplifying effect in the study of geometry.

In complex algebraic geometry, a circle is a projective line expressed in a projective plane, and the degree of choice of presentation is within projective transformations. That is, the presentation can be an ellipse, a parabola, or a hyperbola. Furthermore, this notion can be extended to a higher-dimensional projective space. By looking for an ultimate generalization, one reaches the notion of Grothendieck's scheme theory. The figure (or locus) in n-dimensional complex affine space $\mathbb{C}^n$ defined by

$$f_\alpha(z_1, \dots, z_n) = 0, \qquad \alpha \in A, \tag{1}$$

is considered as a presentation of an original figure. What is important is neither the equations, nor the ideal $J = (f_\alpha, \alpha \in A)$ in the polynomial ring $\mathbb{C}[z_1, \dots, z_n]$, but rather the commutative ring $R = \mathbb{C}[z_1, \dots, z_n]/J$. This commutative ring structure determines the nature of the geometry. A figure defined as in (1) can be considered as the figure determined by presenting R as the quotient ring of the ring $\mathbb{C}[z_1, \dots, z_n]$ of polynomials. A quotient ring of the polynomial ring can be presented in various ways. Hence, various figures can be considered as different presentations of a geometric object. Thus,



Correspondence between a unit circle and a line. The tangent line at $(-1,0)$ intersects the line $x = 1$ at a point at infinity.

it is not unnatural that one should begin to construct geometry from a commutative ring.

This book is an introductory book to scheme theory. The theory of schemes requires the knowledge of commutative rings, sheaf theory and homological algebra. We have tried to begin from as elementary a level as possible. New concepts are introduced one after another in this book; this may cause the reader to have difficulty in finding the essence of the geometry. This volume, *Algebraic Geometry* 1, will focus on the notion of a scheme as a local ringed space by using sheaf theory. This method enables us to present the theory of varieties (manifolds) in a systematic way. In particular, this method connects to the theory of complex analytic spaces. This connection will be treated in *Algebraic Geometry* 3.

The most important step in understanding this book is to make sheaf theory be second nature, and then convince yourself that an affine scheme can be defined by introducing the sheaf of commutative rings over the prime spectrum of a commutative ring. Then you may consider that you understand schemes.

A goal of algebraic geometry is not the introduction of schemes, but the use of schemes freely to study geometry. In order to take full advantage of scheme theory, one needs to study various properties of schemes in detail. The major part of scheme theory will be treated in *Algebraic Geometry* 2, and this book will serve as a foundation. As a preparation for Book 2, we describe some fundamental properties of schemes in Chapter 3, using the language of categories and functors.

This book should be considered as preparation for *Algebraic Geometry* 2 and 3. The definition of a scheme per se does not take up all the pages of this book. Rather than get mountain sickness by taking a lift directly to the top, we have decided to hike up the mountain step by step. Even with our choice, the path may appear to be steep. The reader is recommended to find his or her own examples when a new concept is introduced. We also provide various problems throughout this book, helping the reader to think through these concepts. We recommend that you read this book without undue haste.

We list a few notational assumptions that will be used throughout our series *Algebraic Geometry* 1, 2 and 3.

(i) A commutative ring is assumed to have an identity, denoted by either 1 or 1_R.

(ii) A ring homomorphism $f : R \to S$ is assumed to satisfy $f(1_R) = 1_S$.

(iii) We assume that for any element m of an R-module M we have $1_R m = m$.

(iv) When an arbitrary element of an R-module M can be written as a linear combination of finitely many elements $m_1, \ldots, m_n$ in M with coefficients in R, then M is said to be a *finite R-module*. Namely, if there exists an epimorphism from the finite direct sum $R^{\oplus n}$ onto M, then M is a finite R-module.

(v) For a commutative ring R, if an R-module S is a commutative ring such that for arbitrary $r \in R$ and $a, b \in S$ we have $r(ab) = (ra)b = a(rb)$, then S is said to be an *R-algebra*.

(vi) When an R-algebra S is finitely generated over R, i.e., when there exists an epimorphism of R-algebras from a polynomial ring $R[x_1, \ldots, x_n]$ onto S, then S is said to be a *finite* (or *finitely generated*) *R-algebra*.

Solutions to Problems

Chapter 1

Problem 1. An arbitrary ideal in $k[x]$ is generated by a single element. Hence, for $I = (f(x)) \neq 0$, the set

$$V(I) = \{a \in k | f(a) = 0\}$$

is finite.

Problem 2. $\mathbb{R}[x]$ is a principal ideal domain. $f(x)$ is irreducible if and only if $I = (f(x))$ is a maximal ideal. An irreducible polynomial in $\mathbb{R}[x]$ is either a linear polynomial or a quadratic polynomial.

Problem 3. Note that an element $x \in R$ is integral over S if and only if the subring $S[x]$ generated by x over S is a finite S-module. We prove this statement as follows. If x is integral over S, then we have $x^n + a_1x^{n-1} + \cdots + a_n = 0$, $a_j \in S$. Since $x^{n+r} = -a_1x^{n+r-1} - \cdots - a_nx^r$, an arbitrary element of $S[x]$ can be expressed as $\alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n$, $\alpha_j \in S$. Namely, $S[x]$ is a finite S-module. Conversely, when $S[x]$ is a finite S-module, any element of $S[x]$ can be written as a linear combination of $z_1, \ldots, z_l \in S[x]$ with coefficients in S, i.e.,

$$z_j = b_{j0} + b_{j1}x + \cdots + b_{jk_j}x^{k_j}, \qquad j = 1, \ldots, l.$$

Put $n = \max_j(k_j) + 1$. Then x^n can be expressed as a linear combination of $1, x, x^2, \ldots, x^{n-1}$. Therefore, R is integral over S.

