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American Mathematical Society

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Translated by
Kinetsu Abe

American Mathematical Society
Providence, Rhode Island

幾何学的変分問題

KIKAGAKUTEKI HENBUN MONDAI

by Seiki Nishikawa

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Preface to the English Edition

This book, published originally in Japanese, is an outgrowth of lectures given at Tohoku University and at the 1995 Summer Graduate Program of the Institute for Mathematics and Its Applications, University of Minnesota. In these lectures, through a discussion on variational problems of the length and energy of curves and the energy of maps, I intended to guide the audience to the threshold of the field of geometric variational problems, that is, the study of nonlinear problems arising in geometry and topology from the point of view of global analysis.

It is my pleasure and privilege to express my deepest gratitude to Professor Kinetsu Abe who generously devoted considerable time and effort to the translation. I would also like to take this opportunity to express my deep appreciation to Professor Phillipe Tondeur who invited me to join the 1995 Summer Graduate Program, and to my friend Andrej Treibergs for making his notes [26] available to the organization of the last chapter.

Seiki Nishikawa

April 2001

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Preface

It is said that techniques for surveying were developed from the need to restore lands after frequent floods of the Nile River in ancient Egypt. Geometry is the area of mathematics whose name originates from this method of surveying; namely, "to measure lands" (geo = lands, metry = measure). As such, it is an ancient practice to study figures from the view of practical applications. It is also said that the ancient Greeks already knew of the method of indirect surveying using the congruence conditions of triangles.

A minimal length curve joining two points in a surface is called a geodesic. One may trace the origin of the problem of finding geodesics back to the birth of calculus.

Many contemporary mathematical problems, as in the case of geodesics, may be formulated as variational problems in surfaces or in the more generalized form of manifolds. One may characterize the geometric variational problems as a field of mathematics that studies the global aspects of variational problems relevant in the geometry and toplogy of manifolds. For example, the problem of finding a surface of minimal area spanning a given frame of wire originally appeared as a mathematical model for soap films. It has also been actively investigated as a geometric variational problem. With recent developments in computer graphics, totally new aspects of the study on the subject have begun to emerge.

This book is intended to be an introduction to some of the fundamental questions and results in geometric variational problems, studying the variational problems on the length of curves and the energy of maps.

The first two chapters approach variational problems of length and energy of curves in Riemannian manifolds with an in-depth discussion of the existence and properties of geodesics viewed as the solution to variational problems. In addition, a special emphasis is

placed on the fact that the concepts of connection and covariant differentiation are naturally induced from the first variation formula of this variational problem, and that the notion of curvature is obtained from the second variational formula.

The last two chapters treat the variational problem on the energy of maps between two Riemannian manifolds and its solutions, namely harmonic maps. The concept of harmonic maps includes geodesics and minimal submanifolds as examples. Its existence and properties have successfully been applied to various problems in geometry and topology. This book takes up the existence theorem of Eells-Sampson, which is considered to be the most fundamental among existence theorems for harmonic maps. The proof uses the inverse function theorem for Banach spaces. It is presented to be as self-contained as possible for easy reading.

Each chapter of this book may be read independently with minimal preparation for covariant differentiation and curvature on manifolds. The first two chapters, through the discussion of connections and covariant differentiation, are designed to provide the reader with a basic knowledge of Riemannian manifolds. As prerequisites for reading this book, the author assumes a few elementary facts in the theory of manifolds and functional analysis. They are included in the form of appendices at the end of the book. Details in functional analysis may be skipped. The reader, however, is encouraged to do the exercise problems at the end of each chapter by himself or herself first. The solutions may be consulted if necessary, since many of the exercise problems complement the contents of the book.

This book is an outgrowth of lectures delivered at Tohoku University and the 1995 Summer Graduate Programs held at The Institute for Mathematics and Its Applications, University of Minnesota. The first half of the book aims at a junior and senior level, and the last half at a first and second year graduate level. Each half roughly consists of the amount of topics that may be covered in one semester. In the actual lectures, the author also discusses the harmonic maps between Riemann surfaces. This portion is not included in this book due to the limited space. The reader who is interested in the study of harmonic maps is advised to first study the harmonic maps between Riemann surfaces.

It would be this author's wish as well as pleasure if this book could interest many readers in variational problems in geometry.

Last but not least, the author expresses his sincere gratitude to the editorial staff of Iwanami Shoten for their valuable help in the publication of this book.

Seiki Nishikawa

December 1997

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Outlines and Objectives of the Theory

Among geometric variational problems, the extreme value problem regarding the length of curves is as old as those in calculus. Chapter 1 of this book is devoted to discussions of variational problems of curves in manifolds. As is well known, the length of a curve joining two points in a plane is given by integrating the magnitude of tangent vectors. Similarly, one can define the length and energy for curves in more general Riemannian manifolds by measuring the magnitude of the tangent vectors using Riemannian metrics.

