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Infinite-Dimensional Lie Algebras

Minoru Wakimoto

American Mathematical Society

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無限次元Lie環

INFINITE-DIMENSIONAL LIE ALGEBRAS

by Minoru Wakimoto

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Preface

Over 30 years ago, in 1967 the so-called Kac-Moody algebras were discovered as infinite-dimensional Lie algebras. The representation theories of Lie algebras can be studied without taking account of their Lie groups. Starting with a simple finite-dimensional Lie algebra, we extend it to an infinite-dimensional Lie algebra as a loop algebra over the one-dimensional torus (i.e. the circle S^1), and its central extension is an affine Lie algebra. It has structures quite similar to those of simple finite-dimensional Lie algebras. Its Dynkin diagram is the diagram obtained by adding one vertex to the Dynkin diagram of a simple finite-dimensional Lie algebra, and its Weyl group is also the affine Weyl group of a simple finite-dimensional Lie algebra. All these had already appeared within the framework of theories of simple finite-dimensional Lie algebras, so these were not new. The character formulae of affine Lie algebras have the same form as in the case of simple finite-dimensional Lie algebras. Nevertheless, as soon as character formulae of highest weight modules over affine Lie algebras were discovered, the representation theories of affine Lie algebras made great progress, which one could not imagine from the case of simple finite-dimensional Lie algebras.

The fact that an affine Lie algebra is the infinite-dimensional extension of a simple finite-dimensional Lie algebra as a loop algebra over the circle S^1 gives rise to the following remarkable features.

i. An affine Lie algebra contains an infinite-dimensional Heisenberg Lie algebra as subalgebra; namely, a Cartan subalgebra of a simple finite-dimensional Lie algebra grows up to be a Heisenberg Lie algebra by the central extension of its loop algebra. In this way, a Heisenberg Lie algebra gets into an affine Lie algebra.
ii. The vector fields on S^1 act on the derived subalgebra of an affine Lie algebra as derivations.

(ii) is the reason why an affine Lie algebra works harmoniously with the Virasoro algebra. One might say that the above two features are the reason why representation theories of affine Lie algebras are so important.

As is well known, the irreducible representation of a Heisenberg algebra is equivalent to the Schrödinger representation on a space of functions. To construct the action of an affine Lie algebra in practice, one needs vertex operators in order to describe the action of elements not in the Heisenberg subalgebra, and by them representation theories of an affine Lie algebra become much richer than those of a Heisenberg algebra. One can see from an affine root system which sort of and how many vertex operators are necessary to construct a representation of an affine Lie algebra, and a character formula tells us how one can decompose a module over an affine Lie algebra as modules over a Heisenberg Lie algebra contained in the affine Lie algebra.

The Schrödinger representation of a Heisenberg algebra consists of creation and annihilation operators, and the former act freely. Therefore a representation of an affine Lie algebra contains free action with respect to that of the creation operators of a Heisenberg subalgebra. This will be the key of the modular invariance of characters of an affine Lie algebra. In this sense, I think that the heroes (behind the scenes) of the representation theories of affine Lie algebras are a Heisenberg Lie algebra and vertex operators.

But without such arguments, one can derive useful identities, called the Macdonald identities, by fiddling around with a character formula as an algebraic formula. When Macdonald [**Mac**] discovered the Macdonald identities by considering affine root systems, the "mysterious factors" arising as a stumbling block in the formulae are in fact the factors that have their origin in the free action of the creation operators of a Heisenberg Lie algebra.

What is interesting in an affine Lie algebra is that it may be non-trivial to apply general formulae to some concrete cases. For example, writing the denominator identity for $\hat{\mathfrak{sl}}(2,\mathbb{C})$ we obtain the Jacobi triple product identity, and writing hierarchies for $\hat{\mathfrak{sl}}(2,\mathbb{C})$ we obtain the KdV equation and the non-linear Schrödinger equation. Further, a representation theory describes not only equations but also their solutions. Namely, if one finds a solution, one can construct other solutions successively by letting vertex operators act on it. Since the action on the set of solutions is transitive, one can start from the easiest solution, i.e. the "trivial solution". But we do not touch upon these soliton equations in this book.

In any case, many topics in affine Lie algebras are interesting enough when we specialize them to the case of $\hat{\mathfrak{sl}}(2, \mathbb{C})$ and just look at them. And in such a way, one may feel that one understands the topics. Conversely, once we find a phenomenon for one of the simplest affine Lie algebras such as $\hat{\mathfrak{sl}}(2, \mathbb{C})$, we can lift it up to general affine Lie algebras by making a fair copy of it in terms of Lie algebras, and similar phenomena will be mass-produced "at one stroke". Even if it is said that we must not do things "at one stroke" in these days, this "stroke" is quite exciting!

Thus in representation theories of affine Lie algebras, the basic tools for any purpose are character formulae, and we can say that they are the starting point of the representation theories.

So in this book, at first I will state the structure of Lie algebras (i.e. their root systems) and character formulae in Chapter 2. In this chapter, everything will be treated in a quite general setting, since one can expect that these facts will be developed and applied further. From Chapter 3 on, I will explain the modular invariance of the characters of affine Lie algebras. This is one of the most picturesque scenes, and I will sometimes pause to illuminate it further by concrete examples.

There is a book by Kac [**K4**] on Kac-Moody Lie algebras. Although the first edition was published 16 years ago, it still seems to me that this is the best book, both as an introductory book and as a side book. But I have heard that it is hard to start one's study with this book, because it begins from the theory of generalized Cartan matrices. Hence I started writing the present book as a guidebook to Kac's book. I hope this book will be an *hors d'œuvre* to the great feast of infinite-dimensional Lie algebras.

This book is not an encyclopedia. It does not contain everything that could be written, or even everything important, about infinite-dimensional Lie algebras, but only the facts needed for my development of the theory.

To ease the reader's way through this book, I will explain the derivation of each formula in full detail. The reader may mostly just follow along with his or her eyes, taking time out for occasional calculations. I can comfortably read this book like a novel. The facts appearing in mathematical nature are more beautiful and mysterious than any fiction. I am anxious about whether I have been able to depict them successfully.

Prof. Michio Jimbo encouraged me to write this book, and moreover he advised me, politely but strongly, on the manuscript. I would like to express my deepest gratitude to him. Dr. Kenji Iohara read through the manuscript, and pointed out and corrected lots of mistakes. Dr. Hiroyuki Tagawa has constructed the TeX environment in my office from the very beginning, and moreover he not only taught me how to use TeX and to draw figures personally but also produced some of the figures in this book himself. I also would like to thank the editors, who gave me much valuable advice on the description and on the contents, and those in the Iwanami publishing house who took care of me until I could complete the manuscript of this book.

The computation of the partition numbers shown in an example in §4.2 is carried out by the mathematical system"Reduce 3.6". I first became acquainted with "Reduce" in February 1986 when I had some computations to do. Thanks to the instruction of Prof. Seigo Okamoto, "Reduce" immediately became an important supplementary tool for my research. In addition, it was Prof. Okamoto who introduced me to Kac-Moody Lie algebras. I would like to express my heartiest gratitude to him for his great care.

Finally, I would like to express my appreciation to both of my parents, who let me achieve my dream of studying mathematics.

With gratitude to my wife, Yasuko

Minoru Wakimoto

Preface to the English Edition

This book was originally written in Japanese to provide an outlook and to serve as a guide to the theory of infinite-dimensional Lie algebras, which has grown up in quite recent decades in connection with various areas in mathematics and mathematical physics.

I tried to give an exposition with much detailed explanation, so that the theory may become more familiar to readers, though the topics treated in this book are quite limited for want of space. It will greatly please me if this book may raise its readers' interest in this area and be a help for further research and development of the theory.

For the publishing of this English edition, I thank Iwanami Shoten Publishers, the editors of the AMS, the translator and, in particular, Professor Katsumi Nomizu for a lot of kind attention.

July 2000, in Fukuoka

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Overview

When we consider the structure of a simple finite-dimensional Lie algebra $\mathfrak{g}$, we first choose a Cartan subalgebra and make the root space decomposition. Next we classify the roots into positive and negative ones. The minimal roots among the positive roots with respect to the decomposition (that is, the positive roots which cannot be decomposed into a sum of roots) are called **simple roots**. Denote them by $\alpha_1, \cdots, \alpha_l$. Then the matrix of inner products $\left(\frac{2\langle\alpha_i, \alpha_j\rangle}{\langle\alpha_i, \alpha_i\rangle}\right)$ is called the **Cartan matrix** of $\mathfrak{g}$. To reconstruct a Lie algebra from the Cartan matrix, we have only to impose conditions that can be extracted from the Cartan matrix on the free Lie algebra generated by $2l$ letters, say $e_1, \cdots, e_l; f_1, \cdots, f_l$. In this way, we can reproduce the original Lie algebra, and we have the one-to-one correspondence:

$$\text{Simple Finite-Dimensional Lie Algebras} \quad \longleftrightarrow \quad \text{Cartan Matrices.}$$

Namely, if two Lie algebras are isomorphic, then their Cartan matrices are the same; otherwise their Cartan matrices differ. Therefore the simple finite-dimensional Lie algebras are classified in terms of their Cartan matrices. This is the well-known Cartan-Killing theory, and it appeared about one hundred years ago. But after that, the Cartan-Killing theory on simple finite-dimensional Lie algebras fell asleep peacefully as a completed theory, i.e. as a "theory for textbooks".

But after two thirds of this century has passed, two princes suddenly approached this "Sleeping Beauty". The outrageous princes who kissed the forbidden lips were Kac and Moody. Since Cartan matrices classify the simple finite-dimensional Lie algebras,

> Relaxing conditions on Cartan matrices a bit, one might be able to obtain infinite-dimensional Lie algebras!

We call such modified Cartan matrices generalized Cartan matrices (GCM for short). A new class of infinite-dimensional Lie algebras, i.e., **Kac-Moody Lie algebras**, was born. From its origin, one can argue similarly as in the case of finite dimension. What is necessary to explicate its structure is a representation theory of $\mathfrak{sl}(2,\mathbb{C})$ only. Furthermore, when a GCM is symmetrizable, the corresponding Lie algebra possesses an inner product, and one can define the Casimir element. As an application, character formulae can be proved. They have the same form as in the finite-dimensional case. The points that are different from the finite-dimensional cases are

i. the Weyl group becomes an infinite group, and
ii. imaginary roots appear.

Nonetheless, these two facts have great consequences.

The most lovely class of infinite-dimensional Lie algebras is the affine Lie algebras. An affine Lie algebra is the central extension of the loop algebra of a simple finite-dimensional Lie algebra, so we know the structure of its root system very well, and its Weyl group is the semi-direct product of the Weyl group of a simple finite-dimensional Lie algebra and a $\mathbb{Z}$-lattice. In the character formulae the summation over the $\mathbb{Z}$-lattice gives us theta functions, and hence the character of integrable representations over an affine Lie algebra is a modular function. I will explain this in Chapter 4, and in Chapter 5 I will explain fusion coefficients as a topic related to the transformation matrices of modular transformation of the characters.

After this brief awakening, the 'Beauty' went to sleep again. This 'Beauty' sleeps well. The next prince who disturbed her sleep was Borcherds. Anyone who is woken from a first sleep will be offended. In addition, the task of hauling Cartan matrices in is so daring that it might change the properties of simple roots. But the 'Beauty' is also sweet-tempered. She never loses her temper. Hence Borcherds built up a fine theory.

Needless to say, Borcherds started his research on infinite-dimensional Lie algebras not from the viewpoint of Cartan matrices. He discovered the vertex operator algebras from his research on representations of the Monster simple group. From the simple roots of the infinite-dimensional Lie algebra associated to a vertex operator algebra, he had an insight into the fact that the following situation is significant:

> We do not have to mind even if some strangers (i.e. simple imaginary roots) mix up within the set of simple roots.

A Cartan matrix allowing the existence of such simple roots is called a **GKM matrix** (generalized Kac-Moody matrix), and the corresponding infinite-dimensional Lie algebra is called a **GKM Lie algebra**. Even in this setting, analogous arguments work well to prove character formulae. In this case, the forms of character formulae are not entirely the same as before, but slightly more complicated, and they have a new effect. The formula obtained by specializing character formulae to the trivial representation is called the **denominator identity**. This is a formula of the form "Infinite Product = Infinite Sum". Since imaginary simple roots create a new environment in the case of GKM Lie algebras, their denominator identities show us new equalities for a much wider class of modular functions than those arising from Kac-Moody Lie algebras.

But concerning GKM algebras, the only formula that has some applications up to now is the denominator identity, and the character formulae themselves of the other integrable representations have not been studied yet. Character formulae are too good to apply only to the trivial representation. I would like to see some nice applications.

In this book, for the reason to be explained in §1.4, we use the term BKM algebra instead of GKM algebra. The character formulae for BKM Lie algebras will be important for applications. For this purpose, in Chapter 2, I will describe in detail the root system and the character formulae for BKM algebras and BKM superalgebras in a general setting.

Therefore the main theme of this book are in Chapter 2, "Character formulae for BKM Lie superalgebras", and in Chapter 4, "Modular transformation of characters of affine Lie algebras". Representation theories of affine Lie algebras have applications to many fields. One could explain each topic little by little, but I have preferred to restrict myself to the theme on modular transformations of affine Lie algebras after Chapter 4. But when I looked back after writing up till Chapter 5, I felt something was insufficient. For example, without any description of how one can use the matrix elements of modular transformations presented as examples in Chapter 4, they are simply enumerations of data and may not show the readers their beauty. After I began writing the manuscript of this book, for a while, I was going to omit the Virasoro algebra, because of the rich variety of affine Lie algebras. Representation theories of the Virasoro algebra are so ample that I cannot mention them briefly. It is impossible to fit them into this book. But representation theories of affine Lie algebras without the Virasoro algebra became dreary, like stale beer. So Chapter 6 was added, to give a brief sketch of how representation theories of the Virasoro algebra work (the chapter entitled "In Lieu of Postscript").

Although representation theories of affine Lie algebras are related to several fields, I cannot write about them, owing to the lack of space and of my knowledge. These other fields include, in no particular order, conformal field theory and KZ equations, lattice models, soliton equations, and so on. The first two are covered in the book *Field Theory and Topology* in this series.

It seemed to me that Prof. Ryogo Hirota's use of the theory of bilinear differential operators to find the exact solutions is a marvelous method, like magic. Through investigating it, the group of Prof. Mikio Sato, Prof. Etsuro Date, Prof. Michio Jimbo, Prof. Masaki Kashiwara and Prof. Tetsuji Miwa in the Research Institute of Mathematical Science, Kyoto University, discovered that the symmetry hidden inside the non-linear differential equations called soliton equations is an affine Lie algebra. In May 1982, when the lecture course taught by Kac for the spring semester drew to a close, their theory was introduced in his lecture as a "Work of Japanese School". I got into the lecture-hall without permission or payment for this lecture, and the theory developed on the board was so beautiful that my heart burst into flames. Everyone who is alone in a foreign country will be nationalistic. In addition I am timid. It was the Kyoto school that gave me the motivation to leap at infinite-dimensional Lie algebras, forgetting everything I had been working on before. There are some explanations of the theory, e.g. in [**Sa**].