Hence $S[w_1]$ is a finite S-module, and $S[w_1, w_2]$ is a finite $S[w_1]$-module. Therefore, $S[w_1, w_2]$ is a finite S-module. Consequently, $R = S[w_1, \ldots, w_l]$ is a finite S-module.

Finally, we prove that any element $x \in R = S[w_1, \ldots, w_l]$ is integral over S. Assume that as an S-module, R is generated by

$s_1, \ldots, s_n$. Then we get

$$xs_i = \sum_{j=1}^{n} a_{ij}s_j, \qquad a_{ij} \in S,\ i = 1, \ldots, n.$$

From this we obtain

$$\det(x\delta_{ij} - a_{ij}) = 0, \qquad \delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$$

That is, x is integral over S.

Problem 4. This is clear from Problem 1.

Problem 5. The map that sends $f(x, y)$ to $f(x, x^2)$ gives an isomorphism of rings $k[x, y]/(y - x^2) \to k[x]$.

Problem 6. Put $k[\mathbb{A}^1] = k[t]$. Then $\varphi^{\#-1}(t) = \varnothing$.

Problem 7. Put $z_j = x_j - a_j$, $j = 1, \ldots, m$, and $h(z_1, \ldots, z_m) = f(a_1 + z_1, \ldots, a_m + z_m)$. Then $h(0, \ldots, 0) = f(a_1, \ldots, a_m)$, and $h(z_1, \ldots, z_m) - h(0, \ldots, 0)$ belong to the ideal $(z_1, \ldots, z_m)$ consisting of terms of degree greater than one in $z_1, \ldots, z_m$. Namely,

$$f(x_1, \ldots, x_m) - f(z_1, \ldots, z_m) \in (x_1 - a_1, \ldots, x_m - a_m).$$

Problem 8. We have $a = (a_1, \ldots, a_n) \in V(J)$ if and only if $\mathfrak{m}_a = (x_1 - a_1, \ldots, x_n - a_n) \supset J$.

Problem 9. For $f \in I$, we have $D(f) \subset D(I)$. On the other hand, for $\mathfrak{m} \in D(I)$, we get $I \not\subset \mathfrak{m}$, i.e., we can find $f \in I$ such that $f \notin \mathfrak{m}$. Hence $\mathfrak{m} \in D(f)$ and $D(I) \subset \bigcup_{f \in I} D(f)$. Let J be the ideal generated by f_α, $\alpha \in A$. WE can show also that $\bigcup_{\alpha \in A} D(f_\alpha) = D(J)$.

Problem 10. Let $z = x - \alpha_j$. Notice that an inverse element exists in the ring $k[[z]]$ of formal power series for

$$g = b_0 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots$$

with the condition $b_0 \neq 0$. This is because, for

$$h = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots,$$

if $1 = gh = b_0c_0 + (b_0c_1 + b_1c_0)z + (b_0c_2 + b_1c_1 + b_2c_0)z^2 + \cdots$, we get $c_0 = 1/b_0$, $c_1 = -(b_1c_0)/(b_0) = -b_1(b_0)^2, \ldots$. Since $f(\alpha_j + z) = z^n g(z)$, $g(0) \neq 0$, we have $g(z)^{-1} \in k[[z]]$. That is, $(f(\alpha_j + z)) = (z^{n_j})$.

Problem 12. (1) 2, (2) 5, (3) 2 for $\alpha \neq 0$, 3 for $\alpha = 0$.

Problem 14. Put $I = (f_1, \dots, f_l)$, and let $f_{i1}, \dots, f_{in_i}$ be the homogeneous components of f_i. Then $f_{ij} \in I$, and $I = (f_{11}, \dots, f_{ln_l})$. Conversely, if I is generated by homogeneous polynomials, then I is a homogeneous ideal.

Problem 15. Let $f_{d_1} + \cdots + f_{d_n}$ be the homogeneous component decomposition of $f \in \sqrt{I}$. One can find a positive integer m to satisfy $f^m \in I$. Then the homogeneous component of the least degree of f^m is $f_{d_1}^m$. Since I is a homogeneous ideal, we have $f_{d_1}^m \in I$. Therefore $f_{d_1} \in I$. We have $f - f_{d_1} = f_{d_2} + \cdots + f_{d_n} \in \sqrt{I}$. Similarly, we get $f_{d_2} \in \sqrt{I}$. Then repeat this argument.

