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Mitsuru Ikawa

American Mathematical Society

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Volume 189

Hyperbolic Partial Differential Equations and Wave Phenomena

Mitsuru Ikawa

Translated by
Bohdan I. Kurpita

American Mathematical Society
Providence, Rhode Island

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PARTIAL DIFFERENTIAL EQUATIONS 2

by Mitsuru Ikawa

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Preface to the English Edition

This book is a translation into English of a book that I wrote in Japanese. This book is based on my lectures at various universities.

When I originally set pen to paper, I had only a Japanese audience in mind. So, I am deeply pleased and honored that by means of this English translation the material in this book will reach a wider audience.

I would like to express my appreciation to the American Mathematical Society for publishing this translation. My heartfelt thanks go to Bohdan I. Kurpita for his translation of my book from Japanese into English.

August 1999

Mitusuru Ikawa

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Preface to the Japanese Edition

In this book we discuss initial boundary value problems for second order hyperbolic equations. The term hyperbolic equation refers to members of a specific class of partial differential equations. The most representative examples of this class are the partial differential equations that describe wave phenomena. In essence, the study of hyperbolic equations and the mathematical investigation of wave phenomena can be thought of as one and the same thing. For the purposes of this book, we will restrict our attention to linear equations. What this means for wave phenomena is that the oscillations cannot be particularly large.

Most likely, the first image the mind conjures up when one hears or reads the word "wave" is of water lapping back and forth. More precisely, the surface of the water undulates in a periodic manner with the effect being passed on to the surroundings. This type of phenomenon can also be seen when a string vibrates or a membrane oscillates.

Another form of "wave" behaviour that readily comes to mind is that of sound propagating through air or some other substance. Another example is an electromagnetic wave. A less sanguine example is that of an earthquake, which is a wave that radiates outwards from an epicenter and, on occasion, the subsequent oscillations have a devastating effect causing widespread destruction and bringing misery and death.

As the above examples illustrate, we encounter many wave phenomena on a daily basis, some wave phenomena can be readily seen and some not. At first sight, these wave phenomena seem disparate since they arise from different physical circumstances, however, if we write the mathematical formulae that govern these phenomena, then the phenomena all conform to partial differential equations of the same form; namely, *hyperbolic equations.*

The added benefit of expressing these types of phenomena mathematically is that investigating the partial differential equations leads to detailed insight into wave phenomena. In addition, the similarity among the partial differential equations for various wave phenomena suggests a deep commonality between the phenomena.

In the case of the equation for the propagation of sound, the unknown function is a scalar function. On the other hand, in Maxwell's equations, which govern the transmission of electromagnetic waves, the unknown function is vector valued. This implies that even though the hyperbolic equations are of the same type, the complexity of the associated phenomena will differ according to whether the unknown function is scalar valued or vector valued. Thus, by focusing our deliberations on the characteristics of each of the equations, we will unearth properties of both sound and electromagnetism.

Throughout this book, linear hyperbolic equations of second order with a scalar-valued unknown function will be central to our discussions. The reason is that this type of equation is common and fundamental for all of the equations that describe wave phenomena. With this in mind, our overarching aim is to consider particular linear hyperbolic equations of second order and elucidate the properties of the phenomena governed by this second order equation. In this way, we will discover properties that are common to various wave phenomena.

In summary, we will concern ourselves with a wave transmitted in a space with boundary, and, having been given the initial condition(s) and the state on the boundary, we will investigate how the solution develops with time. Towards this end, we first need to give a mathematical proof of the existence of a solution for the given problem. Then, our core problems are to clarify mathematically observations such as the direction of motion, the reflection of the wave at a boundary, the refraction of the wave at the surface of two substances that are in contact, etc. The clarification of these observations will be obtained by determining properties of the solution.

In closing, I wish to express my deep gratitude to the various members of the editorial board who encouraged me to write this book. In particular, I would like to extend my heartfelt appreciation to Professor Aomoto who read the original manuscript and offered many helpful suggestions.

Also, I wish to thank the editorial staff of the publisher, Iwanami. I have great admiration for all their efforts.

January 1997

Mitùsuru Ikawa

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Outline of the Theory and Objectives

As explained in the preface, the primary objective of this book is to understand wave phenomena in mathematical terms.

So immediately in Chapter 1, we develop the mathematical laws that govern the behaviour for several wave phenomena. To be more precise, the mathematical expressions that we derive in this first chapter consist of a relationship or relationships between the partial derivatives of the unknown function(s); such a relationship(s) is usually referred to as a *partial differential equation* (*or system of equations*). Among the phenomena to be considered in Chapter 1 are the vibration of a string, the oscillation of a spring and the propagation of sound.

As illustrative examples, we look at the vibration of a string and the oscillation of a spring. In the case of a string, if the string is disturbed from its state of rest, then the tension of the string causes a force to arise that pulls the disturbed part towards its rest position. In fact, this force gives a certain acceleration to the string, and the vibration is passed on to the surroundings. Thus a wave is formed.

While for the case of a spring, instead of tension we need to investigate the effect of elasticity. Briefly, what we mean is that when we apply a force to some part of the spring, we effect a contraction on this part of the spring, which, in turn, passes an oscillation to the surrounding parts.

Even the casual observer should be able to perceive a certain dichotomy between the nature of the vibration and the oscillation described above. In the case of the vibration of a string, each part of the string vibrates perpendicular to the direction in which the wave moves. In other words, each part of the string vibrates in a direction that is perpendicular to its position at rest, and with time the vibration is transmitted along the stretched string in the form of a wave. Hence, this kind of wave is known as a *transverse wave*.