CHAPTER 6

In Lieu of Postscript – Virasoro Algebra –

In Chapter 4 and 5, I have explained the representation theory of affine Lie algebras and some of its applications. Even if we restrict ourselves to affine Lie algebras, there still are a lot of topics that have to be mentioned, for example, the theory of the branching functions of representations, soliton equations, and so on. Concerning soliton equations, an excellent expository book has already been published by those who contributed a lot to the development of the theory, so if the reader is interested in it, see, e.g., [**Hir**], [**MJD**].

The branching functions of representations are important theories, and one can naturally develop them as a continuation of Chapter 4. Although this problem itself lies in the framework of affine Lie algebras, the representation of the Virasoro algebra plays a great role when we consider this problem and develop its theory.

The Virasoro algebra is also one of the infinite-dimensional Lie algebras, but belongs to a different class from Kac-Moody algebras or BKM algebras. The Virasoro algebra is also still young. Her figure appeared and disappeared in physics in the latter half of the 1960's, but its representation theory began to be studied only in the 1980's. The fact that it coincides with the period when the theory of affine Lie algebras developed, is not an accident.

In the mid 1980's, The Virasoro algebra fell in passionate love with affine Lie algebras. The love of an affine Lie algebra for the Virasoro algebra, and the love of the Virasoro algebra for affine Lie algebras $\cdots$ each representation theory has cooperated, elevated each other, and grown up beautifully. Namely, the development of the theory of affine Lie algebras and that of the Virasoro algebra is the same event. One evidence of their affection has already appeared in this book, namely, the list of $a(\lambda, \mu)$ in §4.3.

$a(\lambda, \mu)$ itself is the summation over the Weyl group, and is defined by (4.74). One can derive many properties of $a(\lambda, \mu)$ from this formula, but this is not always suitable for a practical computation of its value. For example, the number of elements of the Weyl group of the exceptional Lie algebra E_8 is

$$\prod_{\substack{1 \leq p \leq 29 \\ \gcd(p,30)=1}} (p+1) = 2 \cdot 8 \cdot 12 \cdot 14 \cdot 18 \cdot 20 \cdot 24 \cdot 30 = 696729600.$$

No one would try to compute (4.74) directly when one wants to know the values of $a(\lambda, \mu)$ for level 2 or 3 in the case of the affine Lie algebra of type $E_8^{(1)}$. Much of the numerical data in this list is obtained through the analysis of branching functions using the Virasoro algebra.

In this way, the Virasoro algebra is so important that the representation theory of an affine Lie algebra without the Virasoro algebra becomes too dreary. So, in lieu of postscript, I will briefly introduce a profile of the Virasoro algebra.

The **Virasoro algebra** is the vector space

$$\mathrm{Vir} = \bigoplus_{j \in \mathbb{Z}} \mathbb{C}L_j \oplus \mathbb{C}Z$$

with basis L_j ($j \in \mathbb{Z}$) and Z, and is the Lie algebra satisfying the following bracket products:

$$[L_j, L_k] = (j-k)L_{j+k} + \frac{j^3 - j}{12}\delta_{j+k,0}Z, \tag{6.1}$$

$$[L_j, Z] = 0. \tag{6.2}$$

By setting

$$\mathrm{Vir}_+ := \bigoplus_{j>0} \mathbb{C}L_j, \quad \mathrm{Vir}_- := \bigoplus_{j<0} \mathbb{C}L_j, \quad \mathrm{Vir}_0 := \mathbb{C}L_0 \oplus \mathbb{C}Z,$$

Vir decomposes as

$$\mathrm{Vir} = \mathrm{Vir}_- \oplus \mathrm{Vir}_0 \oplus \mathrm{Vir}_+, \tag{6.3}$$

and by the Poincaré-Birkhoff-Witt Theorem 1.16, it follows that

$$\mathfrak{U}(\mathrm{Vir}) = \mathfrak{U}(\mathrm{Vir}_-) \cdot \mathfrak{U}(\mathrm{Vir}_0) \cdot \mathfrak{U}(\mathrm{Vir}_+). \tag{6.4}$$

Vir_0 plays the role of a Cartan subalgebra of Vir, and one can regard $\mathbb{C}L_j$ as a root space for each j, but there is no root which plays the role of the fundamental root.

By using the triangular decomposition (6.3), one can define a highest weight representation of the Virasoro algebra in the same way as before:

DEFINITION 6.1. Let (π, V) be a representation of Vir.

1. If the linear transformation $\pi(L_0)$ on V is diagonalizable, and each eigenspace is of finite dimension, then the representation (π, V) is said to be **L_0-diagonalizable**.
2. If (π, V) is L_0-diagonalizable, and one can choose complex numbers h, z and $0 \neq v_0 \in V$ satisfying the following three conditions
   (i) $\pi(Z)v_0 = zv_0$,
   (ii) $\pi(L_0)v_0 = hv_0$, $\pi(\mathrm{Vir}_+)v_0 = \{0\}$,
   (iii) $\pi(\mathfrak{U}(\mathrm{Vir}))v_0 = V$,
   then (π, V) is called a **highest weight representation**, (h, z) is called its **highest weight**, and v_0 is called a **highest weight vector**. z is called its **central charge**, and h is called its **conformal weight**.

If (π, V) is a highest weight representation, then its highest weight is uniquely determined, but its highest weight vector is determined only up to a scalar. We denote the irreducible highest weight module with highest weight (h, z) by $\mathfrak{V}(h, z)$.

We define the character of an L_0-diagonalizable representation (π, V) by

$$\mathrm{ch}_V(q) := \sum_{\lambda \in \mathbb{C}} \dim V_\lambda \cdot q^\lambda.$$

Here, V_λ $(\lambda \in \mathbb{C})$ is the eigenspace of $\pi(L_0)$ with eigenvalue λ:

$$V_\lambda := \{v \in V; \pi(L_0)v = \lambda v\}.$$

If V is the irreducible highest weight module $\mathfrak{V}(h, z)$, then we have

$$V = \bigoplus_{\lambda \in h + \mathbb{Z}_{\geq 0}} V_\lambda,$$

and hence it follows that

$$\mathrm{ch}_{\mathfrak{V}(h,z)}(q) = q^h \sum_{n=0}^{\infty} \dim V_{h+n} \cdot q^n.$$

We may simply denote the character $\mathrm{ch}_{\mathfrak{V}(h,z)}$ of an irreducible representation by $\mathrm{ch}_{(h,z)}$.

Below, we may omit π, and simply write $L_j v$ instead of $\pi(L_j)v$.

DEFINITION 6.2. Let V be a Vir-module. If there exists a positive definite Hermitian inner product $\langle\ ,\ \rangle$ satisfying the two conditions

i. $\langle L_j u, v\rangle = \langle u, L_{-j} v\rangle \quad (u, v \in V, j \in \mathbb{Z})$,

ii. $\langle Zu, v\rangle = \langle u, Zv\rangle \quad (u, v \in V)$,

then this representation is said to be **unitarizable**.

It is easy to see that for the irreducible highest weight representation $\mathfrak{V}(h, z)$, one has

$$\text{unitarizable} \Longrightarrow h, z \geq 0. \tag{6.5}$$

Moreover, one can show that

$$h \geq 0, z \geq 1 \Longrightarrow \mathfrak{V}(h, z) \text{ is unitarizable} \tag{6.6}$$

by constructing concretely a "Fock module" of the Virasoro algebra on a space of functions $\mathbb{C}[x_1, x_2, \cdots]$. From the above two assertions (6.5) and (6.6), the following question might arise:

What happens when $h \geq 0$ and $0 \leq z < 1$?

Those who considered this problem and opened a door to solve it were physicists. The Shapovalov determinant of the Virasoro algebra is called the Kac determinant. By analyzing the Kac determinant [**K2**], Friedan, Qiu and Shenker showed in [**FQS1**] and [**FQS2**] that the only possible places in this domain where a unitarizable representation exists are

$$z = z^{(m)} := 1 - \frac{6}{(m+2)(m+3)} \quad (m \in \mathbb{Z}_{\geq 0}), \tag{6.7}$$

$$h = h_{r,s}^{(m)} := \frac{[(m+3)r - (m+2)s]^2 - 1}{4(m+2)(m+3)} \tag{6.8}$$

$$(r, s \in \mathbb{N} \quad \text{s.t.} \quad 1 \leq r \leq m+1,\ 1 \leq s \leq m+2),$$

and the fact that they are, in fact, unitarizable was shown by using a **Goddard-Kent-Olive construction** [**GKO1**],[**GKO2**] (GKO construction for short) of integrable representations of an affine Lie algebra and the Virasoro operators. A GKO construction is also called a coset construction. With this opportunity, remarkable applications of the representation theory of the Virasoro algebra started.

As one can easily see, since $h_{r,s}^{(m)}$ satisfies

$$h_{r,s}^{(m)} = h_{(m+2)-r,(m+3)-s}^{(m)},$$

one can restrict the range of r and s to

$$1 \leq s \leq r \leq m+1, \tag{6.9}$$

and also to

$$1 \le r \le m+2,\ 1 \le s \le m+3 \text{ and } r \equiv s \mod 2. \tag{6.10}$$

By imposing such a condition, one can establish the one-to-one correspondence between the range of the parameters (r,s) and representations.

In other words, one can say as follows. For $m \in \mathbb{Z}_{\ge 0}$, set

$$\widetilde{\mathrm{VIR}}^{(m)} := \{(r,s) \in \mathbb{N}\times\mathbb{N}; 1 \le r \le m+1, \quad 1 \le s \le m+2\},$$

and denote the equivalence classes of this set, where the equivalence relation is defined by

$$(r,s) \sim (m+2-r, m+3-s),$$

by $\mathrm{VIR}^{(m)}$. We denote the element of $\mathrm{VIR}^{(m)}$ defined by $(r,s) \in \widetilde{\mathrm{VIR}}^{(m)}$, i.e., the equivalence class of (r,s), by the same symbol (r,s). (6.9) and (6.10) are just choices of representatives of $\mathrm{VIR}^{(m)}$.

A unitary representation with $0 \le z < 1$ is called the **discrete series** representation of the Virasoro algebra. Its character $\mathrm{ch}^{(m)}_{r,s} := \mathrm{ch}_{\mathfrak{V}(h^{(m)}_{r,s}, z^{(m)})}$ can be obtained as an application of results of Feigin and Fuchs [**FeFu1**],[**FeFu2**], and is given by the following formula:

$$\mathrm{ch}^{(m)}_{r,s}(q) = \frac{1}{\varphi(q)} \sum_{n\in\mathbb{Z}} \left\{ q^{\frac{[2(m+2)(m+3)n+(m+3)r-(m+2)s]^2-1}{4(m+2)(m+3)}} - q^{\frac{[2(m+2)(m+3)n+(m+3)r+(m+2)s]^2-1}{4(m+2)(m+3)}} \right\}.$$

So, if we multiply $q^{-\frac{1}{24}z^{(m)}}$ by the character and denote it by $\Xi^{(m)}_{r,s}(\tau)$ $(q = e^{2\pi i\tau})$, then $\Xi^{(m)}_{r,s}(\tau)$ can be expressed by using the modular functions defined in (4.31) in §4.2 as follows:

$$\Xi^{(m)}_{r,s}(\tau) = \frac{1}{\eta(\tau)}\{f_{(m+3)r-(m+2)s,(m+2)(m+3)}(\tau) - f_{(m+3)r+(m+2)s,(m+2)(m+3)}(\tau)\}. \tag{6.11}$$

Therefore, the (normalized) character of a discrete series representation of the Virasoro algebra is a modular function, and its modular transformations can be computed by Theorem 4.6. Moreover, the leading power of $\Xi^{(m)}_{r,s}(\tau)$ is $h^{(m)}_{r,s} - \frac{1}{24}z^{(m)}$, and it has the following form:

$$\Xi_{r,s}^{(m)}(\tau) = q^{h_{r,s}^{(m)} - \frac{1}{24}z^{(m)}}\{1 + (\text{formal series in positive powers of } q)\}.$$

Suppose that a representation of the Virasoro algebra appears by accident, and its central charge is less than 1. In addition, we assume that we may realize from the situation of its birth that it is a unitarizable representation. In such a case, z and h of the representation should coincide with the list of the values (6.7) and (6.8). This is called the "conformal invariance constraint" .

What connects the Virasoro algebra and an affine Lie algebra are the differential operators

$$d_j := t^{j+1}\frac{d}{dt} \quad (j \in \mathbb{Z})$$

on $\mathbb{R}$. Since d_j becomes

$$d_j = -ie^{ij\theta}\frac{d}{d\theta}$$

by the transformation $t = e^{i\theta}$, d_j can be also regarded as a vector field on the one-dimensional torus S^1. The commutation relations as differential operators are

$$[d_j, d_k] = (k - j)d_{j+k},$$

i.e.,

$$[-d_j, -d_k] = (j - k) \cdot (-d_{j+k}),$$

and the bracket products of elements L_j which form a basis of the Virasoro algebra are those obtained by adding a central element to the bracket products of $-d_j$'s. Therefore, the Virasoro algebra can be regarded as the central extension of the Lie algebra of polynomial vector fields on the one-dimensional torus S^1, and this point of view is sometimes important.

Now, let $\mathfrak{g}$ be the affine Lie algebra over $\overline{\mathfrak{g}}$. What is going to be stated in this section can be applied for any affine Lie algebra, but I will restrict myself to a split type for simplicity. Then, first, we have $h^\vee = g$ for the dual Coxeter number. As is stated in §3.3, $\mathfrak{g}$ can be realized as

$$\mathfrak{g} = \sum_{n\in\mathbb{Z}}(t^n \otimes \overline{\mathfrak{g}}) \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

The **derived algebra** $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$ is the one obtained by slicing off the part $\mathbb{C}d$:

$$\mathfrak{g}' = \sum_{n \in \mathbb{Z}} (t^n \otimes \overline{\mathfrak{g}}) \oplus \mathbb{C}c.$$

If we define the action of d_j on $\mathfrak{g}'$ by

$$[d_j, X(n)] := t^{j+1} \frac{d}{dt}(t^n \otimes X) = nX(n+j) \quad (X \in \overline{\mathfrak{g}}, n \in \mathbb{Z}),$$
$$[d_j, c] := 0,$$

then the d_j ($j \in \mathbb{Z}$) act on $\mathfrak{g}'$ as derivations. Notice that as an action on $\mathfrak{g}'$, $d = d_0$ holds. Namely, the element d sliced off from $\mathfrak{g}$ simply changed its seat.

Note. d is an important element and is also called the **energy operator**. If we let d_j act on d, then it becomes a scalar multiple of d_j, and goes outside of $\mathfrak{g}$. The reason why we considered the derived subalgebra $\mathfrak{g}'$ is to avoid such problems. As it may slightly wander from the subject, we may sometimes regard a representation of $\mathfrak{g}$ as a representation of $\mathfrak{g}'$ below. These two points of view are equivalent for highest weight modules, and one can naturally define the action of d on a highest weight $\mathfrak{g}'$-module, and make it a $\mathfrak{g}$-module. And the irreducibility regarded either as a $\mathfrak{g}$-module or as a $\mathfrak{g}'$-module is equivalent. □

We consider a highest weight representation (π, V) of $\mathfrak{g}$. We do not require it to be irreducible, nor that it have non-dominant integral highest weight. For this representation, we consider the following question:

Do d_j's act on this representation space?