Problem 16. Write $f \in I(V)$ as a sum $f = f_d + f_{d+1} + \cdots + f_m$ of homogeneous polynomials. For $(a_0 : a_1 : \cdots : a_n) \in V$ and $\beta \neq 0$, we have $f(\beta a_0, \dots, \beta a_n) = 0$. Namely,

$$\begin{aligned}\beta^d f_d(a_0, \dots, a_n) &+ \beta^{d+1} f_{d+1}(a_0, \dots, a_n) \\ &+ \cdots + \beta^m f_m(a_0, \dots, a_n) = 0,\end{aligned}$$

which implies $f_d(a_0, \dots, a_n) = 0, \dots, f_m(a_0, \dots, a_n) = 0$. Consequently, $f_d, f_{d+1}, \dots, f_m \in I(V)$.

Problem 17. Repeat the proof of Proposition 1.4, replacing polynomials by homogeneous polynomials.

Problem 18. For the sake of simplicity, consider the case where $i = 0$. Suppose that for polynomials $g(z_1, \dots, z_n)$ and $h(z_1, \dots, z_n)$ of degrees l and m, respectively,

$$g\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) h\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \in I_0.$$

Then let

$$G = x_0^l g\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \quad \text{and} \quad H = x_0^m h\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right).$$

We get $GH \in I$, i.e., either $G \in I$ or $H \in I$. Hence, $g \in I_0$ or $h \in I_0$.

Problem 20. $V(F)$ is irreducible if and only if $I(V(F))$ is a prime ideal. Since $I(V(F)) = \sqrt{(F)}$, from the hypothesis we conclude that $\sqrt{(F)} = (F)$.

Chapter 2

Problem 1. (1) This is obvious from Proposition 2.1.

(2) $D(f) = \varnothing$ is equivalent to the statement that f is contained in all the prime ideals. The equality (2.11) in the proof of Lemma 2.10 implies that the above is equivalent to $f \in \sqrt{0}$.

Problem 2. If $g(x)$ and $h(x)$ in $\mathbb{Z}[x]$ satisfy $g(x)h(x) \in (f(x))$, then there is $r(x) \in \mathbb{Z}[x]$ satisfying $g(x)h(x) = f(x)r(x)$. Then, since $f(x)$ is irreducible in $\mathbb{Q}[x]$, and $f(x)$ divides either $g(x)$ or $h(x)$, the primitive $f(x)$ must divide either of those in $\mathbb{Z}[x]$.

Problem 3. For a prime ideal $\mathfrak{p}$, $f^m \in \mathfrak{p}$ is equivalent to $f \in \mathfrak{p}$.

Problem 4. Since $0 \in \mathfrak{p}$, we have $0 \notin R \setminus \mathfrak{p}$. For s and t in $R \setminus \mathfrak{p}$, if $st \in \mathfrak{p}$, then $\mathfrak{p}$ would not be a prime ideal. Namely, we get $st \in R \setminus \mathfrak{p}$.

Problem 6. If $\varphi_S(r) = 0$, then $tr/t = 0$, $t \in S$. Put $s = ut$. We have $sr = 0$. Conversely, if there is $s \in S$ to satisfy $sr = 0$, then we must have $\varphi_S(r) = 0$.

Problem 7. The inverse image of a maximal ideal in R_S is a maximal element in $\mathcal{S}$ under the natural homomorphism $\varphi_S : R \dot{\to} R_S$. Let $\mathfrak{m}$ be the ideal generated by $\varphi_S(\mathfrak{a})$ for a maximal element $\mathfrak{a}$ in $\mathcal{S}$. If $\mathfrak{m}$ is not a maximal ideal, then there is a maximal ideal $\mathfrak{n}$ such that $\mathfrak{m} \subsetneqq \mathfrak{n}$. Then $\mathfrak{b} = \varphi_S^{-1}(\mathfrak{n})$ would contain $\mathfrak{a}$. Since $\mathfrak{a}$ is maximal, $\mathfrak{a} = \mathfrak{b}$ or $\mathfrak{b} = R$. But those two cases contradict $\mathfrak{m} \subsetneqq \mathfrak{n}$ and $\mathfrak{n} \neq R_S$. Therefore, $\mathfrak{m}$ is a maximal ideal of R_S.

Problem 10. We clearly have $\sqrt{\mathfrak{a}} \subset \bigcap_{\mathfrak{a} \subset \mathfrak{p} \in \operatorname{Spec} R} \mathfrak{p}$. Suppose for $h \in \bigcap_{\mathfrak{a} \subset \mathfrak{p} \in \operatorname{Spec} R} \mathfrak{p}$ we have $h \notin \sqrt{\mathfrak{a}}$. Then put $\mathcal{S} = \{\mathfrak{b} | \mathfrak{b}$ is an ideal of R satisfying $\mathfrak{a} \subset \mathfrak{b}, h^m \notin \mathfrak{b}, m = 1, 2, \ldots\}$. Then $\mathcal{S} \neq \varnothing$. Let $\mathfrak{q}$ be a maximal element in $\mathcal{S}$. Then $\mathfrak{q}$ is a prime ideal. We have $\mathfrak{a} \subset \mathfrak{q}$. Hence $h \in \mathfrak{q}$ must hold. By the definition of $\mathcal{S}, h \notin \mathfrak{q}$, a contradiction. Therefore, $h \in \sqrt{\mathfrak{a}}$.