In contrast, each part of a spring oscillates in the direction of the extended spring. So, this kind of wave is called a *longitudinal wave*.

From the discussion above, it is clear that the respective waves arise from quite different physical conditions, and the vibration and oscillation, themselves, are substantially different. However, as we will see in Chapter 1, if we derive the respective partial differential equations for these two phenomena, what we see is that the differential equations are exactly the same. This means that even though the vibration/oscillation are perceived differently, because these two waves have the same equation, a property for one of the waves also becomes a property of the other wave. For example, the speed of propagation to its surroundings is constant for each of them, reflection at its end point occurs for each of them, etc.

Continuing on from the introduction of the equation of the wave for the string and spring, we next turn our attention to the equation for the propagation of sound. Even if the dimension of the space is high, the astute reader will not be surprised to learn that the partial differential equation which will be found has the same form as that for a string or a spring. Therefore, an immediate consequence is that the propagation of sound must share the same properties, two of which were just mentioned, as the vibration of a string and the oscillation of a spring.

Having established the above, we will next write down, but not derive, *Maxwell's equations* that describe the propagation of electromagnetic waves and *elastic waves* that describe how a wave is transmitted through an elastic body, one example of such a wave comes from the observation of earthquakes. In both of these cases, the equations have unknown functions that are vector valued. Therefore, the oscillating phenomena that are governed by these equations include rather complicated terms. However, by careful consideration of these equations, it is possible to see that they consist of several equations each of which gives an expression as for the propagation of sound, and so without equivocation we can call them *wave equations*. In fact, if we restrict ourselves to relatively simple conditions, then we are able to see that each part of the unknown function becomes a solution of the wave equation. Conversely, by using the solution of the wave equation, we can construct solutions of the above equations. In the above sense, as stated in the Preface, the study of a single hyperbolic equation forms a basis for the investigation of a wave.

Also in Chapter 1, we define what we mean by a linear hyperbolic equation of second order. Then, in Chapter 2, we discuss the basic properties of the solution of this equation. As one of the more prominent properties of the solution, we will find that the speed of propagation of the solution is finite. For the wave equation we know that the solution propagates with a fixed speed, and thus we can conclude that a common property of hyperbolic equations is that the speed with which the oscillation/vibration, given at some part, propagates to its surroundings is finite.

Having completed the above, we will prove that the solution for the initial boundary value problem exists and is smooth.

In Chapter 3, we consider the asymptotic solution of the initial boundary value problem for hyperbolic equations. As stated above, in Chapter 2 we prove the existence of the solution. The proof depends on methods from functional analysis. However, when the state is fixed, these methods are not very suitable for studying the detailed behaviour of the solution. On the other hand, using the methods of asymptotic solutions, we can explicitly construct a good approximation to the solution, the existence of which has already been guaranteed. Thus, we can investigate in detail the way the solution propagates under some particular condition for one of the phenomena governed by the equations. In Chapter 3 we look at the propagation of a particularly important wave with a high frequency.

Light is a good example of a high frequency wave. If we take a minute to consider everyday occurrences, then what stands out is that light travels in a straight line and, also, when it hits a boundary it is reflected. Another light phenomenon that probably is familiar to us from experience is the refraction of light that occurs at the boundary of air and water. Properties such as those just mentioned for light (i.e., a slew of properties of waves that are commonly, if unconsciously, observed by us) in fact, can be explained mathematically by making use of the asymptotic solution.

By similar means, when the solution has a discontinuity, we can study how this discontinuity is transmitted.

In Chapter 4, with respect to the decay of the local energy, we will investigate the propagation of a wave outside a bounded obstacle. If there is no loss of energy due to, say, friction, then the total energy for the wave does not change with time. However, the positions that bear the energy, i.e., the positions of vibration/oscillation, will gradually recede into the distance with the passage of time.

Now, suppose that an observer is standing at some particular point. With the person stationary, let us assume that a vibration or oscillation arises somewhere and subsequently this leads to the transmission of a wave. Further, this wave passes by the observer and on hitting some solid object the wave is reflected repeatedly. After it has passed by the stationary observer a number of times, the effect, with time, will begin to diminish, and eventually in the environs of the observer the waves will die out. In fact, the consideration of such a phenomenon is the same as the investigation of the decay of local energy.

The reader should keep in mind that the problems that we consider in Chapters 3 and 4 relate only to very particular characteristics of waves. In contrast to a wave with a high frequency, a wave with an extremely low frequency, that is, one with a very high wavelength, has virtually no aftereffect even if it does hit some obstacle. Thus, what we can say is that the wave passes by without any reflection. In this book, we will not consider such a phenomenon. Also, we will not cover how the obstacle's shadow effects the wave. This problem is quite a difficult one. However, from the perspectives of both mathematics and physics, this problem is a very interesting one and worthy of future research.

If we try to encapsulate the formal aims of this book, then, first, it is to prove in the strict mathematical sense the existence of a solution, and, in addition, it is to describe the basis of the mathematical means that explain the most typical properties of a wave. To aid the reader's comprehension of our fulfillment of these aims, we will on several occasions repeat similar arguments.

Nomenclature used in this book

We denote the n-dimensional Euclidean space by $\mathbb{R}^n$ and a point in $\mathbb{R}^n$ by $x = (x_1, x_2, \ldots, x_n)$ with $x_j \in \mathbb{R}$ $(j = 1, 2, \ldots, n)$.