This means 'Can we define the element $\pi(d_j)$ of $\mathrm{End}(V)$ for each j satisfying the following three conditions?':

i. $[\pi(d_j), \pi(X(n))] = n\pi(X(n+j)) \quad (\forall X \in \overline{\mathfrak{g}},\ \forall n \in \mathbb{Z})$,
ii. $\pi(d_0) = \pi(d)$,
iii. $[\pi(d_j), \pi(d_k)] = (k-j)\pi(d_{j+k})$.

The answer to this question is 'Yes?' The reason why we added the '?' is that if we add some terms (the term coming from the central extension and shifting d by a scalar operator) to the right hand side of condition (iii) viz., if we allow some concessions or modifications to condition (iii), then the answer can be 'Yes'. Physicists call this 'additional term' a conformal anomaly. It is not an obstructive term, but a meaningful and important term. Representations of the central extension of d_j's, i.e., the Virasoro algebra, naturally appear here.

Concerning these facts, let us proceed more concretely. For this purpose, if we choose two sets $\{u_i\}$ and $\{u^i\}$ of bases of $\overline{\mathfrak{g}}$ satisfying $(u_i, u^j) = \delta_{ij}$, then the Casimir element of $\mathfrak{g}$ can be expressed as follows:

$$\Omega = 2(c+g)d + 2\overline{\rho} + \sum_i u^i u_i + 2\sum_{n=1}^{\infty}\sum_i u^i(-n)u_i(n). \tag{6.12}$$

As was stated in §2.1, any element of $\mathfrak{g}$ commutes with Ω, but since d_j $(0 \neq j \in \mathbb{Z})$ is not an element of $\mathfrak{g}$, one cannot say that $[d_j, \Omega]$ is 0. For $j \in \mathbb{Z}$, let us study their properties. For each $j \in \mathbb{Z}$, set

$$S_j := \left[d_j, \sum_{n=1}^{\infty}\sum_i u^i(-n)u_i(n) \right]. \tag{6.13}$$

As we saw in §2.1, this is an element of the completion $\widehat{\mathfrak{U}}(\mathfrak{g})$ of the universal enveloping algebra. By (6.12) and (6.13), it follows that

$$\begin{aligned}[d_j, \Omega] &= 2(c+g)\underbrace{[d_j, d]}_{-jd_j} + 2\underbrace{[d_j, \overline{\rho}]}_{0} + \underbrace{[d_j, \sum_i u^i u_i]}_{0} + 2S_j \\ &= 2\{S_j - j(c+d)d_j\}.\end{aligned} \tag{6.14}$$

And for $X \in \overline{\mathfrak{g}}$ and $n \in \mathbb{Z}$, by simple computations using the Jacobi identity, one obtains

$$[[d_j, \Omega], X(n)] = [d_j, \underbrace{[\Omega, X(n)]}_{0}] - \underbrace{[\Omega, \underbrace{[d_j, X(n)]}_{\in\mathfrak{g}}]}_{0} = 0, \tag{6.15}$$

$$[[d_j, \Omega], d] = [d_j, \underbrace{[\Omega, d]}_{0}] - [\Omega, \underbrace{[d_j, d]}_{-jd_j}] = j[\Omega, d_j] = -j[d_j, \Omega]. \tag{6.16}$$

So, by applying (6.14) to (6.15) and (6.16), it follows that

$$\begin{aligned}&[S_j - j(c+g)d_j, X(n)] = 0, \\ &[S_j - j(c+g)d_j, d] = -j\{S_j - j(c+g)d_j\},\end{aligned}$$

and hence we obtain

$$[S_j, X(n)] = j(c+g)[d_j, X(n)] = jn(c+g)X(n+j), \tag{6.17}$$

$$[S_j, d] = -j^2(c+g)d_j - j\{S_j - j(c+g)d_j\} = -jS_j. \tag{6.18}$$

Since S_j is an element of $\widehat{\mathfrak{U}}(\mathfrak{g})$ and acts on a highest weight module as stated above, one can put π on S_j, and obtain formulae as a linear operator on V as follows:

$$[\pi(S_j), \pi(X(n))] = jn(\pi(c) + g)\pi(X(n + j)), \tag{6.19}$$

$$[\pi(S_j), \pi(d)] = -j\pi(S_j). \tag{6.20}$$

If we set the level of the representation (π, V) to be m, then $\pi(c)$ is the scalar operator $m \cdot \mathrm{Id}_V$. So for $m \neq -g$, if we set

$$\pi(d_j) := \frac{1}{j(m+g)}\pi(S_j) \tag{6.21}$$

for each j $(0 \neq j \in \mathbb{Z})$, then condition (i) is satisfied. And for $m \neq g$, by computing the commutation relations among the $\pi(S_j)$'s using (6.17) and (6.18), the commutation relations among $\pi(d_j)$'s can be computed as follows:

$$\begin{aligned}[\pi(d_j), \pi(d_k)] = (k - j)&\left\{\pi(d_{j+k}) - \frac{\pi(\Omega)}{2(m+g)}\delta_{j+k,0}\right\} \\ &+ \frac{j^3 - j}{12}\delta_{j+k,0} \cdot \frac{m}{m+g}\dim\overline{\mathfrak{g}} \cdot \mathrm{Id}_V.\end{aligned} \tag{6.22}$$

So, if we set

$$\pi(L_j) := -\pi(d_j) + \frac{\pi(\Omega)}{2(m+g)}\delta_{j,0} \quad (j \in \mathbb{Z}), \tag{6.23}$$

$$\pi(Z) := \frac{m}{m+g}\dim\overline{\mathfrak{g}} \cdot \mathrm{Id}_V, \tag{6.24}$$

then a representation space V of a highest weight representation of an affine Lie algebra becomes a representation space of the Virasoro algebra, and in this case, the central charge z and the conformal weight h of the representation of the Virasoro algebra are given by

$$z = z_\Lambda = \frac{m}{m+g}\dim\overline{\mathfrak{g}}, \tag{6.25}$$

$$h = h_{\overline{\Lambda};m} = h_\Lambda - (\Lambda|d) = \frac{(\Lambda|\Lambda + 2\rho)}{2(m+g)} - (\Lambda|d), \tag{6.26}$$

where $\Lambda \in \mathfrak{h}^*$ is the highest weight of V. Here z_Λ, $h_{\overline{\Lambda};m}$ and h_Λ are defined by the same formulae as (4.64), (4.68) and (4.67) in §4.3 (even in the case of generic $\Lambda \in \mathfrak{h}^*$). If it is necessary to indicate that a quantity is associated to $\mathfrak{g}$, then we use $\mathfrak{g}$ as a supercript, and, for example, denote z_Λ by $z_\Lambda^{\mathfrak{g}}$. Moreover, we set

$$h_\Lambda^{\mathfrak{g}} = h_\Lambda - (\Lambda|d). \tag{6.27}$$

z_Λ depends only on the level of Λ, and h does not depend on a choice of representative Λ of $\Lambda \mod \mathbb{C}\delta$. It is interesting that the quantities appearing in the topic of modular transformations reappear in relation to representations of the Virasoro algebra. In the paper that discovered these facts (appendix 3 of §2.5 in [**KP**]), it was said 'The material of this section has no apparent relevance.....the unexpected mysterious coincidence of a constant...'. It was the 'predawn' of the analysis of representations using the modular invariance and the conformal invariance.

DEFINITION 6.3. $\pi(L_j)$ (or regarded it as an element of $\widehat{\mathfrak{U}}(\mathfrak{g})$) is called a **Virasoro operator** of the affine Lie algebra $\mathfrak{g}$, and is denoted by $L_j^{\mathfrak{g}}$.

In particular, for $\Lambda \in P_+$, $L(\Lambda)$ is unitarizable as a representation of the affine Lie algebra, and it can be easily seen from the form of $L_j^{\mathfrak{g}}$ that it is also unitarizable as a representation of the Virasoro algebra. But its central charge (except for the trivial representation) is always greater than 1, and we cannot use the technique of conformal invariance constraint. So, let us consider the following situation.

Let us take a reductive subalgebra $\overline{\mathfrak{p}}$ of $\overline{\mathfrak{g}}$ so that it satisfies

$$\text{a Cartan subalgebra of } \overline{\mathfrak{p}} \subset \text{a Cartan subalgebra of } \overline{\mathfrak{g}},$$

and let $\mathfrak{p}$ be the affinization of $\overline{\mathfrak{p}}$. Then, since the Virasoro operators $L_j^{\mathfrak{g}}$ and $L_j^{\mathfrak{p}}$ satisfy

$$(6.28) \qquad [L_j^{\mathfrak{g}}, X(n)] = -[d_j, X(n)] = -nX(n+j) \quad (X \in \overline{\mathfrak{g}}, n \in \mathbb{Z}),$$

$$(6.29) \qquad [L_j^{\mathfrak{p}}, X(n)] = -[d_j, X(n)] = -nX(n+j) \quad (X \in \overline{\mathfrak{p}}, n \in \mathbb{Z}),$$

it follows that in particular for $X \in \overline{\mathfrak{p}}$, one has

$$(6.30) \qquad [L_j^{\mathfrak{g}} - L_j^{\mathfrak{p}}, X(n)] = 0,$$

and thus

$$L_j^{\mathfrak{g};\mathfrak{p}} := L_j^{\mathfrak{g}} - L_j^{\mathfrak{p}}$$

commute with any element of the derived subalgebra $\mathfrak{p}' := [\mathfrak{p}, \mathfrak{p}]$ of $\mathfrak{p}$. The commutation relation between $L_j^{\mathfrak{g}}$ and $L_k^{\mathfrak{p}}$ can be computed in the same way as $[L_j^{\mathfrak{g}}, L_k^{\mathfrak{g}}]$ using (6.28) instead, and it is

$$[L_j^{\mathfrak{g}}, L_k^{\mathfrak{p}}] = (j-k)L_{j+k}^{\mathfrak{p}} + \frac{j^3 - j}{12}\delta_{j+k,0} \cdot \frac{\dot{c}}{\dot{c} + \dot{g}} \dim \overline{\mathfrak{p}}.$$

Here, a dot over a symbol signifies a quantity associated to $\mathfrak{p}$. Namely, $\dot{g}$ is the dual Coxeter number of $\mathfrak{p}$, and $\dot{c}$ signifies a central element of $\mathfrak{p}$. Thus, one can immediately obtain the commutation relations among the $L_j^{\mathfrak{g};\mathfrak{p}}$:

$$(6.31) \qquad [L_j^{\mathfrak{g};\mathfrak{p}}, L_k^{\mathfrak{g};\mathfrak{p}}] = (j-k)L_{j+k}^{\mathfrak{p}} + \frac{j^3-j}{12}\delta_{j+k,0} \cdot (z_m - \dot{z}_{\dot{m}}).$$

Here, m is the level of the representation (π, V) of $\mathfrak{g}$, and $\dot{m}$ is the level of the same representation regarded as a $\mathfrak{p}$-module. $L_j^{\mathfrak{g};\mathfrak{p}}$ is called a coset Virasoro operator. Its central charge is $z = z_m - \dot{z}_{\dot{m}}$.

Let us consider decomposing a representation (π, V) of $\mathfrak{g}$ regarded as $\mathfrak{p}$-module. For this purpose, we denote the space of singular vectors of V with respect to $\mathfrak{p}$ by

$$V^{\mathfrak{p}_+} := \{v \in V; xv = 0 \ (\forall x \in \mathfrak{p}_+)\}.$$

Here, $\mathfrak{p}_+$ is the positive part of $\mathfrak{p}$ with respect to a triangular decomposition. Since $L_j^{\mathfrak{g};\mathfrak{p}}$ commutes with any element of $\mathfrak{p}'$ by (6.30), $V^{\mathfrak{p}_+}$ is invariant under the action of $L_j^{\mathfrak{g};\mathfrak{p}}$. Namely, $V^{\mathfrak{p}_+}$ is a representation space of the Virasoro algebra! This representation is called a **coset representation** of the Virasoro algebra, and its representation space $V^{\mathfrak{p}_+}$ is called a **coset Virasoro module**.

When (π, V) is an integrable representation of $\mathfrak{g}$, i.e., $V = L(\Lambda)$ ($\Lambda \in P_+^m, m \in \mathbb{Z}_{\geq 0}$), it is unitarizable as a representation of $\mathfrak{g}$, and $V^{\mathfrak{p}_+}$ becomes a unitarizable representation of the Virasoro algebra. If its central charge

$$z_m^{\mathfrak{g};\mathfrak{p}} := z_m - \dot{z}_{\dot{m}} = z_m^{\mathfrak{g}} - z_{\dot{m}}^{\mathfrak{p}}$$

is smaller than 1, then a trick of the conformal invariance constraint works. Such a lucky thing, i.e, $z_m^{\mathfrak{g};\mathfrak{p}} < 1$, really happens. We refer the reader to [**KW1**] for detailed arguments, and here we show some simple examples.

EXAMPLE 6.4. Since by assumption that $\overline{\mathfrak{p}}$ is a reductive Lie subalgebra, we can take the Cartan subalgebra $\overline{\mathfrak{h}}$ of $\overline{\mathfrak{g}}$ as $\overline{\mathfrak{p}}$. Then, the "string functions" of $L(\Lambda)$ can be computed by using the conformal invariance as follows. Since we cannot denote the affinization of $\overline{\mathfrak{g}}$ by $\mathfrak{h}$ ($\because$ it is the same symbol as a Cartan subalgebra of $\mathfrak{g}$), here we denote it by

$$\widetilde{\mathfrak{h}} := \sum_{n\in\mathbb{Z}} (t^n \otimes \overline{\mathfrak{h}}) \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

In this case, the dual Coxeter number of $\widetilde{\mathfrak{h}}$ is $\dot{g} = 0$, and being independent of the level, one has

$$z_{\dot{m}}^{\widetilde{\mathfrak{h}}} = \dim \overline{\mathfrak{h}} = l.$$

And since the dimension of a simple finite-dimensional Lie algebra can be expressed in terms of the Coxeter number h as

$$\dim \overline{\mathfrak{g}} = l(h+1)$$

(cf. [**Kos1**]), the central charge of this coset Virasoro module is

$$z_m^{\mathfrak{g};\widetilde{\mathfrak{h}}} = z_m^{\mathfrak{g}} - l = \frac{m}{m+g} \cdot l(h+1) - l.$$

For simplicity, let us consider the case where $\overline{\mathfrak{g}}$ is a symmetric simple finite-dimensional Lie algebra, i.e., of type A, D or E. Since we have $g = h$ in this case, it follows that

$$z_m^{\mathfrak{g}} = \frac{m}{m+g} \cdot l(g+1), \tag{6.32}$$

and in particular, if $m = 1$, then

$$z_{\dot{m}}^{\widetilde{\mathfrak{h}}} = l - l = 0.$$

Namely, for $\Lambda \in P_+^1$, the representation of the Virasoro algebra on $V^{\mathfrak{p}+}$ is the direct sum of the trivial representations. Combining this fact, the distribution of the weights of $L(\Lambda)$ and the Weyl group invariance of ch_Λ, we obtain

$$\mathrm{ch}_\Lambda = \left(\sum_{\alpha \in M} e^{t_\alpha \Lambda} \right) \times \left(\text{the character of an irreducible } \widetilde{\mathfrak{h}}\text{-module} \right).$$

By the way, since the action of the negative part of $\widetilde{\mathfrak{h}}$ on $L(\Lambda)$ is free by Proposition 11.9 in Chapter 11 of [**K4**], we can summarize the result obtained in this example as a proposition:

PROPOSITION 6.5. *If $\mathfrak{g}$ is a symmetric affine Lie algebra (viz., it is an affine Lie algebra of split type of type A, D or E), and $\Lambda \in P_+^1$, the character of $L(\Lambda)$ can be expressed as follows:*

$$\mathrm{ch}_\Lambda = \frac{1}{\varphi(q)^l} \sum_{\alpha \in M} e^{t_\alpha \Lambda}.$$

Here, l is the rank of $\overline{\mathfrak{g}}$, and $\varphi(q)$ is the Euler function defined in (3.104).