Problem 11. Let f_y and g_y be the germs at y determined by $f \in \mathcal{F}(U)$ and $g \in \mathcal{F}(V)$. Then, $\tilde{f} = \rho_{U \cap V, U}(f)$ and $\tilde{g} = \rho_{U \cap V, V}(g)$ determine f_y and g_y, respectively. Define $f_y + g_y$ to be the germ at y determined by $\tilde{f} + \tilde{g}$, and $f_y g_y$ to be the germ at y determined by $\tilde{f}\tilde{g}$.

Problem 12. The natural homomorphism $\varphi : R \to R_{\mathfrak{p}}$ induces a homomorphism $\overline{\varphi} : R/\mathfrak{p} \to R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. It is clear that $\overline{\varphi}$ is injective. Since $R/\mathfrak{p}$ is an integral domain and $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ is a field, $\overline{\varphi}$ can be extended to an injective homomorphism $\tilde{\varphi}$ from the quotient field $Q(R/\mathfrak{p})$ of $R/\mathfrak{p}$ to $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. When $a = r/s$, $s \notin \mathfrak{p}$, does not belong to $\mathfrak{p}R_{\mathfrak{p}}$, then $r \notin \mathfrak{p}$ holds and $\overline{r}/\overline{s} \in Q(R/\mathfrak{p})$, where $\overline{r}$ and $\overline{s}$ are the residue classes in $R/\mathfrak{p}$ of r and s, respectively. Since we have $\tilde{\varphi}(\overline{r}/\overline{s}) = a \pmod{\mathfrak{p}R_{\mathfrak{p}}}$, we conclude that $\tilde{\varphi}$ is surjective.

Problem 13. For $s = \{s_{\mathfrak{p}}\} \in \prod_{\mathfrak{p}\in U} R_{\mathfrak{p}}$, $s^{-1} = \{s_{\mathfrak{p}}^{-1}\}$ is an element of $\Gamma(U, \mathcal{O}_X)$.

Problem 14. For a multiplicatively closed set S in R, consider an R-bilinear map of R-modules

$$R_S \times M \to M_S$$

defined by $(r/s, a) \mapsto ra/s$. The universal mapping property of a tensor product implies the existence of an R-module homomorphism

$$\psi : R_S \otimes_R M \to M_S.$$

Clearly, ψ is surjective. In $R_S \otimes_S M$, we have

$$\sum_{j=1}^{l} \frac{r_j}{s_j} \otimes a_j = \sum_{j=1}^{l} \frac{t_j r_j}{s} \otimes a_j = \sum_{j=1}^{l} \frac{1}{s} \otimes t_j r_j a_j = \frac{1}{s} \otimes m,$$

where $s_j \in S$, $r_j \in R$, $a_j \in M$, and

$$s = s_1 \cdots s_l, \qquad t_j = s_1 \cdots s_{j-1} s_{j+1} \cdots s_l,$$

and $m = \sum_{j=1}^{l} t_j r_j a_j$. Namely, an element of $R_S \otimes_R M$ can be written as $\frac{1}{s} \otimes m$, $s \in S$, $m \in M$. If $\psi(\frac{1}{s} \otimes m) = \frac{m}{s} = 0$, there exists $t \in S$ such that $tm = 0$. On the other hand, in R_S we have $1/s = t/st$, and also

$$\frac{1}{s} \otimes m = \frac{t}{st} \otimes m = \frac{1}{st} \otimes tm = \frac{1}{st} \otimes 0 = 0.$$

That is, ψ is injective. Therefore ψ is an isomorphism.

Problem 15.

$$\Gamma(U, \widetilde{M}) = \left\{ \{m_{\mathfrak{p}}\} \in \prod_{\mathfrak{p}\in U} M_{\mathfrak{p}} \;\middle|\; \begin{array}{l} \text{choosing an open covering} \\ \{X_{f_\beta}\}_{\beta\in B} \text{ of } U,\ \beta \in B, \text{ and} \\ m_\beta \in M_{f_\beta} \text{ properly so that, for} \\ \mathfrak{p} \in X_{f_\beta}, \text{ the germ determined by} \\ m_\beta \text{ at } \mathfrak{p} \text{ may coincide with } m_{\mathfrak{p}} \end{array} \right\}$$

Problem 16. The proof is similar to that of Proposition 2.1 and Example 2.2.

Problem 17. If $f(x_0, \ldots, x_n)$ is a homogeneous polynomial of degree d, then $f(a_0, \ldots, a_n) \in S_d$. Hence, for $g \in \operatorname{Ker} \varphi$ such that $g = g_d + g_{d+1} + \cdots + g_m$, i.e., the sum of homogeneous polynomials, we have $g(a_0, \ldots, a_n) = g_d(a_0, \ldots, a_n) + g_{d+1}(a_0, \ldots, a_m) + \cdots + g_m(a_0, \ldots, a_m) = 0$. That is, $g_d(a_0, \ldots, a_n) = 0, g_{d+1}(a_0, \ldots, a_n) = 0, \ldots, g_m(a_0, \ldots, a_n) = 0$, i.e., $g_d, g_{d+1}, \ldots, g_m \in \operatorname{Ker} \varphi$.

Problem 18. For $f \in S_d$, $S_f^{(0)}$ is an R-algebra. Hence we have $\operatorname{Spec} S_f^{(0)} \to \operatorname{Spec} R$. Since $\operatorname{Proj} S$ is obtained by glueing $\operatorname{Spec} S_f^{(0)}$, a scheme morphism is obtained by glueing the above morphisms.