(1) Suppose D is an open set in $\mathbb{R}^n$. Then we will write the partial derivative of first order of a function f defined on D as

$$\frac{\partial f}{\partial x_j} \quad \text{or} \quad f_{x_j}.$$

For the exponent $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_j \in \{0, 1, 2, \ldots\}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$, the partial derivatives of higher order,

$$\frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},$$

will be denoted by

$$\partial_x^\alpha f(x) \quad \text{or} \quad f^{(\alpha)}(x).$$

(2) We denote by $C(D)$ the space consisting of all the continuous functions that are defined on D. Then, for $m = 0, 1, 2, \ldots$ we define $C^m(D)$ by $C^0(D) = C(D)$ and otherwise by

$$C^m(D) = \{f \in C(D);\ \text{for all } |\alpha| \leqslant m \quad f^{(\alpha)} \in C(D)\}.$$

Further, we set $C^\infty(D) = \bigcap_{m=1}^{\infty} C^m(D)$.

(3) We set $\mathcal{B}(D) = \{f \in C(D); f \text{ is bounded in } D\}$; then for $m = 1, 2, \ldots$ we define

$$\mathcal{B}^m(D) = \{f \in C^m(D);\ \text{for all } |\alpha| \leqslant m \quad f^{(\alpha)} \in \mathcal{B}(D)\}.$$

In addition, we set $\mathcal{B}^\infty(D) = \bigcap_{m=1}^{\infty} \mathcal{B}^m(D)$.

(4) Let E be a linear space. Then $C^m(D; E)$ denotes all the functions defined on D that take their values in E and are functions with continuous partial derivatives of mth order with respect to the topology on E.

(5) We will denote Sobolev spaces by $H^m(D)$ $(m = 0, 1, 2, \ldots)$; a definition of these spaces is given in §2.2(a).

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Perspectives on Current Research in Mathematics

In July of 1990, I had the good fortune to attend an international research conference entitled *25 Years of Microlocal Analysis,* which was held very close to the borders of Austria and Switzerland. At this conference, Prof. L. Gårding, one of the preeminent researchers in the theory of partial differential equations, delivered a lecture on the topic of (the) *History of the Mathematics of Double Refraction.* The phenomenon of "double refraction" can be typically observed by looking at what happens to a letter once it passes through a piece of calcite to its underside. As the name suggests, the letter that appears on the underside is seen in double, as if the observer has a case of double vision. In terms of a ray of light entering this piece of calcite, this phenomena is the result of this light ray being divided into several rays.

Prof. Gårding's lecture concerned itself with the history of the mathematical research behind double refraction, starting with Huygens' "Traité de la lumière", published in 1690, and continuing to the current day. The lecture resonated deeply with me, and so I asked for a hard copy of the lecture. The paper of the lecture was subsequently published in *Archive for History of Exact Sciences,* **40** (1989), 355–385.

To be honest, I am quite unfamiliar with the history of the sciences or even with mathematics. Having said that, I was quite struck and surprised by the contents of the lecture. Professor Gårding in his history of research into double refraction noted that contributions have been made by: Huygens, Fresnel, Hamilton, Lamé, Sonya Kovalevskaya, Volterra, Grünwald, Fredholm, Zeilen, Herglotz, Petrovsky; moreover, there also exists collaborative work on this theme by Atiyah, Bott and Gårding.

As a matter of interest, Huygens was born 10 years before Newton. If we take a close look at the above list of mathematicians, who lived before and after the discovery of differential and integral calculus, then what is interesting to see is not that so many great mathematicians carried out some research into light, but rather that the extraordinary phenomena contingent on light continually attracted a considerable number of mathematicians. Further, in their endeavours to try and explain from a mathematical point of view these phenomena, they inexorably created new segments of mathematics.

In fact, Professor Gårding's survey showed that the progress of mathematical research into double refraction stimulates progress in analysis, in particular, advances in the theory of partial differential equations, and *vice versa.* Now, the study of light precedes the theory of partial differential equations, so it is to be expected that this would incessantly give rise to new questions in the theory of partial differential equations, which in turn lead to new insights and developments. It is my opinion that this will continue for some time to come. To all intents and purposes, we know that the relationship between the study of wave phenomena, of which light is an exemplary representative, and the theory of partial differential equations is difficult to sever.

Continuing the above line of thought, we very briefly mention a few areas of research that are presently evolving within mathematics and which are based on problems discussed in this book.

1. Microlocal analysis.

On examining the propagation of a wave, if the wave has a singularity, then this above all else is the most interesting thing for us to consider. Therefore, we must first see how this singularity is transmitted. As we discussed in Chapter 3, §3.4(b), if the singularity is on some smooth surface, then with time this singularity will be propagated in a normal direction to the surface.

In the investigation of the singularity, it is important to know not only the position of the singularity but also the direction of the singularity. For a distribution on $\mathbb{R}^n$, we consider the set of the position x of the singularity and the direction ξ of the singularity at x, so we can write $(x, \xi) \in T^*(\mathbb{R}^n)$. Then the analysis of $(x, \xi) \in T^*(\mathbb{R}^n)$ is a pivotal part of the study of the partial differential equations. The analysis that concerns itself with such $(x, \xi) \in T^*(\mathbb{R}^n)$ is called *microlocal analysis.*

If we consider partial differential operators in a microlocal sense, it is possible to obtain a more precise analysis. Currently, microlocal analysis is used in all aspects of analysis.

2. Fourier integral operator.

By using the Fourier transform, a function, defined on $\mathbb{R}^n$ and with certain properties, can be expressed as

$$f(x) = \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{ix\xi} \widehat{f}(\xi) d\xi.$$

So, by means of the above expression, the linear problem basically reduces to the case with data $e^{ix\xi}$.