Note. If λ is a weight of $L(\Lambda)$, then $\lambda + n\delta$ is a weight for $n \in \mathbb{Z}_{\geq 0}$ by Proposition 11.9 of [**K4**], but since it is a highest weight module, $\lambda + n\delta$ cannot always be a weight for $n \in \mathbb{Z}_{>0}$. So, there exists a weight λ of $L(\Lambda)$ such that $\lambda + \delta$ is not a weight of $L(\Lambda)$. Such a weight of $L(\Lambda)$ is called a **maximal weight**. When λ is a maximal weight of $L(\Lambda)$, the generating series of the dimension of the weight spaces $L(\Lambda)_{\lambda - n\delta}$ $(n \in \mathbb{Z}_{\geq 0})$

$$c_\lambda^\Lambda(q) := \sum_{n=0}^{\infty} \dim L(\Lambda)_{\lambda - n\delta} \cdot q^n$$

is called the **string function** of $L(\Lambda)$ through λ. Proposition 6.5 implies that if $\mathfrak{g}$ is a symmetric affine Lie algebra and $\Lambda \in P_+^1$, then

1. $c_\lambda^\Lambda(q) = \dfrac{1}{\varphi(q)^l}$, and
2. the set of maximal weights of $L(\Lambda) = \{t_\alpha \Lambda; \alpha \in M\}$.

□

EXAMPLE 6.6. Let us consider the decomposition of the tensor product representations $L(\Lambda') \otimes L(\Lambda'')$ of an affine Lie algebra. For $\Lambda' \in P_+^{m'}$ and $\Lambda'' \in P_+^{m''}$, the central charge of the tensor product representation

$$\pi(L_j) \otimes \mathrm{Id} + \mathrm{Id} \otimes \pi(L_j) \quad \text{and} \quad \pi(Z) \otimes \mathrm{Id} + \mathrm{Id} \otimes \pi(Z)$$

of the Virasoro algebra is $z_{m'} + z_{m''}$. On the other hand, if we take one of the irreducible components, say $L(\Lambda)$ $(\Lambda \in P_+^{m'+m''})$, then the central charge is $z_{m'+m''}$ on it. So, the central charge of the coset representation of the Virasoro algebra is

$$\begin{aligned} z = z_{m' \otimes m''} &= z_{m'} + z_{m''} - z_{m'+m''} \\ &= \left\{ \frac{m'}{m'+g} + \frac{m''}{m''+g} - \frac{m'+m''}{m'+m''+g} \right\} \dim \bar{\mathfrak{g}} \end{aligned} \tag{6.33}$$

Let us consider the simplest case, i.e., $\bar{\mathfrak{g}} = \mathfrak{sl}(2, \mathbb{C})$ and $m'' = 1$. Since $\dim \bar{\mathfrak{g}} = 3$ and $g = 2$ in this case, the above formula becomes

$$z_{m' \otimes 1} = \left\{ \frac{m'}{m'+2} + \frac{1}{3} - \frac{m'+1}{m'+3} \right\} \times 3 = 1 - \frac{6}{(m'+2)(m'+3)}.$$

This is just the central charge of discrete series representations of the Virasoro algebra! In fact, as we see in Example 6.12, if Λ' varies all of the elements of P_+, the discrete series representations of the Virasoro algebra all appear in this decomposition of the tensor product. Since they are realized on the space of unitary representations of an affine

Lie algebra, they are naturally unitarizable as representations of the Virasoro algebra. This was one used method to solve the unitarity problem when $0 \le z < 1$.

Proposition 6.5 is the formula expressing the character ch_Λ in terms of string functions. By using this expression, one can derive the formula decomposing the tensor product of a level 1 representation and a level m representation. Here, for simplicity, we state the result restricting ourselves to the 'basic representation' $L(\Lambda_0)$ for a level 1 representation:

PROPOSITION 6.7. *If $\mathfrak{g}$ is a symmetric affine Lie algebra, and $\Lambda \in P_+^m$ $(m \in \mathbb{Z}_{\ge 0})$, then*

$$\mathrm{ch}'_{\Lambda_0} \cdot \mathrm{ch}'_\Lambda = \sum_{\substack{\Lambda' \in P_+^{m+1} \ s.t. \\ \Lambda' - (\Lambda_0 + \Lambda) \in Q}} b_{\Lambda'}^{\Lambda_0 \otimes \Lambda}(\tau) \cdot \mathrm{ch}'_{\Lambda'},$$

where

(6.34)

$$\begin{aligned} b_{\Lambda'}^{\Lambda_0 \otimes \Lambda}(\tau) &= \frac{1}{\eta(\tau)^l} \sum_{w \in W} \varepsilon(w) q^{\frac{(m+g)(m+g+1)}{2} \left| \frac{w(\Lambda'+\rho)}{m+g+1} - \frac{\Lambda+\rho}{m+g} \right|^2} \\ &= \frac{1}{\eta(\tau)^l} \sum_{\alpha \in M} \sum_{\overline{w} \in \overline{W}} \varepsilon(\overline{w}) q^{\frac{(m+g)(m+g+1)}{2} \left| \alpha + \frac{\overline{w}(\Lambda'+\rho)}{m+g+1} - \frac{\Lambda+\rho}{m+g} \right|^2}. \end{aligned}$$

PROOF. Since

$$\mathrm{ch}'_{\Lambda_0} = \frac{1}{\eta(\tau)^l} \sum_{\gamma \in M} e^{t_\gamma(\Lambda_0)}$$

by Proposition 6.5, and

$$\mathrm{ch}'_\Lambda = \frac{A_{\Lambda+\rho}}{A_\rho}$$

by (4.72), multiplying these, one obtains

$$\mathrm{ch}'_{\Lambda_0} \cdot \mathrm{ch}'_\Lambda = \frac{1}{\eta(\tau)^l} \cdot \frac{1}{A_\rho} \cdot A_{\Lambda+\rho} \sum_{\gamma \in M} e^{t_\gamma(\Lambda_0)}. \tag{6.35}$$

So, set

$$(\mathrm{I}) := A_{\Lambda+\rho} \sum_{\gamma \in M} e^{t_\gamma(\Lambda_0)}, \tag{6.36}$$

and let us compute this. Since

$$A_{\Lambda+\rho} = e^{-\frac{|\Lambda+\rho|^2}{2(m+g)}\delta} \sum_{w\in W} \varepsilon(w) e^{w(\Lambda+\rho)}$$

by (4.58), it follows that

$$\begin{aligned} e^{\frac{|\Lambda+\rho|^2}{2(m+g)}\delta} \times (\mathrm{I}) &= \sum_{w\in W}\sum_{\gamma\in M} \varepsilon(w) e^{w(\Lambda+\rho)+t_\gamma(\Lambda_0)} \\ &= \sum_{\overline{w}\in\overline{W}}\sum_{\beta\in M}\sum_{\gamma\in M} \varepsilon(\overline{w}) e^{t_\beta\overline{w}(\Lambda+\rho)+t_\gamma(\Lambda_0)}. \end{aligned}$$

Here if we substitute $\gamma = \beta + \gamma'$, this formula becomes

$$e^{\frac{|\Lambda+\rho|^2}{2(m+g)}\delta} \times (\mathrm{I}) = \sum_{\overline{w}\in\overline{W}}\sum_{\beta\in M}\sum_{\gamma'\in M} \varepsilon(\overline{w}) e^{t_\beta(\overline{w}(\Lambda+\rho)+t_{\gamma'}(\Lambda_0))}.$$

By the way, since we have

$$t_{\gamma'}(\Lambda_0) = \overline{w} t_{\overline{w}^{-1}\gamma'} \cdot \underbrace{\overline{w}^{-1}(\Lambda_0)}_{\Lambda_0} = \overline{w} t_{\overline{w}^{-1}\gamma'}(\Lambda_0),$$

the above formula is

$$e^{\frac{|\Lambda+\rho|^2}{2(m+g)}\delta} \times (\mathrm{I}) = \sum_{\overline{w}\in\overline{W}}\sum_{\beta\in M}\sum_{\gamma'\in M} \varepsilon(\overline{w}) e^{t_\beta\overline{w}(\Lambda+\rho+t_{\overline{w}^{-1}\gamma'}(\Lambda_0))}.$$

Here, if we set $t_\beta\overline{w} = w$, and substitute $\gamma := \overline{w}^{-1}\gamma'$, then, since

$$t_\gamma(\Lambda_0) = \Lambda_0 + \gamma - \frac{1}{2}|\gamma|^2\delta,$$

the above formula takes the form

$$(6.37) \qquad e^{\frac{|\Lambda+\rho|^2}{2(m+g)}\delta} \times (\mathrm{I}) = \sum_{\gamma\in M}\sum_{w\in W} \varepsilon(w) e^{w(\Lambda+\rho+\Lambda_0+\gamma)} e^{-\frac{1}{2}|\gamma|^2\delta}.$$

Since the signature function $\varepsilon(w)$ is in the right hand side of this formula, if $\Lambda+\rho+\Lambda_0+\gamma$ is not a regular element, then the summation of this term over $w \in W$ vanishes. Thus, we have only to take the summation over γ so that $\Lambda + \rho + \Lambda_0 + \gamma$ becomes regular. In this case, i.e., when $\Lambda + \rho + \Lambda_0 + \gamma$ is a regular element, one can transfer it to the interior of the fundamental chamber via the action of the Weyl group. Namely, by Lemma 3.32, there uniquely exist $\sigma \in W$ and $\Lambda' \in P_+^{m+1}$ (mod $\mathbb{C}\delta$) satisfying

$$(6.38) \qquad \begin{aligned} &\Lambda + \rho + \Lambda_0 + \gamma \equiv \sigma(\Lambda' + \rho) \qquad \bmod \mathbb{C}\delta, \\ &\Lambda' - (\Lambda + \Lambda_0) \in Q. \end{aligned}$$

If we write this σ as $\sigma = t_\alpha \overline{\sigma}$ ($\alpha \in M, \overline{\sigma} \in \overline{W}$), then $\sigma(\Lambda' + \rho)$ looks like

$$\begin{aligned}\sigma(\Lambda' + \rho) &= t_\alpha \cdot \overline{\sigma}(\Lambda' + \rho) \\ &= \overline{\sigma}(\Lambda' + \rho) + (m+g+1)\alpha \\ &\quad - \left\{ (\overline{\sigma}(\Lambda' + \rho)|\alpha) + \frac{1}{2}|\alpha|^2(m+g+1) \right\} \delta.\end{aligned} \tag{6.39}$$

So, from (6.38) and (6.39), we obtain

$$\Lambda + \rho + \Lambda_0 + \gamma = \overline{\sigma}(\Lambda' + \rho) + (m+g+1)\alpha, \tag{6.40}$$

i.e.,

$$\gamma = \overline{\sigma}(\Lambda' + \rho) + (m+g+1)\alpha - (\Lambda + \rho + \Lambda_0). \tag{6.41}$$

Let us rewrite (6.37) by using (6.40) and (6.41). By setting

$$\begin{aligned}F(\Lambda', \overline{\sigma}, \alpha) :=& (\overline{\sigma}(\Lambda' + \rho)|\alpha) + \frac{1}{2}|\alpha|^2(m+g+1) \\ & - \frac{1}{2}|\overline{\sigma}(\Lambda' + \rho) + (m+g+1)\alpha - (\Lambda + \Lambda_0 + \rho)|^2,\end{aligned}$$

we get (6.37) in the form

$$\begin{aligned}& e^{\frac{|\Lambda+\rho|^2}{2(m+g)}\delta} \times (\mathrm{I}) \\ &= \sum_{w \in W} \sum_{\overline{\sigma} \in \overline{W}} \sum_{\alpha \in M} \sum_{\substack{\Lambda' \in P_+^{m+1} \text{ s.t.} \\ \Lambda' - (\Lambda + \Lambda_0) \in Q}} \varepsilon(w) e^{w \cdot t_\alpha \cdot \overline{\sigma}(\Lambda' + \rho)} e^{F(\Lambda', \overline{\sigma}, \alpha)\delta} \\ &= \sum_{\substack{\Lambda' \in P_+^{m+1} \text{ s.t.} \\ \Lambda' - (\Lambda + \Lambda_0) \in Q}} \sum_{\overline{\sigma} \in \overline{W}} \sum_{\alpha \in M} \varepsilon(\overline{\sigma}) \underbrace{\left\{ \sum_{w \in W} \varepsilon(w) e^{w(\Lambda' + \rho)} \right\}}_{e^{\frac{|\Lambda'+\rho|^2}{2(m+g+1)}\delta} \cdot A_{\Lambda'+\rho}} e^{F(\Lambda', \overline{\sigma}, \alpha)\delta} \\ &= \sum_{\substack{\Lambda' \in P_+^{m+1} \text{ s.t.} \\ \Lambda' - (\Lambda + \Lambda_0) \in Q}} \left\{ e^{\frac{|\Lambda'+\rho|^2}{2(m+g+1)}\delta} \cdot A_{\Lambda'+\rho} \sum_{\overline{\sigma} \in \overline{W}} \sum_{\alpha \in M} \varepsilon(\overline{\sigma}) e^{F(\Lambda', \overline{\sigma}, \alpha)\delta} \right\}.\end{aligned}$$

Thus, (I) of (6.36) becomes

(6.42)

$$\sum_{\substack{\Lambda' \in P_+^{m+1} \text{ s.t.} \\ \Lambda' - (\Lambda + \Lambda_0) \in Q}} A_{\Lambda'+\rho} \left\{ \sum_{\overline{\sigma} \in \overline{W}} \sum_{\alpha \in M} \varepsilon(\overline{\sigma}) e^{\frac{|\Lambda'+\rho|^2}{2(m+g+1)}\delta} e^{-\frac{|\Lambda+\rho|^2}{2(m+g)}\delta} e^{F(\Lambda',\overline{\sigma},\alpha)\delta} \right\}.$$

Therefore, putting the coefficients of δ in the exponent of e in this formula by $G(\Lambda', \overline{\sigma}, \alpha)$, we get