In Chapter 1, Euler's equation is calculated. It characterizes the critical points of the length and energy of curves when they are considered as functionals defined in the space of curves. Consequently, the equation of geodesics is obtained. The concepts of connections and covariant differentiation are naturally induced from the equation of geodesics in a manifold. Covariant differentiation, an essential tool for studying variational problems in manifolds, is an operation that defines the derivative of a vector field by a vector field in a manifold.

The most fundamental connection, called the Levi-Civita connection, is uniquely determined in a manifold equipped with a Riemannian metric, i.e., a Riemannian manifold. The notion of parallel transport is induced from this connection. The discovery of the notion of parallel transport in Riemannian manifolds (1917) and Einstein's use of geometry based on a four-dimensional indefinite metric for his general relativity (1916) greatly advanced the study of Riemannian geometry.

Geodesics in Riemannian manifolds correspond to straight lines in the plane and they are locally characterized as the curves of minimal length between points. One can construct a special local coordinate system, called a normal coordinate system, using these minimal geodesics about each point in a Riemannian manifold. Parallel transport and normal coordinate systems are the most basic tools in

comparing the geometry of a Riemannian manifold with the geometry of a model space (for example, Euclidean space).

In Chapter 2, using covariant differentiation, the first variation formula (Euler's equation) for the variational problem regarding the energy of curves in Riemannian manifolds is computed in the general case where the image of a curve is not always contained in a local coordinate neighborhood. The second variation formula is subsequently computed. Just as the notion of connections is derived from the first variation formula, it is seen that the second variation formula possesses an intimate relationship to the notion of curvature in Riemannian manifolds. In other words, the notions of curvature tensor and the curvature of a Riemannian manifold are naturally induced from the second variation formula for the energy of curves.

Given two points in a Riemannian manifold, the distance between these two points is given by the least upper bound of the lengths of piecewise smooth curves connecting them. Whether a Riemannian manifold becomes a complete metric space with respect to this distance is an important question. It was relatively recently (1931) that Hopf-Rinow gave necessary and sufficient conditions for the question. The results by Hopf and Rinow are significant not only in making the notion of completeness succinct, but also in showing that this completeness is the condition that guarantees the existence of a minimal geodesic joining two given points.

As stated above, the second variation formula for the energy of curves is closely related to the curvature of Riemannian manifolds. Using this, one can study the effects of the curvature of a Riemannian manifold on its topological structure. Myers' theorem and Synge's theorem are discussed as typical examples of such applications. The former states that the fundamental group of a compact and connected, Riemannian manifold of positive curvature is a finite group, and the latter states that an even-dimensional compact, connected and orientable Riemannian manifold of positive curvature is simply connected. Research on Riemannian manifolds using existence and properties of geodesics is being actively pursued.

In Chapter 3, harmonic maps and the energy of maps are discussed. They generalize the variational problem of the energy of curves in Riemannian manifolds. Namely, a functional called the energy of maps is defined in the mapping space consisting of smooth maps between Riemannian manifolds, and harmonic maps given as its

critical points are investigated. The energy of maps is a natural generalization of the energy of curves. Examples of harmonic maps appear in various aspects of differential geometry. Harmonic functions, geodesics, minimal submanifolds, isometric maps, and holomorphic maps are a few typical examples.

The first variation formula, which characterizes the critical points of the energy functional, can be obtained by essentially the same approach as in the case of geodesics. However, the computations become unnecessarily complicated and only yield results of a local nature without use of the covariant differentiation that is naturally induced from the Levi-Civita connection of Riemannian manifolds. To alleviate these difficulties, it is designed in this chapter to derive, through discoveries in the process, the computational rules for the covariant differentiation that is induced from the Levi-Civita connection in tangent bundles and their tensor products over Riemannian manifolds. This route may not be the most direct one, but the author believes that it is more effective in familiarizing the reader with the definition and the rules of computations for covariant differentiation than the axiomatic approach. At first, the reader may feel uneasy, especially about the portion of the induced connections. Nonetheless, actual computations help promote understanding of the notion. The fastest way to grasp the rules of computation involving covariant differentiation is actually to engage in the computations. The computations of the first variation formula for the energy functional of maps yield a vector field called the tension field. It is given as the trace of the second fundamental form of the maps. A harmonic map is then characterized as a map whose tension field is identically 0.