For example, we look at the initial value problem for a hyperbolic operator of second order, P. Given $u(0,x) = 0$ and $u_t(0,x) = f(x)$, if we can construct a G such that

$$PG(t,x,\xi) = 0, \quad G(0,x,\xi) = 0, \quad G_t(0,x,\xi) = e^{ix\xi},$$

then the desired solution u can be written (at least formally) as

$$\begin{aligned} u(t,x) &= \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} G(t,x,\xi)\widehat{f}(\xi) d\xi \\ &= \left(\frac{1}{2\pi}\right)^{n} \int_{\mathbb{R}^n \times \mathbb{R}^n} G(t,x,\xi) e^{-iy\xi} f(y) dy d\xi. \end{aligned}$$

If we now use the methods for asymptotic solutions discussed in Chapter 3, then by making $|\xi|$ sufficiently large, we can form an approximate solution of G to a good degree of precision. Since the approximate solution of G is explicitly formed, the approximate solution, in turn, becomes useful in the study of the exact properties of the solution u. Also, in the study of the singularity of u it suffices to examine only the component for which ξ is large.

The generalization of the operator formed in the above fashion is the *Fourier integral operator*. To be precise, it is the operator A defined by

$$Af(x) = \int_{\mathbb{R}^n \times \mathbb{R}^N} \exp(i\varphi(x,\theta,y)) a(x,\theta,y) f(y) dy d\theta,$$

where θ is a variable in $\mathbb{R}^N$. In particular, if $\varphi(x,\theta,y) = (x-y)\cdot\theta$ ($\theta \in \mathbb{R}^n$), then the operator is called the *pseudo-differential operator*.

The methods that use Fourier integral operators are extremely strong. An example of this is the tremendous progress that has been made in the study of the asymptotic distribution (Weyl formula) of the eigenvalues of an elliptic operator in a bounded domain.

A particular enhancement of the Fourier integral operator is the FBI operator. In fact, by means of the FBI operator it is possible to derive an accurate analysis of the diffraction of a wave. Until the appearance of the FBI operator, the mathematical study of diffraction phenomena was quite limited to the extent that the most intrinsic parts could not be resolved. However, the results of Lebeau, which made use of the FBI operator, showed that for light that impacts on a convex object, understood in a narrow sense, the intensity of the light in the part that is under shadow is

$$\exp\left(-k^{1/3}\int_0^l \alpha(s)ds\right) \times \text{ the intensity of the original source,}$$

where $\alpha(s)$ is a positive-valued function that is determined from the curvature of the object, l denotes the length of geodesic from the point at which light comes into contact with the object and up to the observed point, and k is the frequency. In the cases of high frequencies, the extent to which a very small amount of light is transmitted to the part in shadow has been examined in detail. Such a precise analysis is made possible by the use of the FBI operator.

3. Scattering theory.

In Chapter 4, we briefly discussed the concept of a wave hitting some object and thereupon being variously dispersed. By investigating the nature of this scattering, we can gain insight into the object that causes the scattering. Not surprisingly, *scattering theory* is the name given to the theory that deals with the relationship between the dispersion of a wave and the circumstances that are the primary factors of this scattering. An example of an application of scattering theory is the investigation of the way a wave is transmitted within the earth. The subsequent analysis allows us to better understand the inner parts of the earth.

As the above example illustrates, such questions are very interesting problems not only as research topics in mathematics, where the relevant branch of mathematics is vibrant and expanding, but they are also important problems from the standpoint of practical applications in engineering. It is my intention to study scattering theory with respect to wave equations in my subsequent book, *Scattering theory*.

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(Theorem 4.8 and Lemma 4.10 are the substance of the above two papers.)

[6] P.D. Lax and R.S. Philips, *Scattering theory* (1989), Academic Press, New York (revised edition).

(Theorem 4.2 is proved by Theorem 2.1 of Chapter 5 of the above book. As mentioned in the text of the present book, the decay of local energy is extremely important in scattering theory; in the section *Notes and Remarks*, at the end of Chapter 5 of the above book, a brief history of the research in this area is given.)

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(The above book is an introductory text on partial differential equations based on lectures given at Moscow State University by one of the foremost Russian authorities on partial differential equations. Hyperbolic equations are the central focus of the book.)

[2] R. Courant and D. Hilbert, *Methods of Mathematical Physics,* vol. 2 (1962), Interscience, New York.

(This book covers the general theory of partial differential equations in relation to their importance to many concepts in physics. Courant wrote this treatise based on deliberations with his mentor Hilbert; it imparts the sense of research in partial differential equations.)

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(In this book, also on the general theory of partial differential equations, methods of dealing with hyperbolic equations, based on the named author's work, are developed.)

[4] S. Mizohata, *The theory of partial differential equations* (1973), Cambridge University Press, Cambridge, New York.

(In the above, initial boundary value problems are considered for general hyperbolic equations.)

[5] J. Rauch, *Partial differential equations* (1991), Springer-Verlag, New York, Tokyo.

(This book deals with the overall theory of partial differential equations.)

[6] L. Hörmander, *The analysis of linear partial operators,* vols. I, II, III, IV (1985), Springer-Verlag, Berlin, Heidelberg, New York.

(The most current methods and results are presented in these four volumes that cover consummately the theory of partial differential equations.)

Solutions to the Exercises

Chapter 1.

1. (1) We begin with the change of variables $\xi = x - t$ and $\eta = x + t$, and so $x = (\xi + \eta)/2$ and $t = (\eta - \xi)/2$. Then, if we set $\widetilde{u}(\xi, \eta) = u((\xi + \eta)/2, (\eta - \xi)/2)$, we see that

$$\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta}(\xi, \eta) = \frac{1}{4}(\Box u)((\xi + \eta)/2, (\eta - \xi)/2) = 0.$$

Hence, we can write $\widetilde{u}(\xi, \eta) = f(\xi) + g(\eta)$.