(6.43)

$$\begin{aligned}
&G(\Lambda', \overline{\sigma}, \alpha) \\
&:= F(\Lambda', \overline{\sigma}, \alpha) + \frac{|\Lambda'+\rho|^2}{2(m+g+1)} - \frac{|\Lambda+\rho|^2}{2(m+g)} \\
&= -\frac{1}{2}|\overline{\sigma}(\Lambda'+\rho) + (m+g+1)\alpha - (\Lambda+\Lambda_0+\rho)|^2 \\
&\quad + (\overline{\sigma}(\Lambda'+\rho)|\alpha) + \frac{1}{2}|\alpha|^2(m+g+1) + \frac{|\Lambda'+\rho|^2}{2(m+g+1)} - \frac{|\Lambda+\rho|^2}{2(m+g)} \\
&= -\frac{1}{2}\Big\{|\overline{\sigma}(\Lambda'+\rho) + (m+g+1)\alpha - (\Lambda+\Lambda_0+\rho)|^2 \\
&\qquad -2(\overline{\sigma}(\Lambda'+\rho)|\alpha) - (m+g+1)|\alpha|^2 - \frac{|\Lambda'+\rho|^2}{m+g+1} + \frac{|\Lambda+\rho|^2}{m+g}\Big\} \\
&= -\frac{(m+g)(m+g+1)}{2} \cdot \left|\frac{\overline{\sigma}(\Lambda'+\rho)}{m+g+1} + \alpha - \frac{\Lambda+\rho}{m+g}\right|^2.
\end{aligned}$$

Rewriting (6.42) by using (6.43), we obtain the proposition by (6.35) and (6.36). □

Note. The level of the element of $\mathfrak{h}^*$ put on e in (6.34) of this proposition,

$$\alpha + \frac{\overline{w}(\Lambda'+\rho)}{m+g+1} - \frac{\Lambda+\rho}{m+g},$$

is 0, and hence its norm is invariant even if we add a scalar multiple of δ to this element. Therefore, we can replace this part by its $\overline{\mathfrak{h}}^*$ as follows:

(6.44)

$$b_{\Lambda'}^{\Lambda_0 \otimes \Lambda}(\tau) = \frac{1}{\eta(\tau)^l} \sum_{\alpha \in M} \sum_{\overline{w} \in \overline{W}} \varepsilon(\overline{w}) q^{\frac{(m+g)(m+g+1)}{2}\left|\alpha + \frac{\overline{w}(\overline{\Lambda'+\rho})}{m+g+1} - \frac{\overline{\Lambda+\rho}}{m+g}\right|^2}. \quad □$$

Note. In Proposition 6.7, we impose the condition that $\mathfrak{g}$ be symmetric, and that one of the representations be of level 1, but the decomposition theorem of the tensor product of representations in this form is proved in some other cases. In any case, if the character of one of the representations is described in terms of its string functions, one can apply the above method of proof. There is no place to explain in this book, but when one of the representations is an admissible representation that is of a wider class than the integrable representations, one can also compute the decomposition of the character of the tensor product of representations. $b_{\Lambda'}^{\Lambda_0\otimes\Lambda}(\tau)$ also has a meaning as a character of the W-algebra associated to the affine Lie algebra. □

$b_{\Lambda'}^{\Lambda_0\otimes\Lambda}(\tau)$ arising in the above decomposition of the tensor product of the representations is a function only of $q=e^{2\pi i\tau}$. This is because, in the decomposition of $L(\Lambda_0)\otimes L(\Lambda)$, for each $\Lambda'\in P_+^{m+1}$, we are looking at the singular vectors whose weights are the same as $\Lambda'\in P_+^{m+1}$ modulo $\mathbb{C}\delta$, i.e., elements of $\{\Lambda'+n\delta; n\in\mathbb{Z}\}$. In general, when we write the decomposition of $L(\Lambda')\otimes L(\Lambda'')$ as

$$\mathrm{ch}'_{\Lambda'}\cdot\mathrm{ch}'_{\Lambda''}=\sum_{\Lambda\in P_+^{m'+m''}} b_{\Lambda}^{\Lambda'\otimes\Lambda''}(\tau)\cdot\mathrm{ch}'_{\Lambda}, \tag{6.45}$$

$b_{\Lambda}^{\Lambda'\otimes\Lambda''}(\tau)$ is called a **branching function** of the decomposition of the tensor product of representations. Since ch'_{Λ}'s are modular functions, $b_{\Lambda}^{\Lambda'\otimes\Lambda''}(\tau)$ is also a modular function, and its modular transformations can be obtained as follows. First, if we apply S to (6.45), we have

$$\mathrm{ch}'_{\Lambda'}|_S\cdot\mathrm{ch}'_{\Lambda''}|_S=\sum_{\Lambda\in P_+^{m'+m''}} b_{\Lambda}^{\Lambda'\otimes\Lambda''}\left(-\frac{1}{\tau}\right)\cdot\mathrm{ch}'_{\Lambda}|_S,$$

and if we rewrite this formula by using the formula for a modular transformation of ch'_{λ},

$$\mathrm{ch}'_{\lambda}|_S=\sum_{\mu} a(\lambda,\mu)\mathrm{ch}'_{\mu},$$

for Λ',Λ'' and Λ, we obtain

$$\begin{aligned}\sum_{\mu'}\sum_{\mu''}&a(\Lambda',\mu')a(\Lambda'',\mu'')\cdot\mathrm{ch}'_{\mu'}\cdot\mathrm{ch}'_{\mu''}\\ &=\sum_{\Lambda}\sum_{\mu} b_{\Lambda}^{\Lambda'\otimes\Lambda''}\left(-\frac{1}{\tau}\right)\cdot a(\Lambda,\mu)\mathrm{ch}'_{\mu}.\end{aligned} \tag{6.46}$$

By the way, since we have

$$\mathrm{ch}'_{\mu'} \cdot \mathrm{ch}'_{\mu''} = \sum_{\mu} b_{\mu}^{\mu' \otimes \mu''}(\tau) \cdot \mathrm{ch}'_{\mu}$$

from (6.45), we can rewrite (6.46) as follows:

$$\sum_{\mu'} \sum_{\mu''} \sum_{\mu} a(\Lambda', \mu') a(\Lambda'', \mu'') b_{\mu}^{\mu' \otimes \mu''}(\tau) \cdot \mathrm{ch}'_{\mu} = \sum_{\Lambda} \sum_{\mu} b_{\Lambda}^{\Lambda' \otimes \Lambda''}\left(-\frac{1}{\tau}\right) \cdot a(\Lambda, \mu) \mathrm{ch}'_{\mu}.$$

ch'_{μ} is a function of (τ, z, t), and by the linearly independence of these functions, we obtain

$$\sum_{\mu'} \sum_{\mu''} a(\Lambda', \mu') a(\Lambda'', \mu'') b_{\mu}^{\mu' \otimes \mu''}(\tau) = \sum_{\Lambda} b_{\Lambda}^{\Lambda' \otimes \Lambda''}\left(-\frac{1}{\tau}\right) a(\Lambda, \mu) \tag{6.47}$$

for each $\mu \in P_{+}^{m'+m''}$. To pick up the single $b_{\Lambda}^{\Lambda' \otimes \Lambda''}\left(-\frac{1}{\tau}\right)$ in the right hand side of this formula, we use the fact that the matrix $(a(\lambda, \mu))$ is unitary. Namely, since we have

$$\sum_{\mu} a(\Lambda, \mu) a({}^t\mu, \nu) = \delta_{\Lambda, \nu}$$

by (4.81), multiplying by both sides of (6.47) $a({}^t\mu, \nu)$ and summing over μ, we obtain the next formula:

$$b_{\Lambda}^{\Lambda' \otimes \Lambda''}\left(-\frac{1}{\tau}\right) = \sum_{\mu} \sum_{\mu'} \sum_{\mu''} a(\Lambda', \mu') a(\Lambda'', \mu'') a(\Lambda, {}^t\mu) b_{\mu}^{\mu' \otimes \mu''}(\tau).$$

Sine we have $a({}^t\mu, \nu) = \overline{a(\mu, \nu)}$ by (4.82), this formula can be rewritten as follows:

$$b_{\Lambda}^{\Lambda' \otimes \Lambda''}\left(-\frac{1}{\tau}\right) = \sum_{\mu} \sum_{\mu'} \sum_{\mu''} a(\Lambda', \mu') a(\Lambda'', \mu'') \overline{a(\Lambda, \mu)} b_{\mu}^{\mu' \otimes \mu''}(\tau).$$

Moreover, it is easy to compute $b_{\Lambda}^{\Lambda' \otimes \Lambda''}(\tau + 1)$, and this can be computed by Theorem 4.13 as follows. First, by letting $\tau \longrightarrow \tau + 1$ in (6.45), we have

$$e^{2\pi i s_{\Lambda'}} e^{2\pi i s_{\Lambda''}} \cdot \mathrm{ch}'_{\Lambda'} \cdot \mathrm{ch}'_{\Lambda''} = \sum_{\Lambda \in P_{+}^{m'+m''}} b_{\Lambda}^{\Lambda' \otimes \Lambda''}(\tau + 1) \cdot e^{2\pi i s_{\Lambda}} \mathrm{ch}'_{\Lambda},$$

and comparing this formula with (6.45), we obtain

$$b_{\Lambda}^{\Lambda'\otimes\Lambda''}(\tau+1)=e^{2\pi i(s_{\Lambda'}+s_{\Lambda''}-s_{\Lambda})}b_{\Lambda}^{\Lambda'\otimes\Lambda''}(\tau).$$

Let us summarize these as a proposition:

PROPOSITION 6.8. *If $\mathfrak{g}$ is an affine Lie algebra of split type, and $\Lambda'\in P_+^{m'}$, $\Lambda''\in P_+^{m''}$ and $\Lambda\in P_+^{m'+m''}$, modular transformations of the branching function $b_{\Lambda}^{\Lambda'\otimes\Lambda''}(\tau)$ of the decomposition of the tensor product of the representations $L(\Lambda')\otimes L(\Lambda'')$ are given as follows:*

1. $b_{\Lambda}^{\Lambda'\otimes\Lambda''}\left(-\frac{1}{\tau}\right)$
$=\sum_{\mu}\sum_{\mu'}\sum_{\mu''}a(\Lambda',\mu')a(\Lambda'',\mu'')\overline{a(\Lambda,\mu)}b_{\mu}^{\mu'\otimes\mu''}(\tau)$,
2. $b_{\Lambda}^{\Lambda'\otimes\Lambda''}(\tau+1)=e^{2\pi i(s_{\Lambda'}+s_{\Lambda''}-s_{\Lambda})}b_{\Lambda}^{\Lambda'\otimes\Lambda''}(\tau)$.

REMARK 6.9. The decomposition of a tensor product has the form

$$L(\Lambda')\otimes L(\Lambda'')=\bigoplus_{\Lambda\in P_+^{m'+m''}}(V_{\Lambda}^{\Lambda'\otimes\Lambda''}\otimes L(\Lambda)),\tag{6.48}$$

and it is (6.45) that expresses this in terms of their characters. Here, $V_{\Lambda}^{\Lambda'\otimes\Lambda''}$ is a Vir-module. Notice that the right hand side of (6.48) is the decomposition regarded not as $\mathfrak{g}$-modules but as $\mathfrak{g}'$-modules. This is because, $\mathfrak{g}$ contains d, and the values of this element d depend on the weight spaces where the singular vectors, that give $L(\Lambda)$ in $L(\Lambda')\otimes L(\Lambda'')$, lie, and they are the branching functions that measure them. Therefore, the right hand side of (6.48) expresses the decomposition regarded as $(\mathrm{Vir}\oplus\mathfrak{g}')$-module. In general, the Vir-module $V_{\Lambda}^{\Lambda'\otimes\Lambda''}$ is not necessarily irreducible.

If we replace τ by $-\frac{1}{\tau}$ in (1) of the above proposition, we get

$$b_{\Lambda}^{\Lambda'\otimes\Lambda''}(\tau)=\sum_{\mu}\sum_{\mu'}\sum_{\mu''}a(\Lambda',\mu')a(\Lambda'',\mu'')\overline{a(\Lambda,\mu)}b_{\mu}^{\mu'\otimes\mu''}\left(-\frac{1}{\tau}\right).\tag{6.49}$$

Using this formula, we can study the asymptotic behavior of $b_{\Lambda}^{\Lambda'\otimes\Lambda''}(\tau)$ as $\tau\downarrow 0$. In the right hand side of (6.49), the terms that contribute to the principal part of the asymptotic behavior as $\tau\downarrow 0$ are those (μ',μ'',μ) when it is $(m'\Lambda_0,m''\Lambda_0,(m'+m'')\Lambda_0)$ or its image of the action of $\widetilde{W}_+$, i.e., the terms

$$(\mu',\mu'',\mu)=(m'\sigma\Lambda_0,m''\sigma\Lambda_0,(m'+m'')\sigma\Lambda_0)\quad(\sigma\in\widetilde{W}_+).$$

And for $\lambda \in P_+^m$ and $\sigma \in \widetilde{W}_+$, the formula

$$a(\lambda, m\Lambda_0) = a(\lambda, \sigma(m\Lambda_0))$$

does not necessarily hold, but in general, one can prove the following:

LEMMA 6.10. *Let $\mathfrak{g}$ be an affine Lie algebra of split type. If $\lambda', \mu' \in P_+^{m'}$; $\lambda'', \mu'' \in P_+^{m''}$; $\lambda, \mu \in P_+^{m'+m''}$ satisfy*

$$\lambda' + \lambda'' - \lambda, \mu' + \mu'' - \mu \in Q,$$

then the next formula holds for any $\sigma \in \widetilde{W}_+$:

$$a(\lambda', \mu')a(\lambda'', \mu'')a({}^t\lambda, \mu) = a(\lambda', \sigma(\mu'))a(\lambda'', \sigma(\mu''))a({}^t\lambda, \sigma(\mu)).$$

By this lemma, it follows that

$$\begin{aligned} &a(\Lambda', \sigma(m'\Lambda_0))a(\Lambda'', \sigma(m''\Lambda_0))\overline{a(\Lambda, \sigma((m'+m'')\Lambda_0))} \\ &= a(\Lambda', m'\Lambda_0)a(\Lambda'', m''\Lambda_0)\overline{a(\Lambda, (m'+m'')\Lambda_0)} \\ &= a(\Lambda')a(\Lambda'')a(\Lambda), \end{aligned}$$

and hence we obtain

$$b_\Lambda^{\Lambda' \otimes \Lambda''}(\tau) \sim |J| \cdot a(\Lambda')a(\Lambda'')a(\Lambda)e^{\frac{\pi i}{12\tau}(z_{m'} + z_{m''} - z_{m'+m''})} \tag{6.50}$$

as $\tau \downarrow 0$.