Chapter 4 is devoted to the existence problem of harmonic maps between compact Riemannian manifolds. Whether or not a given map is homotopically deformable to a harmonic map is one of the most fundamental questions among geometric variational problems. It may be regarded as a generalization of the existence problem of closed geodesics. To this end, the "heat flow method" is first introduced. This is an effective technique for deforming a given map to a harmonic map. Then, using this technique, it is proved that any map from a compact Riemannian manifold M into a compact Riemannian manifold N of nonpositive curvature is free homotopically deformable to a harmonic map. This theorem was first proved by Eells-Sampson in 1964.

The proof of this theorem using the heat flow method first requires the existence of a time-dependent solution to an initial value problem with any initial map of the parabolic equation for harmonic maps. The original proof uses successive approximations to construct a solution after converting the problem to a problem of integral equations via the fundamental solution of the heat equation. In this book, the solution is constructed through use of the inverse function theorem in Banach spaces in an effort to minimize the amount of preparation.

The existence of time-dependent local solutions is always guaranteed, but the existence of global time-dependent solutions is not self-evident, since the parabolic equation for harmonic maps is nonlinear. In fact, proving the existence of global time-dependent solutions entails some estimates of the growth rate of solutions in time. The curvature of the Riemannian manifold N plays a crucial role in estimating the influence of nonlinear terms. An estimation formula that guarantees the existence and convergence of time-dependent global solutions is obtained using the Weitzenböck formula for the heat operator under the condition that N is of nonpositive curvature.

The Weizenböck formula, in general, gives the relationship between second order partial differential operators naturally acting on tensor fields on Riemannian manifolds and the Laplace or heat operator acting on functions. It is revealed that the Riemann curvature and its Ricci identity play essential roles for existence of solutions to those differential operators. In this chapter, an a priori estimate regarding the growth rate of solutions is obtained using the Weizenböck formula for the energy density of solutions to the parabolic equation for harmonic maps and the heat operator. This idea is originally due to Bochner. It has become an effective and fundamental technique for the proofs of theorems such as the Kodaira vanishing theorem and more recently in gauge theory.

As in the case of geodesics, one can also investigate the structures of Riemannian manifolds using the existence and properties of harmonic maps. The theorem of Preissman, one of the typical applications of harmonic maps, is discussed. The theorem states that a nontrivial Abelian subgroup of the fundamental group of a compact manifold of negative curvature is infinitely cyclic. The research of Riemannian manifolds using the existence and properties of harmonic maps seems to possess a promising future. For example, new proofs from a more analytical point of view for the topological sphere theorem and the Frankel conjecture were recently given by exploiting

the existence theorem of harmonic spheres due to Sacks and Uhlenbeck. A strong rigidity theorem regarding complex structures in Kähler manifolds of negative curvature was also obtained using the existence theorem of Eells and Sampson.

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Solutions to Exercise Problems

Chapter 1

1.1 Easily follows from the definition.

1.2 Let $\{V_\beta\}$ be an open cover of M such that each V_β is a local coordinate neighborhood of M and $\overline{V}_\beta$ is compact. Since M with the second axiom of countability is paracompact, there are a locally finite refinement $\{U_\alpha\}$ and a partition of unity $\{f_\alpha\}$ subordinate to $\{U_\alpha\}$. Since U_α can be identified with an open subset of $\mathbb{R}^m$ through the local coordinate system, it has a Riemannian metric g^α. At each point $x \in M$, define an inner product g_x in T_xM by

$$g_x(v,w) = \sum{}' f_\alpha(x) g_x^\alpha(v,w), \quad v, w \in T_xM.$$

Here, $\sum'$ represents the sum over α with $x \in U_\alpha$. Then $g = \{g_x\}$ is the desired Riemannian metric. There is another way to prove it. Note the theorem of Whitney that states that any m-dimensional C^∞ manifold can be imbedded in $2m+1$-dimensional Euclidean space. Then follow Example 1.2.

1.3 Do in the same manner as deriving (1.14) from (1.13).

1.4 (1) The transformation formula of the natural frame fields

$$\frac{\partial}{\partial \bar{x}^j} = \sum_{q=1}^m \frac{\partial x^q}{\partial \bar{x}^j} \frac{\partial}{\partial x^q}, \qquad \frac{\partial}{\partial \bar{x}^k} = \sum_{r=1}^m \frac{\partial x^r}{\partial \bar{x}^k} \frac{\partial}{\partial x^r}$$

and (1.23) yield

$$\nabla_{\frac{\partial}{\partial \bar{x}^j}} \frac{\partial}{\partial \bar{x}^k} = \sum_{p=1}^m \left\{ \sum_{q=1}^m \frac{\partial x^q}{\partial \bar{x}^j} \left(\frac{\partial^2 x^p}{\partial x^q \partial \bar{x}^k} + \sum_{q=1}^m \Gamma_{qr}^p \frac{\partial x^r}{\partial \bar{x}^k} \right) \right\} \frac{\partial}{\partial \bar{x}^p}.$$

Substitute the following transformation formula in the above equation

$$\frac{\partial}{\partial x^p} = \sum_{i=1}^m \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial}{\partial \bar{x}^i}$$

and compare it with (1.22).