(2) It suffices to choose f and g so that the initial conditions are satisfied. So, since

$$f(x) + g(x) = \varphi(x), \quad -f'(x) + g'(x) = \psi(x),$$

we obtain that $f'(x) = (\varphi'(x) - \psi(x))/2$ and that $g'(x) = (\varphi'(x) + \psi(x))/2$.

Now, integrate and adjust the constant accordingly.

(3) If we assume $u(t, 0) = f(-t) + g(t) = 0$, then we must have that $g(t) = -f(-t)$.

(4) Let f be as in (3). The requirement is that $f(x) - f(-x) = \varphi(x)$ $(x > 0)$ and $f'(x) - f'(-x) = \psi(x)$ $(x > 0)$. Now, if we assume that f is defined on all of $\mathbb{R}$, then $f(x) - f(-x)$ is an odd function. So, if we denote the extensions of the odd functions φ and ψ to all of $\mathbb{R}$ by $\widetilde{\varphi}(x)$ and $\widetilde{\psi}(x)$, respectively, we obtain that

$$f(l) = \frac{1}{2}\left\{\widetilde{\varphi}(l) + \int_0^l \widetilde{\psi}(s)\, ds\right\} \quad (l \in \mathbb{R}).$$

Hence, we have that

$$u(t, x) = \frac{1}{2}\left\{\widetilde{\varphi}(x + t) + \widetilde{\varphi}(x - t) + \int_{x-t}^{x+t} \widetilde{\psi}(s)\, ds\right\}.$$

In the case when $x \geqslant t$, in the above equations, $\widetilde{\varphi}$ and $\widetilde{\psi}$ are equal to φ and ψ, respectively. While, when $0 < x < t$,

$$u(t,x) = \frac{1}{2}\left\{\varphi(x+t) - \varphi(t-x) + \int_{t-x}^{t+x} \psi(s)\,ds\right\}.$$

2. Assume that L is the length of the string and ρ is the density of the string. At the point x, with this distance being measured from the ceiling, the force that is exerted by the string from this point and up to the bottom end is given by $g\rho(L-x)$, where g is the gravitational constant. Therefore, the tension at this point is $g\rho(L-x)$. Then, with deference to the methods in §1.1(a), we obtain that

$$\rho\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(g\rho(L-x)\frac{\partial u}{\partial x}\right) = 0.$$

3. This is just a direct calculation.
4. (1) For $Pu(t,x) = 0$, if we set $\widetilde{u}(t,x) = u(t, \sqrt{a}\,x)$, then we get that $\Box\widetilde{u}(t,x) = 0$, where $\sqrt{a}\,x = (\sqrt{a_{11}}\,x_1, \sqrt{a_{22}}\,x_2, \sqrt{a_{33}}\,x_3)$. Therefore, if we take $\widetilde{u}$ to be the solution of

$$\Box\widetilde{u} = 0, \quad \widetilde{u}(0,x) = 0, \quad \widetilde{u}_t(0,x) = \varphi(\sqrt{a}\,x),$$

then we can define u in terms of the above relations. Now, we express $\widetilde{u}$ by means of (1.41); then by a change of variable we obtain the following expression for u:

$$u(t,x) = \frac{1}{4\pi t\sqrt{|a|}}\int_{d_A(x,\xi)=t} \varphi(\xi)dS_\xi,$$

where $|a| = a_{11}a_{22}a_{33}$,

$$d_A(x,\xi) = \left\{\frac{(x_1-\xi_1)^2}{a_{11}} + \frac{(x_2-\xi_2)^2}{a_{22}} + \frac{(x_3-\xi_3)^2}{a_{33}}\right\}^{1/2},$$

and dS_ξ denotes the area element of the surface $d_A(x,\xi) = t$.

(2) We have that

$$\operatorname{supp} u(t,\cdot\,) \subset \{x;\ \text{for some } |y| \leqslant \varepsilon, \quad d_A(x,y) = t\}.$$

But, this is contained in $\{x; \sqrt{a_{33}}\,t - \varepsilon \leqslant |x| \leqslant \sqrt{a_{11}}\,t + \varepsilon\}$.

5. Set $u = \operatorname{div}\boldsymbol{B}$. Then, since $\Box u = -\operatorname{div}\operatorname{rot}\boldsymbol{j} = 0$ and, moreover, $u(0,x) = u_t(0,x) = 0$, we obtain that $u \equiv 0$. Now, (1.23)

follows from the definition of $\boldsymbol{E}$. Further, because $\operatorname{div}\boldsymbol{B} = 0$, it follows that $\operatorname{rot}\operatorname{rot}\boldsymbol{B} = -\Delta\boldsymbol{B}$, and so we have that

$$\begin{aligned}\operatorname{rot}\boldsymbol{E} &= \int_0^t (-\Delta\boldsymbol{B} + \operatorname{rot}\boldsymbol{j})ds + \operatorname{rot}\boldsymbol{E}_0 \\ &= \int_0^t \left(-\frac{\partial^2\boldsymbol{B}}{\partial t^2}\right) ds - \boldsymbol{B}_t(0,x) = -\frac{\partial\boldsymbol{B}}{\partial t}(t,x).\end{aligned}$$

So, $\partial_l(\operatorname{div}\boldsymbol{E}) = \operatorname{div}\boldsymbol{j} = \partial_t q$. Then, since

$$q(0,x) = \operatorname{div}\boldsymbol{E}(0,x),$$

we see that $\operatorname{div}\boldsymbol{E} = q$.