Next, if we specialize (6.45), in particular, to $z = t = 0$, then we have

$$\mathrm{ch}'_{\Lambda'}(\tau, 0, 0) \cdot \mathrm{ch}'_{\Lambda''}(\tau, 0, 0) = \sum_{\Lambda \in P_+^{m'+m''}} b_\Lambda^{\Lambda' \otimes \Lambda''}(\tau) \cdot \mathrm{ch}'_\Lambda(\tau, 0, 0), \tag{6.51}$$

but since the asymptotic behavior of $\mathrm{ch}'_\lambda(\tau, 0, 0) = \chi_\Lambda(\tau)$ $(\lambda \in P_+^m)$ as $\tau \downarrow 0$ is given by

$$\chi_\lambda(\tau) \sim a(\lambda)e^{\frac{\pi i}{12\tau}z_m} \quad (\lambda \in P_+^m)$$

by Theorem 4.17, it follows from this, (6.50) and (6.51) that

$$a(\Lambda')a(\Lambda'') = \sum_{\substack{\Lambda \in P_+^{m'+m''} \\ \text{s.t.} \Lambda - (\Lambda' + \Lambda'') \in Q}} |J|a(\Lambda')a(\Lambda'')a(\Lambda) \cdot a(\Lambda),$$

and hence we obtain

$$\sum_{\substack{\Lambda \in P_+^{m'+m''} \\ \text{s.t.} \Lambda - (\Lambda' + \Lambda'') \in Q}} a(\Lambda)^2 = \frac{1}{|J|}.$$

Summarizing these, we obtain the following proposition:

PROPOSITION 6.11. *Let $\mathfrak{g}$ be an affine Le algebra of split type.*

1. *For $\Lambda' \in P_+^{m'}, \Lambda'' \in P_+^{m''}$ and $\Lambda \in P_+^{m'+m''}$, the asymptotic behavior as $\tau \downarrow 0$ of the branching function $b_\Lambda^{\Lambda'\otimes\Lambda''}(\tau)$ of the decomposition of the tensor product of representations $L(\Lambda') \otimes L(\Lambda'')$ is given by*

$$b_\Lambda^{\Lambda'\otimes\Lambda''}(\tau) \sim |J| \cdot a(\Lambda')a(\Lambda'')a(\Lambda)e^{\frac{\pi i}{12\tau}(z_{m'}+z_{m''}-z_{m'+m''})}$$

2. *For $m \in \mathbb{Z}_{\geq 0}$, the following formula holds for any $\Lambda \in P_+^m$:*

$$\sum_{\substack{\lambda \in P_+^m \\ s.t. \lambda-\Lambda \in Q}} a(\lambda)^2 = \frac{1}{|J|}.$$

Note. From (1) of this proposition, we can conclude that for $\Lambda' \in P_+^{m'}$ and $\Lambda'' \in P_+^{m''}$, a $\Lambda \in P_+^{m'+m''}$ satisfying $\Lambda - (\Lambda' + \Lambda'') \in Q$ necessarily appears in the decomposition of the tensor product of representations $L(\Lambda') \otimes L(\Lambda'')$. □

Note. From (2) of this proposition and Corollary 4.15 in §4.3, it follows that $|J|$ coincides with the number of conjugacy classes of P_+^m with respect to Q for each fixed $m \in \mathbb{N}$. Moreover, it is known (cf. [**Kos2**]) that $|J|$ is equal to the determinant of the Cartan matrix of the simple finite-dimensional Lie algebra $\bar{\mathfrak{g}}$. □

EXAMPLE 6.12. For $A_1^{(1)} = \hat{\mathfrak{sl}}(2,\mathbb{C})$, since $\dim\bar{\mathfrak{g}} = \dim\mathfrak{sl}(2,\mathbb{C}) = 3$ and $g = 2$, the central charge of the Virasoro algebra on an integrable representation of level m is

$$z_m = \frac{m}{m+g}\dim\bar{\mathfrak{g}} = \frac{3m}{m+2}.$$

Let us consider the decomposition of the tensor product of representations $L(\Lambda_0) \otimes L(\Lambda)$. If we set

$$\lambda_{m,j} := (m-j)\Lambda_0 + j\Lambda_1 = m\Lambda_0 + \frac{j}{2}\alpha_1,$$

in the same way as (4.89) in §4.3, then we have

$$\lambda_{m,j} + 2\rho = (m+4)\Lambda_0 + \left(\frac{j}{2}+1\right)\alpha_1,$$

and

$$(\lambda_{m,j}|\lambda_{m,j}+2\rho) = j\left(\frac{j}{2}+1\right) = \frac{j(j+2)}{2},$$

which implies

$$h^{\mathfrak{g}}_{\lambda_{m,j}} = \frac{1}{2(m+2)} \cdot \frac{j(j+2)}{2} = \frac{j(j+2)}{4(m+2)}.$$

Since the level of a representation of $A_1^{(1)}$ appearing in the decomposition $L(\Lambda_0) \otimes L(\lambda_{m,j})$ is $m+1$, if we put its weight as $\lambda_{m+1,k}$, then we have

$$h^{\mathfrak{g}}_{\lambda_{m+1,k}} = \frac{k(k+2)}{4(m+3)},$$

and hence their difference is

$$\begin{aligned}
&h^{\mathfrak{g}}_{\Lambda_0} + h^{\mathfrak{g}}_{\lambda_{m,j}} - h^{\mathfrak{g}}_{\lambda_{m+1,k}} \\
&= O + \frac{j(j+2)}{4(m+2)} - \frac{k(k+2)}{4(m+3)} \\
&= \frac{j(j+2)(m+3) - k(k+2)(m+2)}{4(m+2)(m+3)} \\
&= \frac{[(j+1)(m+3) - (k+1)(m+2)]^2 - 1}{4(m+2)(m+3)} - \frac{(k-j)^2}{4} \\
&= h^{(m)}_{j+1,k+1} - \frac{(k-j)^2}{4}.
\end{aligned}$$

Namely, we have

$$h^{(m)}_{j+1,k+1} = h^{\mathfrak{g}}_{\Lambda_0} + h^{\mathfrak{g}}_{\lambda_{m,j}} - h^{\mathfrak{g}}_{\lambda_{m+1,k}} + \frac{(k-j)^2}{4}. \tag{6.52}$$

This formula shows that the first place where the singular vector of $L(\lambda_{m+1,k})$ appears in the decomposition of $L(\Lambda_0) \otimes L(\lambda_{m,j})$ is the weight space of weight

$$\Lambda_0 + \lambda_{m,j} - \frac{(k-j)^2}{4}\delta + \frac{k-j}{2}\alpha_1.$$

Here, if $L(\lambda_{m+1,k})$ appears in the decomposition of $L(\Lambda_0) \otimes L(\lambda_{m,j})$, then since it should satisfy

$$\lambda_{m+1,k} - (\Lambda_0 + \lambda_{m,j}) \in \mathbb{Z}\alpha_0 + \mathbb{Z}\alpha_1,$$

it follows that $k \equiv j \mod 2$.

Next, let us compute the branching function

$$b^{\Lambda_0 \otimes \lambda_{m,j}}_{\lambda_{m+1,k}}(\tau) = \frac{1}{\eta(\tau)} \sum_{\alpha \in M} \sum_{\overline{w} \in \overline{W}} \varepsilon(\overline{w}) q^{\frac{(m+2)(m+3)}{2}\left|\alpha + \frac{\overline{w}(\overline{\lambda}_{m+1,k}+\overline{\rho})}{m+3} - \frac{\overline{\lambda}_{m,j}+\overline{\rho}}{m+2}\right|^2}$$

by using (6.44). Since we have

$$M = \mathbb{Z}\alpha_1, \quad \overline{\lambda}_{m,j} + \overline{\rho} = \frac{j+1}{2}\alpha_1, \quad \overline{\lambda}_{m+1,k} + \overline{\rho} = \frac{k+1}{2}\alpha_1,$$

by putting $\alpha = n\alpha_1$ ($n \in \mathbb{Z}$), the above formula becomes

$$\begin{aligned}
(6.53)\quad & b_{\lambda_{m+1,k}}^{\Lambda_0 \otimes \lambda_{m,j}}(\tau) \\
&= \frac{1}{\eta(\tau)} \sum_{n\in\mathbb{Z}} \sum_{\overline{w}\in\overline{W}} \varepsilon(\overline{w}) q^{\frac{(m+2)(m+3)}{2}\left|n\alpha_1 + \frac{k+1}{2(m+3)}\overline{w}(\alpha_1) - \frac{j+1}{2(m+2)}\alpha_1\right|^2} \\
&= \frac{1}{\eta(\tau)} \left\{ \sum_{n\in\mathbb{Z}} q^{(m+2)(m+3)\left[n + \frac{k+1}{2(m+3)} - \frac{j+1}{2(m+2)}\right]^2} \right. \\
&\qquad \left. - \sum_{n\in\mathbb{Z}} q^{(m+2)(m+3)\left[n - \frac{k+1}{2(m+3)} - \frac{j+1}{2(m+2)}\right]^2} \right\} \\
&= \Xi_{j+1,k+1}^{(m)}(\tau).
\end{aligned}$$

Therefore, we obtain the following decomposition for $A_1^{(1)}$:

$$L(\Lambda_0) \otimes L(\lambda_{m,j}) = \sum_{\substack{0\le k\le m+1 \\ \text{s.t.} \\ j\equiv k \bmod 2}} \mathfrak{V}(h_{j+1,k+1}^{(m)}, z^{(m)}) \otimes L(\lambda_{m+1,k}).$$

(6.53) is a formula expressing characters of the Virasoro algebra in terms of the Weyl group of $A_1^{(1)}$. There is a restriction $j \equiv k \bmod 2$ in the decomposition of the tensor product, but by using the relation $h_{j+1,k+1}^{(m)} = h_{m+1-j,m+2-k}^{(m)}$, one can remove this restriction. And for all j, k, we obtain the following formula:

PROPOSITION 6.13. *For $m \in \mathbb{N}$ and $j, k \in \mathbb{Z}_{\ge 0}$ satisfying $0 \le j \le m$ and $0 \le k \le m+1$, $\Xi_{j+1,k+1}^{(m)}(\tau)$ can be expressed in terms of $A_1^{(1)}$ as follows:*

$$\Xi_{j+1,k+1}^{(m)}(\tau) = \frac{1}{\eta(\tau)} \sum_{w\in W} \varepsilon(w) q^{\frac{(m+2)(m+3)}{2}\left|\frac{w(\lambda_{m+1,k}+\rho)}{m+3} - \frac{\lambda_{m,j}+\rho}{m+2}\right|^2}.$$

Note. The characters of the discrete series of the Virasoro algebra enjoy the modular invariance properties, but there is a wider class called "minimal series" (or, BPZ series) known (thanks to [**BPZ**]) to be a class of representations with the modular invariance properties. The characters of the minimal series representations can also be expressed by using the affine Lie algebra $A_1^{(1)}$ as follows (here, we set

$l = 1, g = 2$):

(6.54)
$$\varphi_{\lambda,\mu}(q) := \frac{1}{\eta(\tau)^l} \sum_{w \in W} \varepsilon(w) q^{\frac{(m+g)(m'+g)}{2} \left| \frac{w(\lambda+\rho)}{m+g} - \frac{\mu+\rho}{m'+g} \right|^2}$$
$$(\lambda \in P_+^m,\ \mu \in P_+^{m'}).$$

For a minimal series representation, m and m' are non-negative integers satisfying $\gcd(m+2, m'+2) = 1$, and in particular, it is a discrete series if $m - m' = \pm 1$. When $\mathfrak{g}$ is a symmetric affine Lie algebra, the functions $\varphi_{\lambda,\mu}$ defined by (6.54) are the characters of minimal series representations of the W-algebra, and their modular transformations can be described by the matrices of the modular transformations of the affine Lie algebra. (cf. [**FKW**], [**KW2**]) □

The modular transformations and the asymptotic behavior of the characters of the Virasoro algebra can be computed very easily by using Proposition 6.8, Proposition 6.11, (6.53), (4.90) and the relation associated to the symmetry of the Dynkin diagram of $A_1^{(1)}$,

$$b_{\lambda_{m+1,k}}^{\Lambda_1 \otimes \lambda_{m,j}}(\tau) = b_{\lambda_{m+1,m+1-k}}^{\Lambda_0 \otimes \lambda_{m,m-j}}(\tau) = \Xi_{m+1-j,m+2-k}^{(m)}(\tau),$$

and we obtain the following proposition:

Proposition 6.14.

1. $$\Xi_{r,s}^{(m)}\left(-\frac{1}{\tau}\right) = \sum_{\substack{r',s' \in \mathbb{N} s.t. \\ 1 \le s' \le r' \le m+1}} S_{(r,s),(r',s')}^{(m)} \Xi_{r',s'}^{(m)}(\tau).$$

   Here, $S_{(r,s),(r',s')}^{(m)}$ are defined as follows:

(6.55)
$$S_{(r,s),(r',s')}^{(m)} := (-1)^{(r+s)(r'+s')} \sqrt{\frac{8}{(m+2)(m+3)}} \sin \frac{\pi r r'}{m+2} \sin \frac{\pi s s'}{m+3}.$$

2. $$\Xi_{r,s}^{(m)}(\tau) \underset{\tau \downarrow 0}{\sim} \sqrt{\frac{8}{(m+2)(m+3)}} \sin \frac{\pi r}{m+2} \sin \frac{\pi s}{m+3} \cdot e^{\frac{\pi i}{12\tau} z^{(m)}}.$$

Example 6.15. Let us compute the characters of the Virasoro algebra for $m = 1$. In this case, its central charge is $z^{(1)} = \frac{1}{2}$, and the possible values of (r, s) are

$$(r, s) = (1,1),\ (1,2),\ (2,2);$$

correspondingly, $h^{(1)}_{r,s}$ is

$$h^{(1)}_{1,1} = 0, \quad h^{(1)}_{2,1} = \frac{1}{2}, \quad h^{(1)}_{2,2} = \frac{1}{16}.$$

Thus, the characters of the irreducible representations have the forms

$$\begin{aligned}
\Xi^{(1)}_{1,1}(\tau) &= q^{-\frac{1}{48}}(1+\cdots),\\
\Xi^{(1)}_{2,1}(\tau) &= q^{-\frac{1}{48}+\frac{1}{2}}(1+\cdots),\\
\Xi^{(1)}_{2,2}(\tau) &= q^{-\frac{1}{48}+\frac{1}{16}}(1+\cdots) = q^{\frac{1}{24}}(1+\cdots).
\end{aligned}$$

Arranging (r, s) in this order, the matrix of the transformation of the characters becomes

$$\left(S^{(1)}_{(r,s),(r',s')}\right) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix}.$$

So, if we set

$$\begin{aligned}
f_1(\tau) &:= \Xi^{(1)}_{1,1}(\tau) + \Xi^{(1)}_{2,1}(\tau),\\
f_2(\tau) &:= \Xi^{(1)}_{1,1}(\tau) - \Xi^{(1)}_{2,1}(\tau), \quad f_3(\tau) := \sqrt{2}\cdot\Xi^{(1)}_{2,2}(\tau),
\end{aligned}$$

then these functions have the following forms:

$$\begin{aligned}
f_1(\tau) &= q^{-\frac{1}{48}}\sum_{n=0}^{\infty} a_n q^{\frac{n}{2}} \quad (a_0 = 1),\\
f_2(\tau) &= q^{-\frac{1}{48}}\sum_{n=0}^{\infty}(-1)^n a_n q^{\frac{n}{2}},\\
f_3(\tau) &= q^{\frac{1}{24}}\sum_{n=0}^{\infty} b_n q^n,
\end{aligned}$$

and their modular transformations are

$$f_1\left(-\frac{1}{\tau}\right) = f_1(\tau), \quad f_2\left(-\frac{1}{\tau}\right) = f_3(\tau), \quad f_3\left(-\frac{1}{\tau}\right) = f_2(\tau).$$

Thus, by Lemma 4.9, it follows that $f_i(\tau) = \phi_i(\tau)$ $(i = 1, 2, 3)$. Here, the $\phi_i(\tau)$'s are the functions defined in (4.45) in §4.2. In this way, the characters of the irreducible representations of the Virasoro algebra for $m=1$ are obtained as follows:

$$\begin{aligned}
&\Xi_{1,1}^{(1)}(\tau)+\Xi_{2,1}^{(1)}(\tau)=\frac{\eta(\tau)^2}{\eta\left(\frac{\tau}{2}\right)\eta(2\tau)}=q^{-\frac{1}{48}}\prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}}),\\
(6.56)\qquad &\Xi_{1,1}^{(1)}(\tau)-\Xi_{2,1}^{(1)}(\tau)=\frac{\eta\left(\frac{\tau}{2}\right)}{\eta(\tau)}=q^{-\frac{1}{48}}\prod_{n=1}^{\infty}(1-q^{n-\frac{1}{2}}),\\
&\Xi_{2,2}^{(1)}(\tau)=\frac{\eta(2\tau)}{\eta(\tau)}=q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1+q^{n}).
\end{aligned}$$