Problem 19. If $\Gamma(U, \mathcal{O}_X)$ has a nilpotent element f, then the germ f_x at $x \in U$ is a nilpotent element in $\mathcal{O}_{X,x}$. Conversely, if there is a nilpotent element f_x in $\mathcal{O}_{X,x}$ such that $f_x^m = 0$, then there is an $f \in \Gamma(V, \mathcal{O}_X)$ in a neighborhood V of x satisfying $f^m = 0$. Hence $\Gamma(V, \mathcal{O}_X)$ has a nilpotent element.

Problem 20. One can choose an open covering such that $X = \bigcup_{\lambda=1}^m U_\lambda$, $U_\lambda \in \operatorname{Spec} A$, where A_λ is Noetherian. Let $I(F_i \cap U_\lambda) = J_{\lambda,i}$. Then $V(J_{\lambda,i}) = F_i \cap U_\lambda$, and we have an ascending chain of ideals

$$J_{\lambda,1} \subset J_{\lambda,2} \subset \cdots \subset J_{\lambda,i} \subset J_{\lambda,i+1} \subset \cdots .$$

Since A_λ is a Noetherian ring, there exists i_λ such that

$$J_{\lambda,i_\lambda} = J_{\lambda,i_\lambda+1} = \cdots .$$

By putting $m = \max_\lambda(i_\lambda)$, we have $J_{\lambda,m} = J_{\lambda,m+1} = \cdots$. Namely,

$$F_m = F_{m+1} = \cdots .$$

Chapter 3

Problem 2. Determine φ_4 by $g_4(t) = t - 2t^2$ and $h_4(t) = t + t^2$. For $g_3(t) = t^2$ and $h_3(t) = t$, determine the R_4-valued point by $g_4(t) = t^2 + t^4$ and $h_4(t) = t$.

Problem 5. For a set-theoretic direct product $X \times Y$, we have an isomorphism

$$\operatorname{Hom}(Z, X \times Y) \simeq \operatorname{Hom}(Z, X) \times \operatorname{Hom}(Z, Y).$$

Problem 6. We have

$$\begin{aligned}&\mathrm{Hom}_{(\mathrm{Set})}(W, X \times_Z Y)\\ &\quad = \{F \in \mathrm{Hom}_{(\mathrm{Set})}(W, X \times Y) | q_1 \circ p_1 \circ F = q_2 \circ p_2 \circ F\}\\ &\quad = \{(f,g) \in \mathrm{Hom}_{(\mathrm{Set})}(W, X) \times \mathrm{Hom}_{(\mathrm{Set})}(W, Y) | q_1 \circ f = q_2 \circ g\}.\end{aligned}$$

Problem 7. For $T \in \mathrm{Ob}(\mathcal{C})$, (3.12) becomes

$$G(T) = \{(f,g) \in \mathrm{Hom}_{\mathcal{C}}(T, X) \times \mathrm{Hom}_{\mathcal{C}}(T, X) | q \circ f = g\},$$

which is isomorphic to $\mathrm{Hom}_{\mathcal{C}}(T, X)$. Namely, we get $G(T) \simeq h_X(T)$.

Problem 8. Over the field k, $k[x_1, \ldots, x_m] \otimes_k k[y_1, \ldots, y_n]$ is generated by $x_1 \otimes 1, \ldots, x_m \otimes 1$ and $1 \otimes y_1, \ldots, 1 \otimes y_n$, and is isomorphic to the polynomial ring $k[z_1, \ldots, z_{m+n}]$. $\mathrm{Spec}\, k$ consists of a point, and the underlying space of $\mathbb{A}^1_k \times_{\mathrm{Spec}\, k} \mathbb{A}^1_k$ is the direct product as a set. The closed set $V((f(x,y)))$ in $\mathbb{A}^2_k$ determined by an irreducible polynomial $f(x,y)$ involving two variables x and y is not a closed set for the product topology on $\mathbb{A}^1_k \times_{\mathrm{Spec}\, k} \mathbb{A}^1_k$.

Problem 9. If Δ is a closed set, then Δ^c is open. For $a \neq b$ we have $(a,b) \in \Delta^c$, and there is an open neighborhood W of (a,b) contained in Δ^c. Moreover, for sufficiently small open neighborhoods U and V of a and b, respectively, we get $U \times V \subset W$. Then we have $U \times V \cap \Delta = \varnothing$; that is, $U \cap V = \varnothing$. Conversely, for $a \neq b$, choose open neighborhoods U and V of a and b satisfying $U \cap V = \varnothing$. We have $U \times V \subset \Delta^c$.