(2) Define $\nabla_X Y$ by (1.23) in each coordinate neighborhood. Then we can patch them together using the transformation formulas in (1); consequently, a linear connection ∇ is obtained in M.

1.5 (1) For the second equation, we replace X, Y, Z in $[X, [Y, Z]] = [X, YZ - ZY] = XYZ - XZY - YZX + ZYX$ in the order $X \to Y \to Z \to X$.

(2) From the first equation in (1), we see that the Lie derivative $L_X Y$ satisfies the following rules:

(i) $L_X(Y + Z) = L_X Y + L_X Z$,

(ii) $L_X(fY) = X(f)Y + fL_X Y$,

(iii) $L_{X+Y} Z = L_X Z + L_Y Z$,

(iv) $L_{fX} Y = fL_X Y - (Yf)X$.

Among them, (iv) is different from the rule $\nabla_{fX} Y = f\nabla_X Y$ for covariant differentiation. This implies that the value $L_X Y(x)$ of the Lie derivative is essentially determined by the behavior of both X and Y around x, unlike the covariant derivative, where the value $\nabla_X Y$ is determined by the behavior of Y around x and the value $X(x)$ of X at x. In other words, Lie differentiation is not an operator determined by the direction $X(x)$ of differentiation alone.

Furthermore, the second equation in (1) yields a relation $L_X L_Y - L_Y L_X = L_{[X,Y]}$, but the relation $\nabla_X \nabla_Y - \nabla_Y \nabla_X = \nabla_{[X,Y]}$ does not generally hold for covariant differentiation. This last relation holds only when the Riemannian manifolds are flat (curvature tensor $R = 0$) (see §2.1).

1.6 Let $\{e_1, \dots, e_m\}$ be a base for $T_x M$. Set $E_i(t) = P_t e_i$ $(1 \le i \le m)$. Since $\{E_1(t), \dots, E_1(t)\}$ is a base for $T_{c(t)} M$ for each t, we can write as

$$Y_{c(t)} = \sum_{i=1}^{m} Y^i(t) E_i(t), \quad Y^i \in C^\infty([0, l]).$$

Then, from the properties of linear connections and E_i being parallel, we get

$$\nabla_v Y = \sum_{i=1}^{m} \frac{dY^i}{dt}(0) e_i = \frac{d}{dt}\left(\sum_{i=1}^{m} Y^i(t) e_i(t)\right)\Bigg|_{t=0} = \frac{d}{dt} P_t^{-1} Y_{c(t)}\Bigg|_{t=0}.$$

1.7 If we assume that there is a vector field Φ as stated in the problem, we see that its integral curves $\varphi(t) = (c(t), c'(t))$ are solutions to (1.30) and, hence, unique. As for its existence, we may locally define the vector field Φ by (1.30). From the uniqueness, Φ is defined globally.

1.8 (1) Necessity is clear from the definition. Sufficiency follows from the uniqueness of geodesics regarding the initial condition. (2) follows readily from (1).

1.9 As is well known, a connected C^∞ manifold is arc-wise connected. Hence, there is a continuous curve $c : [a, b] \to M$ joining p and q. Cover the compact set $c([a, b]) \subset M$ by a finite number of coordinate neighborhoods U_α, then replace c by a C^∞ curve in each U_α.

1.10 Given $p \in M$, for sufficiently small ϵ, $\exp_p B_\epsilon(0) = \{q \in M \mid d(p, q) < \epsilon\}$ holds, where $B_\epsilon(0) = \{v \in T_pM \mid |v| < \epsilon\}$. $\exp_p$ is a local homeomorphism and $\exp_p B_\epsilon(0)$'s form a base for the local neighborhood system at p. Consequently, the topology defined by d coincides with the topology of the manifold.

Chapter 2

2.1 This follows readily from the definition.

2.2 A type $(1, s)$ tensor field

$$T = \sum T^j_{i_1 \cdots i_s} (dx^{i_1}) \otimes \cdots \otimes (dx^{i_s})_x \otimes \left(\frac{\partial}{\partial x^j}\right)$$

on T_pM corresponds to an s linear map

$$T\left(\left(\frac{\partial}{\partial x^{i_1}}\right)_x, \ldots, \left(\frac{\partial}{\partial x^{i_s}}\right)_x\right) = \sum_j T^j_{i_1 \cdots i_s} \left(\frac{\partial}{\partial x^j}\right)_x$$

from $T_xM \times \cdots \times T_xM$ into T_xM.