Moreover, by (2) of Proposition 6.14, the asymptotic behavior of $\Xi_{r,s}^{(1)}(\tau)$ as $\tau\downarrow 0$ is

$$\begin{aligned}
&\Xi_{1,1}^{(1)}(\tau)\sim\frac{1}{2}\cdot e^{\frac{\pi i}{24\tau}},\\
(6.57)\qquad &\Xi_{2,1}^{(1)}(\tau)\sim\frac{1}{2}\cdot e^{\frac{\pi i}{24\tau}},\\
&\Xi_{2,2}^{(1)}(\tau)\sim\frac{1}{\sqrt{2}}\cdot e^{\frac{\pi i}{24\tau}}.
\end{aligned}$$

EXAMPLE 6.16. Let us consider the tensor product of level 1 representations of $E_8^{(1)}$. Since the colabels $a_i^\vee$ $(0\le i\le 8)$ of $E_8^{(1)}$ are given by

$$\begin{pmatrix} a_0^\vee & a_1^\vee & a_2^\vee & a_3^\vee & a_4^\vee & a_5^\vee & a_6^\vee & a_7^\vee \\ & & & & & a_8^\vee & & \end{pmatrix}
=\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 4 & 2 \\ & & & & & 3 & & \end{pmatrix},$$

the only weight of level 1 is Λ_0, and there are three weights $2\Lambda_0,\Lambda_1,\Lambda_7$ of level 2. Moreover, since we have $h=g=\sum_{i=0}^{8}a_i^\vee=30$, it follows that $\dim E_8=8\cdot(30+1)=248$. So, for $m=1,2$, each z_m takes

$$z_1=\frac{1}{1+30}\cdot 248=8,\qquad z_2=\frac{2}{2+30}\cdot 248=15+\frac{1}{2},$$

and the central charge of the Virasoro algebra appearing in the decomposition of the tensor product of representations $L(\Lambda_0)\otimes L(\Lambda_0)$ is

$$2z_1-z_2=\frac{1}{2}.$$

h_Λ for weights of level 1 and 2 is

$$h_{\Lambda_0}=h_{2\Lambda_0}=0,\quad h_{\Lambda_1}=\frac{15}{16},\quad h_{\Lambda_7}=\frac{3}{2}.$$

Let us study the decomposition of this tensor product of representations. If we let v_0 be a highest weight vector of $L(\Lambda_0)$, then $v_0 \otimes v_0$ and $f_0 v_0 \otimes v_0 - v_0 \otimes f_0 v_0$ are singular vectors, and their weights are $2\Lambda_0$ and $2\Lambda_0 - \alpha_0 = \Lambda_1$, respectively. We have

$$\begin{aligned} 2h_{\Lambda_0} - h_{2\Lambda_0} &= 0 = h^{(1)}_{1,1}, \\ 2h_{\Lambda_0} - h_{\lambda_1} &= -\frac{15}{16} = \frac{1}{16} - 1 = h^{(1)}_{2,2} - 1, \end{aligned} \tag{6.58}$$

and since there are no other values of h of the discrete series for $z = \frac{1}{2}$ of the Virasoro algebra than those in (6.58), which differ from $2h_{\Lambda_0} - h_{2\Lambda_0}$ or $2h_{\Lambda_0} - h_{\lambda_1}$ by an integer, $L(2\Lambda_0)$ and $L(\Lambda_1)$ can appear only in the forms $L(2\Lambda_0) \otimes \mathfrak{V}(0, \frac{1}{2})$ and $L(\Lambda_1) \otimes \mathfrak{V}(\frac{1}{16}, \frac{1}{2})$. Here, the discrepancy -1 in (6.58) comes from that the fact that the first place where a singular vector of this weight appear is $2\Lambda_0 - \alpha_0$, and that $\langle d, \alpha_0 \rangle = 1$.

Let us see what $L(\Lambda_7)$ looks like. Since

$$2h_{\Lambda_0} - h_{\Lambda_7} = -\frac{3}{2} = \frac{1}{2} - 2 = h^{(1)}_{2,1} - 2,$$

if we denote the first place where a singular vector of this weight appears by $2\Lambda_0 - \alpha$, then $\alpha \in \sum_{i=0}^{8} \mathbb{Z}_{\geq 0} \alpha_i$ should satisfy the following conditions:

i. $\Lambda_7 = 2\Lambda_0 - \alpha$,
ii. $\langle \alpha, d \rangle = 2$, i.e., $\alpha = 2\alpha_0 + \sum_{i=1}^{8} m_i \alpha_i \quad (m_i \in \mathbb{Z}_{\geq 0})$.

For $E_8^{(1)}$, if we express $\sum_{i=0}^{8} m_i \alpha_i$ as

$$\begin{pmatrix} m_0 & m_1 & m_2 & m_3 & m_4 & m_5 & m_6 & m_7 \\ & & & & & m_8 & & \end{pmatrix},$$

then by the above conditions, α looks like

$$\begin{pmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 1 & 0 \\ & & & & & 1 & & \end{pmatrix}.$$

We denote the multiplicity of singular vectors in this weight space $(L(\Lambda_0) \otimes L(\Lambda_0))_{2\Lambda_0 - \Lambda_7}$ by a. What we can say about a at this stage is $a \geq 1$ only. So, we can write the decomposition of $L(\Lambda_0) \otimes L(\Lambda_0)$ as follows:

(6.59)

$$\begin{aligned} L(\Lambda_0) \otimes L(\Lambda_0) = &\left(L(2\Lambda_0) \otimes \mathfrak{V}(h^{(1)}_{1,1}, z^{(1)}) \right) \oplus \left(L(\Lambda_1) \otimes \mathfrak{V}(h^{(1)}_{2,2}, z^{(1)}) \right) \\ &\oplus \left(a \cdot L(\Lambda_7) \otimes \mathfrak{V}(h^{(1)}_{2,1}, z^{(1)}) \right). \end{aligned}$$

Let us prove that $a = 1$.

From (6.59) and the asymptotic behavior (6.57) of the characters of the Virasoro algebra for $m = 1$, the asymptotic behavior of the branching functions is

$$
\begin{aligned}
b_{2\Lambda_0}^{\Lambda_0\otimes\Lambda_0}(\tau) &= \Xi_{1,1}^{(1)}(\tau) \sim \frac{1}{2}\cdot e^{\frac{\pi i}{24\tau}}, \\
b_{\Lambda_1}^{\Lambda_0\otimes\Lambda_0}(\tau) &= \Xi_{2,2}^{(1)}(\tau) \sim \frac{1}{\sqrt{2}}\cdot e^{\frac{\pi i}{24\tau}}, \\
b_{\Lambda_7}^{\Lambda_0\otimes\Lambda_0}(\tau) &= a\cdot\Xi_{2,1}^{(1)}(\tau) \sim \frac{a}{2}\cdot e^{\frac{\pi i}{24\tau}}.
\end{aligned}
\tag{6.60}
$$

On the other hand, since the asymptotic behavior of $b_{\Lambda}^{\Lambda_0\otimes\Lambda_0}(\tau)$ $(\Lambda \in P_+^2)$ is given by

$$
b_{\Lambda}^{\Lambda_0\otimes\Lambda_0}(\tau) \sim a(\Lambda)e^{\frac{\pi i}{24\tau}}
\tag{6.61}
$$

by (1) of Proposition 6.11, one can see from (6.60) and (6.61) that $a(\Lambda)$ $(\Lambda \in P_+^2)$ is given by

$$
a(2\Lambda_0) = \frac{1}{2}, \quad a(\Lambda_1) = \frac{1}{\sqrt{2}}, \quad a(\Lambda_7) = \frac{a}{2}.
$$

So, by using $\sum_{\Lambda\in P_+^2} a(\Lambda)^2 = 1$, we conclude that $a = 1$. Thus we obtain the decomposition of $L(\Lambda_0)\otimes L(\Lambda_0)$ as follows:

(6.62)

$$
\begin{aligned}
L(\Lambda_0)\otimes L(\Lambda_0) = \Big(L(2\Lambda_0)\otimes \mathfrak{V}(h_{1,1}^{(1)}, z^{(1)})\Big) \oplus \Big(L(\Lambda_1)\otimes \mathfrak{V}(h_{2,2}^{(1)}, z^{(1)})\Big) \\
\oplus \Big(L(\Lambda_7)\otimes \mathfrak{V}(h_{2,1}^{(1)}, z^{(1)})\Big).
\end{aligned}
$$

In terms of their characters, it has the following form:

$$
(\mathrm{ch}'_{\Lambda_0})^2 = \mathrm{ch}'_{2\Lambda_0}\cdot\Xi_{1,1}^{(1)}(\tau) + \mathrm{ch}'_{\Lambda_7}\cdot\Xi_{2,1}^{(1)}(\tau) + \mathrm{ch}'_{\Lambda_1}\cdot\Xi_{2,2}^{(1)}(\tau).
\tag{6.63}
$$

Since ch'_{Λ_0} is invariant under the modular transformations, and the modular transformations of the characters of the Virasoro algebra are known, it is quite easy to compute the modular transformations of ch'_{Λ} $(\Lambda \in P_+^2)$ by using (6.63). We showed their modular transformations in §4.3. Some of the concrete values of $a(\lambda,\mu)$ shown in §4.3 are computed in this way.

We define the fusion coefficients $N^{(m)}_{\xi,\xi',\xi''}$ ($\xi,\xi',\xi'' \in \mathrm{VIR}^{(m)}$) of the Virasoro algebra by using the Verlinde formula as follows:

$$N^{(m)}_{\xi,\xi',\xi''} := \sum_{x \in \mathrm{VIR}^{(m)}} \frac{S^{(m)}_{\xi,x} S^{(m)}_{\xi',x} S^{(m)}_{\xi'',x}}{S^{(m)}_{(1,1),x}}.$$

Concerning this **fusion coefficient**, we have the next proposition:

PROPOSITION 6.17 (cf. Theorem 4.3 of [**FKW**]). *Set representatives of* ξ,ξ',ξ'' *in* $\widetilde{\mathrm{VIR}}^{(m)}$ *as* $(j+1,k+1), (j'+1,k'+1), (j''+1,k''+1)$. *The fusion coefficient* $N^{(m)}_{\xi,\xi',\xi''}$ *can be expressed by using that of the affine Lie algebra* $A_1^{(1)}$ *as follows:*

$$N^{(m)}_{\xi,\xi',\xi''} = \begin{cases} N_{\lambda_{m,j},\lambda_{m,j'},\lambda_{m,j''}} \cdot N_{\lambda_{m+1,k},\lambda_{m+1,k'},\lambda_{m+1,k''}} \\ \quad (\textit{if one can choose representatives of} \\ \quad \xi,\xi',\xi'' \textit{ satisfying} \\ \quad j+j'+j'' \in 2\mathbb{Z},\ k+k'+k'' \in 2\mathbb{Z}), \\ 0 \quad (\textit{otherwise}). \end{cases}$$

Since we know the fusion coefficients for $A_1^{(1)}$ from §5.1, by using them, the fusion coefficients of the Virasoro algebra can be concretely described as follows:

COROLLARY 6.18. *If we set representatives of* ξ,ξ',ξ'' *in* $\widetilde{\mathrm{VIR}}^{(m)}$ *as* $(j+1,k+1), (j'+1,k'+1), (j''+1,k''+1)$, *then the fusion coefficient* $N^{(m)}_{\xi,\xi',\xi''}$ *is given by the following formula:*

$$N^{(m)}_{\xi,\xi',\xi''} = \begin{cases} 1 \quad (\textit{if one can choose representatives of} \\ \quad \xi,\xi',\xi'' \textit{ satisfying} \\ \quad j+j'+j'', k+k'+k'' \in 2\mathbb{Z}, \\ \quad j+j'+j'' \le 2m, \\ \quad k+k'+k'' \le 2(m+1), \\ \quad |j'-j''| \le j \le j'+j'', \\ \quad |k'-k''| \le k \le k'+k''), \\ 0 \quad (\textit{otherwise}). \end{cases}$$

Note. There exists a similar formula for a fusion coefficient of the minimal series of the W-algebra associated to a symmetric affine Lie algebra (cf. [**FKW**]). □

The Virasoro algebra plays an important role in conformal field theories and vertex operator algebras.

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Further Developments

Those who scrape together a living day by day clear have no prospect of the future. All day and every day, they only manage to finish their jobs.

There are too many fields related to infinite-dimensional Lie algebras for me to be able to cover them all. Automorphic forms, combinatorial identities, link invariants, soliton equations, quantum groups, solvable lattice models in statistical physics, theory of superstrings in particle physics, conformal field theory, gravitational field theory, etc.- the related branches in both mathematics and physics are broad, and the course of the development of the representation theory of infinite-dimensional Lie algebras has been affected by all of them. Because of these systematic and intricately intertwined connections, the representation theory of infinite-dimensional Lie algebras has grown to include even deeper contents, has absorbed new lives constantly, and has maintained its freshness.

In fact, there are more researchers who are studying the representation theory of infinite-dimensional Lie algebras for such applications than there are those studying it as a representation theory. Such things were a motive for extending the representation theory of Lie algebras in the right direction. For example, it seems that conformal field theories cannot be discovered without affects from physics. Those who told us the importance of Virasoro algebra, Neveu-Schwarz algebra, Ramond algebra and vertex operator algebras are physicists. Even now, the researchers who are studying superalgebras 'practically' are physicists. The fact that physicists are interested in those things guarantees that it is a nice 'object' to study, and the author is delighted. We mathematicians can concentrate on inquiring deeply, on pursuing beauty, and on searching for deep significance without sticking to its applications. It is good for research on the representation theory of Lie algebras to be in such a favored environment. Without retiring into oneself, nature tells us what we should do.

There always is a symmetry underlying a physical phenomenon, and if we accept that non-artificial 'good mathematics' possesses a symmetry, as the universal principle, then the fundamental algebraic systems describing its symmetry might be groups or Lie algebras. Of course, they are not all of the symmetry. But even quantum groups are born in the framework of Lie algebras, and it seems that Lie algebras are still the mother of quantum groups. Other "mothers" of vertex operator algebras include the Monster simple group and infinite-dimensional Lie algebras.

In this way, the influence on groups and Lie algebras as algebraic systems describing symmetry must continue as long as the sciences try to elucidate nature, whether mathematical or physical. Thus, I think that the representation theory of Lie algebras will enlarge its contents and make sound development without bursting like bubbles or going bankrupt.

So, I cannot imagine the direction after now. Who on earth could have predicted the present situation 30 years ago, i.e., when Kac-Moody algebras appeared? Even 20 years ago?