Chapter 2.

1. Define $\widetilde{u}$ by $u(t,x) = u(t' + \Phi(x'), x') = \widetilde{u}(t', x')$. From the equality $u(t,x) = \widetilde{u}(t - \Phi(x), x)$, we have that

$$\frac{\partial u}{\partial t} = \frac{\partial \widetilde{u}}{\partial t'}, \quad \frac{\partial u}{\partial x_j} = \frac{\partial \widetilde{u}}{\partial t'}\left(-\frac{\partial\Phi}{\partial x_j}\right) + \frac{\partial\widetilde{u}}{\partial x_j}.$$

Therefore,

$$\begin{aligned}P[u] = &\left\{1 - 2\sum_{j=1}^n h_j\frac{\partial\Phi}{\partial x_j} - \sum_{j,l=1}^n a_{jl}(t,x)\left(-\frac{\partial\Phi}{\partial x_j}\right)\left(-\frac{\partial\Phi}{\partial x_l}\right)\right\}\frac{\partial^2\widetilde{u}}{\partial t'^2} \\ &+ 2\sum_{j=1}^n\left\{h_j - \sum_{l=1}^n a_{jl}\left(-\frac{\partial\Phi}{\partial x_l}\right)\right\}\frac{\partial^2\widetilde{u}}{\partial t'\partial x'_j} - \sum_{j,l=1}^n a_{jl}\frac{\partial^2\widetilde{u}}{\partial x'_j\partial x'_l} \\ &+ \text{ an order operator of first order.}\end{aligned}$$

So, the characteristic equation is

$$\begin{aligned}\widetilde{p}_0(t',x',\lambda',\xi') = &\, p_0\left(t',x',1,-\frac{\partial\Phi}{\partial x}\right)\lambda'^2 \\ &+ 2\sum_{j=1}^n\left\{h_j - \sum_{l=1}^n a_{jl}\left(-\frac{\partial\Phi}{\partial x_l}\right)\right\}\lambda'\xi'_l - \sum a_{jl}\xi'_j\xi'_l \\ = &\, p_0\left(t',x',\lambda',\xi' - \lambda'\frac{\partial\Phi}{\partial x}\right).\end{aligned}$$

The roots of the quadratic equation with respect to λ' are the characteristic roots.

In the region for which $\Phi(x)$ satisfies

$$p_0(t', x', 1, -\Phi_x(x')) > 0,$$

the transformed operator $\widetilde{P}$ is hyperbolic.

2. Take the inner product of both sides with respect to $\overline{\boldsymbol{u}}_t$ and then integrate over $(0, t) \times \mathbb{R}^3$. So, we have that

$$\begin{aligned}
&\mathrm{Re}\,\{\alpha\Delta \boldsymbol{u}\cdot\overline{\boldsymbol{u}}_t + \beta \operatorname{grad}\operatorname{div}\boldsymbol{u}\cdot\overline{\boldsymbol{u}}_t\} \\
&= -\frac{1}{2}\frac{\partial}{\partial t}\left\{\alpha\sum_{j=1}^{3}|\nabla u_j|^2\right\} + \mathrm{Re}\sum_{l=1}^{3}\frac{\partial}{\partial x_l}\left\{\sum_{j=1}^{3}\frac{\partial u_j}{\partial x_l}\frac{\partial \overline{u}_j}{\partial t}\right\} \\
&\quad -\frac{1}{2}\frac{\partial}{\partial t}\left\{\beta\left|\sum_{j=1}^{3}\frac{\partial u_j}{\partial x_j}\right|^2\right\} + \mathrm{Re}\,\frac{\partial}{\partial x_l}\left\{\beta\sum_{j=1}^{3}\frac{\partial u_j}{\partial x_j}\frac{\partial u_l}{\partial t}\right\}.
\end{aligned}$$

Now, if we set

$$E(\boldsymbol{u}; t) = \frac{1}{2}\left(\alpha\sum_{j=1}^{3}|\nabla u_j|^2 + \beta\left|\sum_{j=1}^{3}\frac{\partial u_j}{\partial x_j}\right|^2\right),$$

then it follows that

$$E(\boldsymbol{u}; t) \leqslant e^t\left\{E(\boldsymbol{u}; 0) + \int_0^t \|f(s,\cdot)\|^2_{L^2(\mathbb{R}^3)}ds\right\}.$$

With regard to the existence of the solution, for $\lambda > 0$, if we prove the existence of the solution of

$$(*) \qquad \alpha\Delta\boldsymbol{u} + \beta\operatorname{grad}\operatorname{div}\boldsymbol{u} - \lambda\boldsymbol{u} = \boldsymbol{g},$$

then we can apply the Hille-Yosida theorem. The method of proof of the existence of the solution of $(*)$ is essentially the same as that for Δ with a single unknown function.

