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Volume 197

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Sheaves and Cohomology

Kenji Ueno



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Algebraic Geometry 2
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Kenji Ueno

Translated by
Goro Kato



American Mathematical Society
Providence, Rhode Island

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ABSTRACT. This is the second of three books by the author aimed at introducing the reader to Grothendieck's scheme theory as a method of studying algebraic geometry. This book contains definitions and results related to coherent schemes, proper and projective morphisms, and cohomology of sheaves on schemes. As in the first book, the author includes many examples and problems illustrating the topics discussed in the main text.

The book is aimed at graduate and upper-level undergraduate students who want to learn modern algebraic geometry.

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Modern algebraic geometry is built upon two fundamental notions: schemes and sheaves. The theory of schemes is presented in the first part of this book (*Algebraic Geometry I: From Algebraic Varieties to Schemes*, AMS, 1999, Translations of Mathematical Monographs, Volume 185). In the present book, the author turns to the theory of sheaves and their cohomology. Loosely speaking, a sheaf is a way of keeping track of local information defined on a topological space, such as the local algebraic functions on an algebraic manifold or the local sections of a vector bundle. Sheaf cohomology is a primary tool in understanding sheaves and using them to study properties of the corresponding manifolds.

The text covers the important topics of the theory of sheaves on algebraic varieties, including types of sheaves and the fundamental operations on them, such as coherent and quasicoherent sheaves, direct and inverse images, behavior of sheaves under proper and projective morphisms, and Čech cohomology.

The book contains numerous problems and exercises with solutions. It would be an excellent text for the second part of a course in algebraic geometry.

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