2.3 This follows because, in general, $\nabla(X, fY) = (Xf)Y + f\nabla_X Y \neq f\nabla(X, Y)$.

2.4 Since $[\partial/\partial x^i, \partial/\partial x^j] = 0$, from the definition, we have

$$\begin{aligned}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \frac{\partial}{\partial x^k} &= \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} - \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k} \\ &= \sum_{l=1}^m \left\{ \frac{\partial}{\partial x^i} \Gamma^l_{jk} - \frac{\partial}{\partial x^j} \Gamma^l_{ik} + \sum_{r=1}^m (\Gamma^l_{ir} \Gamma^r_{jk} - \Gamma^l_{jr} \Gamma^r_{ik}) \right\} \frac{\partial}{\partial x^l}.\end{aligned}$$

2.5 It can be done readily by noting the properties of R in Proposition 2.11 and that the orthonormal basis $\{v', w'\}$ for σ is given by the Gram-Schmidt orthonormalization as

$$v' = \frac{v}{|v|}, \qquad w' = \frac{w - g_x(v, w)v'}{|w - g_x(v, w)v'|}.$$

2.6 It suffices to be able to determine $R(u,v,w,t)$, $\forall u,v,w,t \in T_xM$, given $R(u,v,v,u)$, $\forall u,v \in T_xM$. First of all, from the relation

$$R(u+t,v,v,u+t) = R(u,v,v,u) + R(t,v,v,t) + 2R(u,v,v,t)$$

follows that $R(u,v,v,t)$ is also determined. From this and the relation

$$\begin{aligned} R(u,v+w,v+w,t) = R(u,v,v,t) &+ R(u,w,w,t) \\ &+ R(u,v,w,t) + R(u,w,v,t), \end{aligned}$$

we have

$$R(u,v,w,t) + R(u,w,v,t) = (*),$$

where $(*)$ is the sum of known terms. By applying the first Bianchi identity to the second term, we get

$$2R(u,v,w,t) - R(w,v,u,t) = (*).$$

By exchanging u and w here, we also get

$$2R(w,v,u,t) - R(u,v,w,t) = (*).$$

These two equations imply that $R(u,v,w,t) = (*)$; hence, $R(u,v,w,t)$ is determined.

2.7 Consider a C^∞ map $u : O \to M$ with $u(0,s) = u(0,0)$ $(-\epsilon < s < \epsilon)$, where O is an open subset of $\mathbb{R}^2$ given by

$$O = \{(t,s) \mid -\epsilon < t < 1+\epsilon,\ -\epsilon < s < \epsilon\}\ (\epsilon > 0).$$

Given $v \in T_xM$, define a C^∞ vector field V along u to satisfy that $V(0,s) = v$ and $V(t,s)$ is the parallel transport of v along each C^∞ curve $t \mapsto u(t,s)$ for $t \neq 0$. Then by Lemma 2.15, we get

$$\frac{D}{\partial s}\frac{D}{\partial t}V = 0 = \frac{D}{\partial t}\frac{D}{\partial s}V + R\left(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}\right)V.$$

Since the parallel transport does not depend on the choice of the curves from the assumption, $V(1,s)$ equals the parallel transport of $V(1,0)$ along the C^∞ curve $s \mapsto u(1,s)$; hence, $\dfrac{D}{\partial s}V(1,0) = 0$. Consequently, from the above equation, we get

$$R\left(\frac{\partial u}{\partial s}(1,0), \frac{\partial u}{\partial t}(1,0)\right)V(1,0) = 0.$$

Since u and v are arbitrarily chosen, we get the desired conclusion.

2.8 Set $'\nabla_X Y = d\varphi^{-1}(\nabla'_{d\varphi(X)} d\varphi(Y))$. Then $'\nabla$ defines a linear connection in M and satisfies conditions (i) and (ii) in Theorem 1.12. From the uniqueness of the Levi-Civita connections, we see that $'\nabla = \nabla$. This implies (1). (2) readily follows from (1).

2.9 For (1), define $\tilde{g} = \bar{\omega}^* g$. (2) readily follows from (2) of Problem 2.8 above.

2.10 If M is orientable, Theorem 2.26 implies that M is simply connected. If M is not orientable, apply Theorem 2.26 to the orientable double covering space $\tilde{M}$.