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Solutions to Exercises

Chapter 1

1.1. (1) Clearly we have $\varphi(\mathbb{A}_k^1) \subset V((x^3 - y^2, y^2 - z))$. For $(a, b, c) \in V((x^3 - y^2, y^2 - z))$, we have $a^3 = b^2$ and $b^2 = c$. If $a = 0$, then $b = 0$ and $c = 0$. Thus we get $\varphi(0) = (0, 0, 0)$, i.e. contained in the image of φ. For $a \neq 0$, write $a = (b/a)^2$. Put $t = b/a$. Then $a = t^2$, $b = at = t^3$, $c = b^2 = t^6$. Then $(a, b, c) = \varphi(b/a)$. We have $\varphi(\mathbb{A}_k^1) = V((x^2 - y^2, y^2 - z))$. Since for $t \neq t'$ we have $\varphi(t) \neq \varphi(t')$, φ is a set-theoretic bijection. On the other hand, φ induces a ring homomorphism

$$\begin{aligned} \varphi^{\#}: \qquad k[x, y, z]/(x^3 - y^2, y^2 - z) &\to k[t], \\ f(x, y, z) \ (\mathrm{mod}(x^3 - y^2, y^2 - z)) &\mapsto f(t^2, t^3, t^6). \end{aligned}$$

Since $\varphi^{\#-1}(t) = \varnothing$, $\varphi^{\#}$ is not an isomorphism.

(2) The proof is similar to (1).

1.2. Since $P_1 = (a_0 : a_1), P_2 = (b_0 : b_1)$ and $P_3 = (c_0 : c_1)$ are distinct points, one can choose $\alpha, \beta \in k$ to satisfy

$$b_0 = \alpha a_0 + \beta c_0, \qquad b_1 = \alpha a_1 + \beta c_1.$$

Then the projective transformation

$$\varphi_P : (x_0 : x_1) \mapsto (\alpha a_0 x_0 + \beta c_0 x_1 : \alpha a_1 x_0 + \beta c_1 x_1)$$

takes $\varphi_P((1 : 0)) = P_1$, $\varphi_P((1 : 1)) = P_2$, $\varphi_P((0 : 1)) = P_3$. Similarly, Q_1, Q_2, and Q_3 induce a projective transformation φ_Q satisfying $\varphi_Q((1 : 0)) = Q_1$, $\varphi_Q((1 : 1)) = Q_2$ and $\varphi_Q((0 : 1)) = Q_3$. Then $\psi = \varphi_Q \circ \varphi_P^{-1}$ is what is needed.

1.3. (1) The proof is clear from the determinants

$$\begin{vmatrix} a_0 & a_1 & a_2 \\ a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} b_0 & b_1 & b_2 \\ a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{vmatrix} = 0.$$

(2) From the above, the equation of the line is given by

$$\begin{vmatrix} x_0 & x_1 & x_2 \\ a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{vmatrix} = 0.$$

1.4. Without loss of generality, by a projective transformation we may assume that the line L is given by the equation $x_2 = 0$. Then x_2 does not divide $F(x_0, x_1, x_2)$. Hence $F(x_0, x_1, 0)$ is a homogeneous polynomial of degree n. Then, with the multiplicity considered, there are n solutions.

1.5. Clearly, $\varphi(\mathbb{P}^1_k) \subset V((x_0x_2 - x_1^2))$. For $(b_0 : b_1 : b_2) \in V((x_0x_2 - x_1^2))$, we have $b_0b_2 - b_1^2 = 0$. If $b_0 \neq 0$, we get

$$\frac{b_2}{b_0} = \left(\frac{b_1}{b_0}\right)^2.$$

Hence

$$(b_0 : b_1 : b_2) = \left(1 : \frac{b_1}{b_0} : \frac{b_2}{b_0}\right) = \varphi\left(\left(1 : \frac{b_1}{b_0}\right)\right).$$

If $b_2 \neq 0$, we get

$$\frac{b_0}{b_2} = \left(\frac{b_1}{b_2}\right)^2.$$

Then

$$(b_0 : b_1 : b_2) = \left(\frac{b_0}{b_2} : \frac{b_1}{b_2} : 1\right) = \varphi\left(\left(\frac{b_1}{b_2} : 1\right)\right).$$

For $b_0 = b_2 = 0$, we get $b_1 = 0$, i.e., not a point on the projective plane. Consequently, $\varphi(\mathbb{P}^1_k) = V((x_0x_2 - x_1^2))$, i.e., φ is surjective. If $((a_0)^2 : a_0a_1 : (a_1)^2) = ((c_0)^2 : c_0c_1 : (c_1)^2)$, it is easy to show that $(a_0 : a_1) = (c_0 : c_1)$, i.e., φ is injective.

1.6. Similar to 1.5.

Chapter 2

2.1. If f is not a nilpotent element in R, then there exists a prime ideal $\mathfrak{p}$ that does not contain f. For $f \in \sqrt{\mathfrak{a}}$, we have $\mathfrak{p} \notin V(\mathfrak{a})$. On the other hand, $\sqrt{\mathfrak{a}}$ does not contain invertible elements of R. Hence $\sqrt{\mathfrak{a}}$ contains only nilpotent elements of R. By the definition, we have $\mathfrak{N}(R) = \sqrt{(0)}$. Then $\mathfrak{N}(R) = \sqrt{(0)} \subset \sqrt{\mathfrak{a}} \subset \mathfrak{N}(R)$, i.e., $\mathfrak{N}(R) = \sqrt{(0)}$.

2.3. We have

$$U = \bigcup_{j=1}^{n} D(x_j).$$

Express $f \in \Gamma(U, \mathcal{O}_{\mathbb{A}^n})$ as $f = f_j/x_j^{m_j}$ on $D(x_j)$ where $f_j \in R$. Then on $D(x_i) \cup D(x_j)$, we get $x_i^{m_i} f_j = x_j^{m_j} f_i$. Namely, $x_i^{m_i}$ must divide f_i. Hence f must be a polynomial.