Therefore, I will guess about the research subjects for only a few years from now, restricting myself to the framework of the representation theory of Lie algebras. It will be a highly biased point of view, owing to my narrow outlook. Concerning the representation theory, I feel that the following directions are possible:

1. The representation theory of simple finite-dimensional Lie superalgebras and of affine Lie superalgebras.
2. The structure and the representation theory of the W-algebra associated to an affine Lie algebra and an affine Lie superalgebra.
3. The representation theory of super-conformal algebras, and of their associated conformal superalgebras. Also the relations among them.
4. The representation theory of toroidal Lie algebras.
5. Their extension to quantum groups, and their applications.
6. Et cetera $\cdots$.

The representation theory of affine Lie superalgebras is, surprisingly, an unexplored field. Even the general cases of finite-dimensional representations of simple finite-dimensional Lie superalgebras are still open. The character formulae of finite-dimensional representations of

$\mathfrak{sl}(m|n)$ were completed only recently (cf. [**Serg1**]).[1] The representation theory of the affinization of this algebra is almost unknown, but it should be a nice object.

One may feel that for a Lie superalgebra, the only difference is that the bracket product becomes the super-bracket product, and many theories work as well as Lie algebras, but this is not true. There are big obstructions. Certainly, there are some technical difficulties, but once we overcome them, we can find completely different aspects. Therefore, this originates essentially in its interior.

First, an obstruction arises in the analysis of Verma modules, but this is not the only problem. For an affine Lie algebra, there is a modular invariance as we saw in this book, and the discrete series of Virasoro algebra played a role. But, for a Lie superalgebra, the unitarity does not hold even for an integrable representation. So we cannot apply the discrete series of Virasoro algebra. It seems that the modular invariance is also broken. Therefore, its representation theory eventually differs from the representation theory of affine Lie algebras.

Even for affine Lie superalgebras, there is a case when modular invariance necessarily appears. That is, the character formula of the trivial representation, i.e., the denominator identity. One side of the denominator identity is infinite products on roots, and it is a modular function that can be expressed as a product and a quotient of the Jacobi theta functions. The denominator identity is a formula expressing it by an infinite sum. If we do this in practice, we obtain something completely different from those for affine Lie algebras.

Let us see this by a simple example. Let us consider the denominator identities of affine Lie superalgebras of type $\widehat{A}(1,0) = \widehat{\mathfrak{sl}}(2,1)$ and $\widehat{B}(1,1) = \widehat{\mathfrak{osp}}(3,2)$ that are affinizations of $A(1,0) = \mathfrak{sl}(2,1)$ and $B(1,1) = \mathfrak{osp}(3,2)$ in Figure 1.2. We name the simple roots of $A(1,0)$ and $B(1,1)$ as in Figure 1.2, and denote a new root appearing by the affinization by α_0. Then, for each case, the fundamental imaginary root looks as follows:

$$\delta = \begin{cases} \alpha_0 + \alpha_1 + \alpha_2 & \text{for } \widehat{A}(1,0), \\ \alpha_0 + 2\alpha_1 + \alpha_2 & \text{for } \widehat{B}(1,1). \end{cases}$$

δ is an even root in each case.

[1] *Translator's note.* See [**Serg2**] for further developments.

First, when we set $u := e^{-\alpha_0}, v := e^{-\alpha_2}, q := e^{-\delta}$, the denominator identity of $\widehat{A}(1,0)$ becomes

$$(1)\qquad \prod_{n=1}^{\infty} \frac{(1-q^n)^2(1-uvq^{n-1})(1-(uv)^{-1}q^n)}{(1+uq^{n-1})(1+u^{-1}q^n)(1+vq^{n-1})(1+v^{-1}q^n)} = \sum_{m,n\geq 0}(-1)^{m+n}u^m v^n q^{mn} - \sum_{m,n<0}(-1)^{m+n}u^m v^n q^{mn}.$$

One can also find this formula in the old analysis text of Whittaker and Watson [**WW**]. The denominator identity for $\widehat{B}(1,1)$ is slightly more complicated, and by setting $u := e^{-\alpha_1}, v := e^{-\alpha_2}, q := e^{-\delta}$, it becomes

(2)

$$\prod_{n=1}^{\infty} \frac{(1-q^n)^2(1-vq^{n-1})(1-v^{-1}q^n)(1-(uv)^2q^{n-1})(1-(uv)^{-2}q^n)}{(1+uq^{n-1})(1+u^{-1}q^n)(1+uvq^{n-1})(1+(uv)^{-1}q^n)(1+uv^2q^{n-1})(1+u^{-1}v^{-2}q^n)}$$

$$= \left\{ \sum_{\substack{m,n\geq 0 \\ \text{s.t.} \\ m\equiv n+1 \bmod 2}} - \sum_{\substack{m,n<0 \\ \text{s.t.} \\ m\equiv n+1 \bmod 2}} \right\} (-1)^{\frac{m-n+1}{2}} u^{\frac{m+n-1}{2}} v^m q^{\frac{mn}{2}}.$$

According to [**Hic**], these are the Ramanujan mock theta functions! Then, one could call the denominator identity of affine Lie superalgebras a generalization of the mock theta functions.

As an application of (1), one can derive the modular transformations of the characters of the minimal series representations of the $N = 2$ super-conformal algebra (cf. [**KW3**]). And by using their modular transformations, one can compute the fusion algebra of the $N = 2$ super-conformal algebra. The reason why the denominator identity of $\widehat{A}(1,0)$ has such an application is that the $N = 2$ super-conformal algebra is the W-algebra of $\widehat{A}(1,0)$. The relations between an affine Lie superalgebra and its W-algebra, in particular between their representations, are very interesting research subjects after this. In the same way as I described the characters and their modular transformations of representations of Virasoro algebra by using $\widehat{\mathfrak{sl}}(2,\mathbb{C})$, one may possibly describe representations and their properties of the W-algebra by using the associated affine Lie superalgebra. For this purpose, one has to study representations and their properties for affine Lie superalgebras.

Similar to the case of Lie algebras and Lie superalgebras, there are some series of super-conformal algebras, and they are classified by Kac [**K1**]. Among them, the simplest are $N = 0$ (Virasoro), $N = 1$ (Neveu-Schwarz and Ramond), $N = 2$, $N = 3$, $N = 4$ super-conformal algebras,[2] and $N = 2$ super-conformal algebras related with the mirror symmetry, and they are mainly studied by algebraic geometers and physicists. From the viewpoint of the structure of $N = 2$ super-conformal algebra, this symmetry is an exchange of two odd fields. For $N = 3$ and $N = 4$, the even part of the super-conformal algebra is $\widehat{\mathfrak{sl}}(2, \mathbb{C})$ and is non-abelian. So their representation theory becomes more interesting and difficult, since they are related to the representation theory of the affine Lie algebra $\widehat{\mathfrak{sl}}(2, \mathbb{C})$. Including these, it is interesting to study the representation theory of super-conformal algebras, the modular transformations of their characters and the associated fusion algebras.

Moreover, the conformal superalgebra (CSA for short) has recently been discovered by Kac, and its definition is given in §2.7 of [**K5**]. Their representation theory has been started in [**CK**]. It is like a *matsutake* mushroom derived from a big tree called a vertex operator algebra, and it is a portable version of a super-conformal algebra and a vertex operator algebra. There is an experimental report saying that it is more delicious to munch a *matsutake* mushroom than its landlord i.e., a Japanese red pine.

Let us also munch it a bit. A portable version of Virasoro algebra is called Virasoro conformal algebra. Using the symbol L, this is the $\mathbb{C}[\partial]$-module $\mathbb{C}[\partial]L$ generated by L with the operation

$$L \underset{\lambda}{\cdot} L = (\partial + 2\lambda)L,$$

and this operation is an algebraic expression of the operator products in the corresponding vertex operator algebra. In the above formula, λ is a parameter, and the symbol $L \underset{\lambda}{\cdot} L$ is the generating function of $L_{(n)}L \in \mathbb{C}[\partial]L$ ($n \in \mathbb{Z}_{\geq 0}$). The representation theory of this algebra as a CSA, in particular the problem of extensions, can be very easily obtained (cf. [**CKW**]) by simply computing polynomials in two variables (regarding ∂ and λ as indeterminates). In other words, this is the cohomology of Virasoro algebra, and one can compare this with

[2] These names were not used yet in [**K1**]. $K_1, K_2 (\cong W_2), K_3$ in [**K1**] are $N = 1, N = 2, N = 3$ respectively; S_2' is $N = 4$; and W_0 is $N = 0$. N represents the number of odd fields whose conformal dimension is $\frac{3}{2}$.

the result stated in item F in Section 3 of Chapter 2 of [**Fu**]. Furthermore, the classification of differential Lie algebras (cf. [**Cas**]) can be very easily obtained by using CSA's.

The representation theory of CSA's is in the process of developing as a new branch of representation theory, under the name 'algebraic conformal field theory'. . It will be quite interesting to see further developments such as which sort of physical or mathematical applications they have and what are the relations with the so-called topological conformal field theory, etc.

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Among these references, let us give some remarks related to each chapter.

Chapter 1:
In [**YS**], representation theories of a group of rotations $SO(3)$ and its universal covering group $SU(2)$ are explained in detail. Going back to some years ago, when I was a student of physics, I was bothered by the words 'a representation of a group' or 'an infinitesimal transformation', which are not defined in a lecture for junior year, and by this book, I felt that the relation between a Lie group and the corresponding Lie algebra or representation theories of Lie algebras, at least for $SU(2)$ and $\mathfrak{sl}(2,\mathbb{C})$, became clear to me. It was a very impressive book in my student days.

When I could understand $\mathfrak{sl}(2,\mathbb{C})$, I wanted to know further theories. Then, the book that I found in a library which seemed easy even for me was [**Iw1**]. After explaining fundamental facts on semi-simple Lie algebras and their general properties, in *II*, for example, the Killing form for each simple Lie algebra is computed case by case, and it was my great pleasure to compute them along with the book. I was taken prisoner by the Lie algebras. But, even if it was interesting, there was nothing left to make a living in the world of simple finite-dimensional Lie algebras. In 1982, when I met with Kac-Moody Lie algebras, it reminded me of my deep emotion on reading these two books in my school days, and I burned with ardor. Even if it was simply a useless game at a stage, it might turn out to be useful. In this sense, although it is my personal impression, [**YS**], [**Iw1**] and [**Iw2**] might be the best introductory books to Lie algebras.[3] Even though they are old publications, their contents have never lost vitality. [**Po**] is also famous as an introductory book to a relevant topic.

Since I had a hard time in my seminar for senior year and master course with an old version of [**Hel**], it was especially memoried, and I think its Chapter 3 (Topics in Cartan subalgebras and root systems) is well-organized.

[**Bour2**] is a detailed exposition of root systems and Coxeter groups, and probably (if the reader does not dislike linear algebra) the reader may learn enough by reading only this volume. The list of root systems in its appendix is also useful.

There is an exposition of Lie superalgebras and related geometry in [**Man**].

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[3] *Translator's note.* [**Iw2**] and [**GW**] treat similar topics.

Concerning Kac-Moody Lie algebras, [**K4**] seems to be the best book both as an introductory book and as a side book. As an introductory book, the first edition of this book (Birkhäuser 1983) is easy to follow, and is rather suitable since it does not explain so many things. I was crazy about reading the manuscript of the first edition of this book. Kac-Moody Lie algebras are similar to and, at the same time, slightly different from finite-dimensional Lie algebras, but because of this small difference, we have such a new world, and I was greatly touched by several beautiful scenes developed in this book. I thought I would like to go on further than this book. This book was really impressive.

Chapter 2:

A proof of the character formula and the structure of the root system of a BKM Lie algebra is explained in §11.13 of [**K4**]. Superizing it, one may find the character formula and the root system of a Lie superalgebra given in an appendix of [**GN1**], but the theory itself can be developed in analogy to the case of Lie algebras, as I explained in Chapter 2 of this book. In particular, it seems that relations with modular forms will be developed further (see, e.g., [**Bor2**], [**Bor3**], [**Bor4**], [**GN1**] and [**GN2**]).

Chapter 3:

Concerning the representation theory of simple finite-dimensional Lie algebras, [**Hum**] is said to be a standard textbook, but it might depend upon the reader's taste. Is seems that [**FH**] is an accessible book. [**CSM**] consists of three chapters Lie algebras written by Carter, Lie groups by Segal and linear algebraic group by Macdonald and it is an well-arranged book with 190 pages. One can find a detailed explanation of the Shapovalov determinant in [**MP**].

Chapter 4:

[**Ser**] and [**Mum**] are readable introductory books on theta functions. [**Z**] is also a well-summarized exposition. It seems to me that [**KP**] is a monument to the representation theory of affine Lie algebras. It is in this paper where the modular invariance of the characters was discovered. For a continuation of this chapter, the reader may refer to Chapter 13 of [**K4**], or [**KW1**].

Chapter 5:

There are many papers on conformal field theories and on fusion algebras, and the published monographs, at the moment, are almost all written by physicists for physicists, and there are too many hard obstructions for me to understand their symbols and the way of descriptions. In such situations it seems that [**FMS**] is accessible also to mathematicians and explains conformal field theories and fusion algebras in detail. An outline of conformal field theories and their related topics is also given in [**Koh**] in this series of courses.

Chapter 6:

Let me introduce some references treating soliton equations which are not touched upon in this book. The articles where one discovered how to describe soliton equations by means of symmetry of affine Lie algebras are [**DJKM1**], [**DJKM2**] and [**DJKM3**]. In connection with that, [**I.Fr**] was also impressive. As an exposition in Japanese, I shall first recommend [**MJ**]. This is a vivid exposition which explains the theory discovered just then. In [**MJ**], after a report by Miwa and Jimbo, they unrolled a round-table talk with Mikio Sato; the excitement and the heated atmosphere when they discovered this magnificent theory are recorded as they were. Now, it has already been a decade, without growing old as we listen to Beethoven's music, it still makes a fresh impression on me when I reread it even now.

[**MJD**] is an exposition by those who discovered the transformation group of soliton equations. On the contrary, the author of [**J1**] did not touch upon affine Lie algebras, but stated the history and the background of soliton equations and related topics in other fields of mathematics. So, it seems more recondite to me.

The bridges that connect soliton equations and affine Lie algebras are bilinear differential operators of Hirota. In [**Hir**], I am pulled in by the skillful technique of treating several soliton equations used by the author. The phrase 'affine Lie algebras' does not appear in this book; it is quite interesting to turn over in the reader's mind through looking at several examples treated in this book.

Studies on hierarchies for toroidal Lie algebras have just started, and it seems they provide us with new types of soliton equations that are different from those for affine Lie algebras (cf. [**Bil**], [**ISW**]).

Representation theories of the Virasoro algebra are well-organized in [**KR**]. This is a document of twelve lectures given by Kac in December 1985 and January 1986 at the Tata Institute in India. It was exactly the period when research on the representation theory of the Virasoro algebra was begun from both the physical and the mathematical side, and was accelerating. When these lectures were given, Kac himself was also getting more and more excited about the relations between Virasoro algebra and affine Lie algebras. Although this book is small, it well depicts the heated atmosphere of that time.

There are some textbooks on vertex operator algebras, but since the ways to treat δ-function are different and hence the ways to describe real calculus, i.e. computation, differ, and they seem to be different at least at a glance, the reader may follow his or her own taste. I personally prefer [**FLM**] as a pioneer of vertex operator algebras. In [**K5**], it says that the essence of a vertex operator algebra is in its "locality", and the author constructs the theory from this point of view. Its description is clear, and it may tell us what a 'pleasure' it is to play with vertex operator algebras.

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