Chapter 3

3.1 Since the support of f is compact, the right hand side is a finite sum. Let $\{V_\beta, \psi_\beta\}_{\beta \in B}$ be another coordinate neighborhood system and denote by $\{\sigma_\beta\}_{\beta \in B}$ a partition of unity. Then at $p \in U_\alpha \cap V_\beta$, we have

$$\sqrt{\det(g^\beta_{kl})}(p) = |\det J(\phi_\alpha \circ \psi_\beta^{-1})(\psi_\beta(p))| \sqrt{\det(g^\alpha_{ij})}(p).$$

The change of variable formula for the integral in $\mathbb{R}^m$ then readily implies

$$\begin{aligned}
&\sum_\beta \int_{\psi_\beta(V_\beta)} \left(\sigma_\beta f \sqrt{\det(g^\beta_{kl})}\right) \circ \psi_\beta^{-1} dx^1_\beta \cdots dx^m_\beta \\
&\quad = \sum_{\alpha,\beta} \int_{\psi_\beta(V_\beta \cap U_\alpha)} \left(\sigma_\beta \rho_\alpha f \sqrt{\det(g^\beta_{kl})}\right) \circ \psi_\beta^{-1} dx^1_\beta \cdots dx^m_\beta \\
&\quad = \sum_{\alpha,\beta} \int_{\psi_\beta(V_\beta \cap U_\alpha)} \left(\sigma_\beta \rho_\alpha f \sqrt{\det(g^\beta_{kl})}\right) \\
&\qquad\qquad \circ \psi_\beta^{-1} |\det J(\phi_\alpha \circ \psi_\beta^{-1})| dx^1_\beta \cdots dx^m_\beta \\
&\quad = \sum_{\alpha,\beta} \int_{\phi_\alpha(U_\beta \cap U_\alpha)} \left(\sigma_\beta \rho_\alpha f \sqrt{\det(g^\alpha_{ij})}\right) \circ \phi_\alpha^{-1} dx^1_\alpha \cdots dx^m_\alpha \\
&\quad = \sum_\alpha \int_{\phi_\alpha(U_\beta)} \left(\rho_\alpha f \sqrt{\det(g^\alpha_{ij})}\right) \circ \phi_\alpha^{-1} dx^1_\alpha \cdots dx^m_\alpha.
\end{aligned}$$

That μ_g is positive definite, namely, $\mu_g(f) \geq 0$ for nonnegative functions f, is clear from the definition. In order to see μ_g being a Radon measure on M, namely, being a bounded linear functional in $C_0(M)$, we must show the following. Given an arbitrary continuous function f whose support is a compact subset K, there is a constant c_K such that

$$|\mu_g(f)| \leq c_K \sup_{p \in M} |f(p)|$$

holds. This is also clear from the definition.

3.2 Choose a coordinate neighborhood U of $x \in M$ and a coordinate neighborhood V of $u(x) \in N$ so that $u(U) \subset V$ holds. Denote by (y^α) a local coordinate system in V. Since $\{(\partial/\partial y^\alpha) \circ u\}$ $(1 \leq \alpha \leq n)$ forms a basis for the fiber $T_{u(x)}N$ over x in $u^{-1}TN$, a section η of $u^{-1}TN$ over U is expressed as $\eta = \sum_\alpha \eta^\alpha (\partial/\partial y^\alpha) \circ u$. Assume that a linear connection $'\nabla$ in the problem exists. Then, from the properties of a connection, we must have

$$'\nabla_v \eta = \sum_\alpha \left\{ v(\eta^\alpha) \frac{\partial}{\partial y^\alpha} \circ u + \eta^\alpha \nabla'_{du_x(v)} \frac{\partial}{\partial y^\alpha} \right\},$$

and, hence, is unique.

As for the existence, we may define $'\nabla$ by the above equation. It is readily verified that $'\nabla$, independent of the choice of coordinate systems, defines a linear connection.

3.3 Each can be verified directly from the definitions. For (2), see the solution to Problem 3.4 below.

3.4 Let $\{\varphi_t\}$ be the one-parameter group of transformations generated by X. Denote by (x^i) a local coordinate system in the coordinate neighborhood (U, ϕ). The measure determined by the pullback $\varphi_t^* g$ of g under φ_t is given as

$$d\mu_{\varphi_t^* g} = \sqrt{\det \left(\sum_{k,l} g_{kl} \circ \varphi_t \frac{\partial \varphi_t^k}{\partial x^i} \varphi_t \frac{\partial \varphi_t^l}{\partial x^j} \right)} \, dx^1 \cdots dx^m.$$