2.4. We only need to show that p is continuous. For an open set U of X, let s_x be an arbitrary point in $p^{-1}(U)$, where $x \in U$. Then there exist an open set V containing x and a section $s \in \Gamma(V, \mathcal{F})$ such that the germ of s at x is s_x. We have $V(s) \subset p^{-1}(U)$. Each point in $p^{-1}(U)$ has an open neighborhood contained in $p^{-1}(U)$, i.e., $p^{-1}(U)$ is open. That is, p is continuous.

(1) For an open set $V(c)$ with $c \in \Gamma(V, \mathcal{F})$, if $a_+^{-1}(V(c))$ is open, then a_+ is continuous.

If $(a_x, b_x) \in a_+^{-1}(V(c))$ for $x \in V$, then we have $a_x + b_x = c_x$. Then in V one can take an open set W containing x so that the germs of a and b in $\Gamma(W, \mathcal{F})$ are a_x and b_x. The germ of $a + b$ at x is c_x. Then, by the definition of a germ, in a neighborhood W_0 of x in W, we have $a + b|_{W_0} = c|_{W_0}$. Then $W_0(a)$ and $W_0(b)$ are open sets of $\mathbb{F}$, and $W_0(a) \times_{W_0} W_0(b)$ is an open set of $\mathbb{F} \times_X \mathbb{F}$. Hence, we obtain $W_0(a) \times_{W_0} W_0(b) \subset a_+^{-1}(V(c))$, i.e., $a_+^{-1}(V(c))$ is an open set. That is, a_+ is a continuous map. The proofs for a_-, 0, and m are similar.

(2) For $s \in \Gamma(U, \mathbb{F})$ and $x \in U$, one can choose an open neighborhood V_x of x and $a^{(x)} \in \Gamma(V_x, \mathcal{F})$ so that the germ of $a^{(x)}$ at x may coincide with $s(x)$. If necessary, choose V_x sufficiently small to satisfy $V_x \subset U$, so that for an arbitrary $y \in V_x$ the germ of $a^{(x)}$ at y may coincide with $s(y)$. Therefore, $\{V_x | x \in U\}$ is an open covering of U, and for $V_x \cap V_y \neq \varnothing$, $a^{(x)} \in \Gamma(V_x, \mathcal{F})$ satisfies $a^{(x)}|_{V_x \cap V_y} = a^{(y)}|_{V_x \cap V_y}$. By the definition of a sheaf, there exists

$a \in \Gamma(U, \mathcal{F})$ satisfying $a|_{V_x} = a^{(x)}$. The correspondence $s \to a$ identifies $\Gamma(U, \mathbb{F})$ with $\Gamma(U, \mathcal{F})$.

2.5. We will show that $\widetilde{\mathcal{G}}$ satisfies (F1) and (F2). The restriction map $\rho_{V,U} : \widetilde{\mathcal{G}}(U) \to \widetilde{\mathcal{G}}(V)$ is obtained from the usual restriction of a map. For $U = \bigcup_{\lambda \in \Lambda} U_\lambda$ and $s \in \widetilde{\mathcal{G}}(U)$, if $\rho_{U_\lambda, U}(s) = 0$ then $s(x) \in \mathcal{G}_x$ is zero for an arbitrary point x in U. That is, $s = 0$, verifying (F1). On the other hand, for $U_\lambda \cap U_\mu \neq \varnothing$, if $s_\lambda \in \widetilde{\mathcal{G}}(U_\lambda)$ satisfies $\rho_{U_\lambda \cap U_\mu, U_\lambda}(s_\lambda) = \rho_{U_\lambda \cap U_\mu, U_\mu}(s_\mu)$, then $s_\lambda(x) = s_\mu(x)$ for $x \in U_\lambda \cap U_\mu$. Therefore, $s_\lambda : U_\lambda \to \widetilde{\mathbb{G}}$, $\lambda \in \Gamma$, determines $s : U \to \widetilde{\mathbb{G}}$ satisfying $s_\lambda = \rho_{U_\lambda, U}(s)$, $s \in \widetilde{\mathcal{G}}(U)$, verifying (F2).

Chapter 3

3.1. Define $\psi : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}(h_X, h_Y)$ as follows. For $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $W \in \mathrm{Ob}(\mathcal{C})$ and $m \in h_X(W)$, define the map by assigning $f \circ m \in h_Y(W)$. Then define $\eta(W)(m) = f \circ m$. For $g \in \mathrm{Hom}_{\mathcal{C}}(Z, W)$, the diagram

$$\begin{array}{ccccc}
h_X(W) & & \xrightarrow{\eta(W)} & & h_Y(W) \\
 & m & \longmapsto & f \circ m & \\
{\scriptstyle h_X(g)}\downarrow & \downarrow & & \downarrow & \downarrow{\scriptstyle h_Y(g)} \\
 & m \circ g & \longmapsto & f \circ m \circ g & \\
h_X(Z) & & \xrightarrow[\eta(Z)]{} & & h_Y(Z)
\end{array}$$