3. First,

$$\begin{aligned}
\int_{\mathbb{R}^n} A_j(x)\frac{\partial \boldsymbol{u}}{\partial x_j}\cdot\overline{\boldsymbol{u}}\,dx &= -\int_{\mathbb{R}^n}\boldsymbol{u}\cdot\left(\frac{\partial}{\partial x_j}{}^tA_j(x)\overline{\boldsymbol{u}}\right)dx \\
&= -\int_{\mathbb{R}^n}\boldsymbol{u}\cdot\overline{{}^tA_j(x)\frac{\partial \boldsymbol{u}}{\partial x_j}}dx \\
&\quad -\int_{\mathbb{R}^n}\boldsymbol{u}\cdot({}^tA_j)_{x_j}\overline{\boldsymbol{u}}\,dx.
\end{aligned}$$

Now, since $\overline{{}^tA_j} = A_j$, we see that

$$2\mathrm{Re}\int_{\mathbb{R}^n} A_j \frac{\partial \boldsymbol{u}}{\partial x_j} \cdot \overline{\boldsymbol{u}}\, dx = -\int_{\mathbb{R}^n} \boldsymbol{u} \cdot ({}^tA_j)_{x_j} \overline{\boldsymbol{u}}\, dx.$$

Also,

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^n} & L[\boldsymbol{u}](s,x) \cdot \overline{\boldsymbol{u}(s,x)}\, ds dx \\ & \leqslant \frac{1}{2}||\boldsymbol{u}(t,\cdot)||^2_{L^2(\mathbb{R}^n)} - \frac{1}{2}||\boldsymbol{u}(0,\cdot)||^2_{L^2(\mathbb{R}^n)} \\ & \quad + C\int_0^t ||\boldsymbol{u}(s,\cdot)||^2_{L^2(\mathbb{R}^n)} ds.\end{aligned}$$

Then, from the above, we obtain the following *a priori* estimate:

$$\begin{aligned}&||u(t,\cdot)||^2_{L^2(\mathbb{R}^n)} \\ &\quad \leqslant e^{Ct}\left\{||\boldsymbol{u}(0,\cdot)||^2_{L^2(\mathbb{R}^n)} + \int_0^t ||\boldsymbol{f}(s,\cdot)||^2_{L^2(\mathbb{R}^n)} ds\right\}.\end{aligned}$$

For sufficiently large λ, if we can prove the existence of the solution of

$$\sum_{j=1}^n A_j(x)\frac{\partial \boldsymbol{v}}{\partial x_j} + A_0(x)\boldsymbol{v} - \lambda \boldsymbol{v} = \boldsymbol{g}$$

and estimate it; then it is possible to apply the Hille-Yosida theorem.

4. If we integrate $\mathrm{Re}\, Pu\overline{u}_t$ over $(t_1, t_2) \times \Omega$, then we get that

$$E(u;t_2) = E(u;t_1) + \int_{t_1}^{t_2} dt \int_\Omega h_0(x)\left|\frac{\partial u}{\partial t}(t,x)\right|^2 dx.$$

Now, if $E(u;t_2) = E(u;t_1)$, for $h_0 > 0$, then we obtain that $u_t(t,x) = 0$. Hence, $u_{tt}(t,x) = 0$, and then since $Pu = 0$, we have $\sum \partial_{x_j}(a_{kj}u_{x_k}) = 0$ in Ω. So, $u_{x_k}(t,x) = 0$, from which it follows that $u \equiv 0$.

Chapter 3.

1. We begin by setting $H(\tau,\xi) = \tau^2 - a_{11}\xi_1^2 - a_{22}\xi_2^2 - a_{33}\xi_3^2$. We also choose $\Phi(t,x) = t - \varphi(x)$ so that $H(\nabla\Phi) = 0$. From the results of §3.3(b), the solution of (3.2) propagates along the characteristic curve. Now, let $-\nabla\varphi(x_0) = \theta_0 = (\theta_{01}, \theta_{02}, \theta_{03})$.

Then, the characteristic curve that passes through the point x_0 is $X(t) = x_0 + ta\theta_0$, where $a\theta_0 = (a_{11}\theta_{01}, a_{22}\theta_{02}, a_{33}\theta_{03})$. This fact shows that at x_0 the wave is propagated with a speed $|a\theta_0|$ in the direction $a\theta_0$ and in a straight manner.

Now, since $a\nabla\varphi \cdot \nabla\varphi = 1$, we get that $a\theta_0 \cdot \theta_0 = 1$. The normal vector at θ_0 of the surface $\{\xi \in \mathbb{R}^3; a\xi \cdot \xi = 1\}$ is given by $a\theta_0$, and it is the direction in which the wave propagates.

Also, if we let the angle between θ_0 and $a\theta_0$ be γ, then we see that $|a\theta_0| = |\theta_0|^{-1}(\cos\gamma)^{-1}$. Since on the axis of the surface $\gamma = 0$, we must have $|a\theta_0| = |\theta_0|^{-1}$. So, if we assume that γ is not very large, the approximate speed of the wave that propagates in the direction $a\theta_0$ is inversely proportional to the size of θ_0 on the surface. That is to say, the size of θ_0 on the surface indicates the magnitude of lateness of the wave that propagates in the direction $a\theta_0$.

2. Since

$$\begin{aligned} &e^{-ik(t-\varphi(x))}\operatorname{div} \boldsymbol{A} \\ &\quad = ik(-\operatorname{grad}\psi \cdot \boldsymbol{v}_0) + (ik)^0(-\operatorname{grad}\psi \cdot \boldsymbol{v}_1 + \operatorname{div}\boldsymbol{v}_0) \\ &\qquad + \dots + (ik)^{-j+1}(-\operatorname{grad}\varphi \cdot \boldsymbol{v}_j + \operatorname{div}\boldsymbol{v}_{j-1}) + \dots, \end{aligned}$$

we choose $\boldsymbol{v}_j(0,x)$ in such a way that $\operatorname{grad}\psi \cdot \boldsymbol{v}_0(0,x) = 0$, $\operatorname{grad}\psi \cdot \boldsymbol{v}_1(0,x) = \operatorname{div}\boldsymbol{v}_0, \dots$ are all satisfied. Then, we construct each component of $\boldsymbol{A}$ in such a way that it is an asymptotic solution of $\square u = 0$.