Hence, we have

$$\frac{d}{dt}\bigg|_{t=0} d\mu_t^* g = \operatorname{div} X \cdot d\mu_g.$$

In fact, when $(a_{ij})(t)$ is a differentiable nonsingular matrix with respect to t, the derivative of the determinant is given by

$$\frac{d}{dt} \det(a_{ij}(t)) = \det(a_{ij}(t)) \sum_{k,l} a^{kl}(t) \frac{d}{dt} a_{kl}(t),$$

where $(a^{ij}(t))$ denotes the inverse matrix of $(a_{ij}(t))$. Furthermore, since $d/dt|_{t=0} (\partial \phi_t^k / \partial x^i) = \partial X^k / \partial x^i$ and $\partial \varphi_0^k / \partial x^i = \delta_i^k$ for

$X = \sum_i X^i \partial/\partial x^i$, we have

$$\begin{aligned}
\text{L.H.S.} &= \left\{ \frac{X \det(g_{ij})}{2\sqrt{\det(g_{ij})}} + \sqrt{\det(g_{ij})} \frac{d}{dt}\Big|_{t=0} \det\left(\frac{\partial \varphi_t^k}{\partial x^i}\right) \right\} dx^1 \cdots dx^m \\
&= \left\{ \frac{1}{2} \sum_{k,l} g^{kl} \left(\sum_i X^i \frac{\partial g_{kl}}{\partial x^i} \right) + \sum_i \frac{\partial X^i}{\partial x^i} \right\} \sqrt{\det(g_{ij})}\, dx^1 \cdots dx^m \\
&= \left(\sum_i \nabla_i X^i \right) \sqrt{\det(g_{ij})}\, dx^1 \cdots dx^m \\
&= \text{R.H.S.}
\end{aligned}$$

Since each φ_t is a diffeomorphism, we see readily from definition that

$$\int_M d\mu_{\phi_t^* g} = \int_M d\mu_g.$$

Consequently, by adding the local forms of the above result using the partition of unity, we get

$$0 = \frac{d}{dt}\Big|_{t=0} \int_M d\mu_{\phi_t^* g} = \int_M \operatorname{div} X d\mu_g.$$

3.5 (1) follows readily from the definition. To see that ∇T becomes an $(r, s+1)$ tensor, it, for instance, suffices to note that ∇T becomes C^∞-linear with respect to C^∞ modules $\Gamma(TM)$, $\Gamma(TM^*)$; namely, the following holds

$$\begin{aligned}
&\nabla T(fX, f_1X_1, \dots, f_sX_s, h_1w_1, \dots, h_rw_r) \\
&\quad = ff_1 \cdots f_s h_1 \cdots h_r \nabla T(X, X_1, \dots, X_s, w_1, \dots, w_r).
\end{aligned}$$

(2) follows from a similar computation to Lemma 3.4, noting (3.16), (3.26).

3.6 From Problem 3.5 above, ∇R is defined by

$$\begin{aligned}
\nabla R(X, Y, Z, V, w) = X \cdot R(Y, Z, V, w) &- R(\nabla_X Y, Z, V, w) \\
&- R(Y, \nabla_X Z, V, w) - R(Y, Z, \nabla_X V, w) \\
&- R(Y, Z, V, \nabla_X^* w).
\end{aligned}$$

Noting that $R(Y, Z, V, w) = w(R(Y, Z)V)$ and $\nabla_X^* w(Y) = Xw(Y) - w(\nabla_X Y)$, we get

$$\begin{aligned}
\nabla R(X, Y, Z, V, w) = w(&\nabla_X \cdot R(Y, Z)V - R(\nabla_X Y, Z)V \\
&- R(Y, \nabla_X Z)V - R(Y, Z)\nabla_X V).
\end{aligned}$$

Hence, it suffices to see that the symmetric sum over X, Y, Z of the expression

$$\begin{aligned}\nabla_X \cdot R(Y,Z)V - R(\nabla_X Y, Z)V - R(Y, \nabla_X Z)V - R(Y,Z)\nabla_X V \\ = [\nabla_X, R(Y,Z)]V - R(\nabla_X Y, Z)V - R(Y, \nabla_X Z)V\end{aligned}$$

equals 0. Here, noting that $\nabla_X Y - \nabla_Y X = [X,Y]$ and $R(X,Y)Z = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, we get

$$\begin{aligned}&[\nabla_X, R(Y,Z)]V - R(\nabla_X Y, Z)V - R(Y, \nabla_X Z)V \\ &\quad + [\nabla_Y, R(Z,X)]V - R(\nabla_Y Z, X)V - R(Z, \nabla_Y X)V \\ &\quad + [\nabla_Z, R(X,Y)]V - R(\nabla_Z X, Y)V - R(X, \nabla_Z Y)V \\ &= [\nabla_X, R(Y,Z)]V + [\nabla_Y, R(Z,X)]V + [\nabla_Z, R(X,Y)]V \\ &\quad - R([X,Y],Z)V - R([Y,Z],X)V - R([Z,X],Y)V \\ &= ([\nabla_X, [\nabla_Y, \nabla_Z]] + [\nabla_Y, [\nabla_Z, \nabla_X]] + [\nabla_Z, [\nabla_X, \nabla_Y]])V \\ &\quad + (\nabla_{[[X,Y],Z]} + \nabla_{[[Y,Z],X]} + \nabla_{[[Z,X],Y]})V.\end{aligned}$$

Hence, we get the desired conclusion from the Jacobi identity for the operator and the vector fields.