commutes. Therefore η defines a morphism from h_X to h_Y. Let $\psi(f) = \eta$. Consider the map $\psi \circ \varphi$. For $\eta \in \mathrm{Hom}(h_X, h_Y)$, let f be $\varphi(\eta) = \eta(X)(\mathrm{id}_X)$. For $m \in h_X(W)$ we have the commutative diagram

$$\begin{array}{ccccc}
h_X(X) & & \xrightarrow{\eta(X)} & & h_Y(X) \\
 & id_X & \longmapsto & f & \\
{\scriptstyle h_X(m)}\downarrow & \downarrow & & \downarrow & \downarrow{\scriptstyle h_Y(m)} \\
 & m & \longmapsto & f \circ m & \\
h_X(W) & & \xrightarrow[\eta(W)]{} & & h_Y(W)
\end{array}$$

obtaining $\eta(W)(m) = f \circ m$. Hence we get $\psi(f) = \eta$, which implies $\psi \circ \varphi(\eta) = \eta$. That is, $\psi \circ \varphi$ is an identity map. Similarly one shows that $\varphi \circ \psi$ is an identity map, proving the bijectivity of φ.

3.2. For a monic polynomial $f(x) \in K[x]$, consider

$$L = K[x]/(f(x)).$$

In $\overline{K}$ we can decompose $f(x) = \prod_{i=1}^{n}(x - \alpha_i)$, $\alpha_i \in \overline{K}$, $n = [L : K]$. There exists a ring isomorphism

$$L \otimes_K \overline{K} = \overline{K}[x]/(f(x)) \simeq \prod_{i=1}^{n} \overline{K}[x]/(x - \alpha_i) \simeq \prod_{i=1}^{n} \overline{K}.$$

Therefore, by Example 2.33,

$$X \times_Z Y = \operatorname{Spec}(L \otimes_K \overline{K}) \simeq \operatorname{Spec}\left(\prod_{i=1}^{n} \overline{K}\right)$$

is the direct sum of n copies of $\operatorname{Spec} \overline{K}$.

3.3. (1) $R = \mathbb{R}[x, y]/(x^2 + y^2)$ is an integral domain.

(2) Since $\mathbb{C}[x, y]/(x^2 + y^2) \simeq \mathbb{C}[x, y]/((x + \sqrt{-1}y)(x - \sqrt{-1}y))$, it follows that $X_0 \times_{\mathbb{R}} \mathbb{C}$ is reduced but not irreducible.

(3) $\mathbb{R}$-valued points correspond in a one-to-one manner with $\mathbb{R}$-homomorphisms

$$\varphi : R = \mathbb{R}[x, y]/(x^2 + y^2) \to \mathbb{R}.$$

Let $\overline{x}$ and $\overline{y}$ be the residue classes in R of x and y, respectively. For $\varphi(\overline{x}) = a$ and $\varphi(\overline{y}) = b$, we have $a^2 + b^2 = 0$. Since $a, b \in \mathbb{R}$, we get $a = b = 0$, i.e., φ is uniquely determined. On the other hand, there is a one-to-one correspondence between $\mathbb{C}$-valued points and $\mathbb{R}$-homomorphisms $R \to \mathbb{C}$. Put $\psi(\overline{x}) = a$ and $\psi(\overline{y}) = b$. Then $a^2 + b^2 = 0$, which implies $b = \pm a\sqrt{-1}$. Conversely, for $a \in \mathbb{C}$ there is an $\mathbb{R}$-homomorphism $\psi_a : R \to \mathbb{C}$ satisfying $\psi_a(\overline{x}) = a$ and $\psi_a(\overline{y}) = a\sqrt{-1}$. Hence there are infinitely many $\mathbb{C}$-valued points.

3.4. Let M be the maximal ideal of the Noetherian local ring R. For $f : \operatorname{Spec} R \to X$, put $f(M) = x$. By the definition of a scheme morphism, we get a local homomorphism $g : \mathcal{O}_{X,x} \to R$. Conversely, for a given (x, y), there exists an affine open set $\operatorname{Spec} A$ containing x. Put $x = \mathfrak{p} \in \operatorname{Spec} A$. Then we have $\mathcal{O}_{X,x} = A_{\mathfrak{p}}$, and we get $\tilde{g} : A \to R$ by composing $g : A_{\mathfrak{p}} \to R$ with $A \to A_{\mathfrak{p}}$. Namely, we obtain a scheme homomorphism $\operatorname{Spec} R \to \operatorname{Spec} A \subset X$, i.e., an R-valued point is obtained.

3.5. For simplicity, assume that a k-rational point is contained in

$$U_0 = \operatorname{Spec} k\left[\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right].$$

Then the k-rational point is expressed by $(b_1, \dots, b_n) \in \mathbb{A}_k^n$, corresponding to the maximal ideal

$$\left(\frac{x_1}{x_0} - b_1, \frac{x_2}{x_0} - b_2, \dots, \frac{x_n}{x_0} - b_n\right).$$

The homogeneous ideal in $k[x_0, \dots, x_n]$ corresponding to this ideal is

$$\mathfrak{p} = (x_1 - b_1 x_0, x_2 - b_2 x_0, \dots, x_n - b_n x_0).$$

For $(a_0 : a_1 : \cdots : a_n) = (1 : b_1 : \cdots : b_n)$, this ideal coincides with $(a_i x_j - a_j x_i,\ 0 \le i, j \le n)$.

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