Next, since $\operatorname{div}\boldsymbol{A}|_{t=0}$ and

$$\square\operatorname{div}\boldsymbol{A} = \operatorname{div}\square\boldsymbol{A} = O(k^{-N+1}),$$

we see that $\operatorname{div}\boldsymbol{A} = O(k^{-N+1})$. Now, by using (1.24) and (1.25), we see that the principal part of $\boldsymbol{B}$ is $\exp(ik - \psi(x)) \cdot (-ik)\operatorname{grad}\psi \times \boldsymbol{v}_0$. Also, the principal part of $\boldsymbol{E}$ is $\exp(ik - \psi(x))(-ik)\boldsymbol{v}_0$. Hence, the oscillations of the principal parts of both $\boldsymbol{B}$ and $\boldsymbol{E}$ are in the direction perpendicular to $\operatorname{grad}\psi$; i.e., in the direction of motion of the wave, and moreover, $\boldsymbol{B}$ and $\boldsymbol{E}$ are mutually orthogonal.

3. Consider the case of the wave that is the asymptotic solution of (3.187) with $\lambda = \lambda_1$ and is transversely incident at the boundary. That is, we assume that $-\Delta\varphi \cdot \nu(x) \geqslant \alpha_0 > 0$. Further, we assume that the reflected wave is formed by means

of

$$\boldsymbol{u}^+ = e^{ik(\lambda_1 t - \varphi^+(x))} \sum_{l=0}^{N} (ik)^{-l} \boldsymbol{v}_l^+(t,x),$$

$$\boldsymbol{w}^- = e^{ik(\lambda_1 t - \psi(x))} \sum_{l=0}^{N} (ik)^{-l} \boldsymbol{r}_l(t,x),$$

where the phase function is chosen so that the following are satisfied:

$$\begin{cases} |\nabla\varphi^+| = 1, \quad |\nabla\psi|^2 = (\alpha+\beta)/\alpha & \text{in } \Omega, \\ \varphi^+(x) = \psi(x) = \varphi(x) \quad \text{on } \Gamma. \end{cases}$$

If such a phase function is found, then it suffices to determine the value of the amplitude function on Γ so that the boundary condition is satisfied.

Now, look at the case of $l = 0$. Since $\boldsymbol{v}_0$ is of the form $p_0\nabla\varphi$, it is sufficient to find a real-valued function p_0 and a vector-valued function $\boldsymbol{r}_0$, which is orthogonal to $\nabla\psi$, such that they satisfy

$$p_0\nabla\varphi = p_0^+\nabla\varphi^+ + \boldsymbol{r}_0 \quad \text{on } \Gamma.$$

By setting $\boldsymbol{r}_0 = q\boldsymbol{h} + r\boldsymbol{j}$, we need to check whether (p^+, q, r) is determined.

4. At time $t = 0$ we assume that the amplitude function v is of the form $v(0,x;k) = m(|x|)p(\omega)$, where $\omega = x/|x|$. Further, suppose that $\operatorname{supp} m = \{l; 1 \leqslant l \leqslant 2\}$. Then, from the transport equation, we obtain that $v_0(t,x) = (|x|+t)/|x| \cdot m(|x|+t)p(\omega)$. By letting $v_1(0,x) = 0$, we can determine v_1. If P is not a harmonic function on the sphere, then there exists a $c_0 > 0$ such that

$$\sup_{|x|=r} |\Box v_0(t,x)| \geqslant c_0|m(t+r)|r^{-3} + O(r^{-2}).$$

Therefore,

$$\sup_{|x|=r} |v_1(t,x)| \geqslant \widetilde{c}_0|m(t+r)|r^{-2} + O(r^{-1}).$$

By continuing this approach, we shall eventually obtain that the following holds true: $\sup_{|x|=r} |v_j(t,x)| \geqslant c_j|m(t+|x|)|r^{-j-1} + O(r^{-j})$.

Hence, for some $c_N > 0$, we see that

$$\sup_{|x|=r} |\Box u(t,x;k)| \geqslant c_N |m(t+|x|)| k^{-N} r^{-N}.$$

Now, if we consider the above estimate and the conditions on the support of m and, in addition, we also consider the upper bound estimate of the above, then in the region $t < 1 - k^{-(1-\varepsilon)}$ $(\varepsilon > 0)$ as $k \to \infty$ we see that $\Box u$ becomes small. However, for $t \geqslant 1 - k^{-1}$, the estimate is difficult.

Clearly, at $t = 1$, the parts that are in $|x| = 1$ at $t = 0$ are clustered at $x = 0$, so we can say $x = 0$ is the focus. The asymptotic solution of the form in (3.96) is also valid in the region that is not at the focus, but close to the focus (the actual distance depends on the frequency k) it is not valid. So far, there is no general theory for the construction of an asymptotic solution in the domain that contains the focus.

Chapter 4.

1. Let us suppose that on the ray that emerges from a point of $\mathcal{R}$ there is a point X that passes through FA and a point Y that also lies on the curve ABC. The segment XY is to the right of the segment FY. Hence, the reflected ray at Y is to the left of $F'Y$, and never passes through the segment FF'.
2. By integrating $Pu \cdot u_t$ over $(0,t) \times \mathbb{R}^n$, we obtain that

$$\int_0^t \alpha \|u(s,\cdot)\|^2_{L^2(\mathbb{R}^n)} ds \leqslant E(0;u,\infty).$$

Then, by choosing a suitable λ and by integrating $Pu \cdot (tu_t + \lambda u)$, we see that

$$\int_0^t sE(s;u,\infty) ds \leqslant CE(0;u,\infty).$$

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