3.7 (1), (2) follow readily from the definitions.

(3) Apply the definition (1.26) of the Levi-Civita connection together with (1), (2).

3.8 One may verify that ϕ satisfies the equation for harmonic maps by a direct computation. One can also show that ϕ is a harmonic map from Proposition 3.17 by noting that $S^3 \to S^2$ is a Riemannian submersion and that $\phi^{-1}(y)$ is a geodesic (a great circle) of S^3 for each $y \in S^2$.

3.9 Since $\varphi^* = e^{2\rho} g$,

$$\sum_{k,l} g_{kl}(\varphi) \frac{\partial \varphi^k}{\partial x^i} \frac{\partial \varphi^l}{\partial x^j} = e^{2\rho} g_{ij}$$

and

$$\sum_{i,j} g_{ij}(\varphi) \frac{\partial \varphi^k}{\partial x^i} \frac{\partial \varphi^l}{\partial x^j} = e^{-2\rho} g^{kl}(\varphi)$$

hold for the components of g with respect to a local coordinate system. From this we get

$$\begin{aligned} e(u\circ\varphi) &= \sum_{i,j,k,l}\sum_{\alpha,\beta} g^{ij}h_{\alpha,\beta}(u\circ\varphi)\frac{\partial u^\alpha}{\partial x^k}\frac{\partial\varphi^k}{\partial x^i}\frac{\partial u^\beta}{\partial x^l}\frac{\partial\varphi^l}{\partial x^j} \\ &= e^{-2\rho}\sum_{k,l}\sum_{\alpha,\beta} g^{kl}(\varphi)h_{\alpha,\beta}(u\circ\varphi)\frac{\partial u^\alpha}{\partial x^k}\frac{\partial u^\beta}{\partial x^l} = e^{-2\rho}e(u)(\varphi), \end{aligned}$$

$$\begin{aligned} \varphi^*(d\mu_g) &= \sqrt{\det\left(\sum_{k,l} g_{kl}(\varphi)\frac{\partial\varphi^k}{\partial x^i}\frac{\partial\varphi^l}{\partial x^j}\right)}\,dx^1dx^2 = e^{2\rho}d\mu_g. \\ &\le C_3\epsilon^{(\alpha-\alpha')/2}|w|_Q^{(\alpha,\alpha/2)}. \end{aligned}$$

From this it follows that follows $\langle\zeta w\rangle_x^{(\alpha'/2)} \le C_3\epsilon^{(\alpha-\alpha')/2}|w|_Q^{(\alpha,\alpha/2)}$ through a simple computation for $X, Y \in \Gamma(TM)$. Consequently, from the definition of tension field and Lemma 3.3, we get that $\tau(\varphi) = (2-m)\operatorname{grad}\rho$, where m is the dimension of M. In particular, when $m = 2$, φ is a harmonic map.

3.10 Let $\tilde{h} = u^*h$ denote the induced metric from h by u and denote by $\tilde{h}_{ij}$ the components of $\tilde{h}$. Let $\tilde{h}^i_j$ denote the components of the type (1,1) tensor field corresponding to the type (0,2) tensor field $\tilde{h}$ of M under the ispomorphism $TM^*\otimes TM^*$. Then we get $\tilde{h}^i_j = \sum_k \tilde{h}_{jk}g^{ki}$. On the other hand, we see that

$$\sqrt{\det(\tilde{h}^i_j)} \le \frac{1}{2}\operatorname{trace}(\tilde{h}^i_j)$$

holds for the (2,2) matrix $(\tilde{h}^i_j)$, and that the equality holds when there is a positive number λ such that $\tilde{h}^i_j = \lambda\delta^i_j$; namely, only when $\tilde{h}^i_j = \lambda g_{ij}$. Exercise 3.10 follows readily from this.

On the other hand, noting that the induced connection $'\nabla$ on the vector bundle $\varphi^{-1}TM$ is compatible with the fiber metric φ^* together with $\varphi^*g = e^{2\rho}g$ and (1.26), we can readily verify that

$$'\nabla_X d\varphi(Y) = d\varphi(\nabla_X Y) + (X\rho)Y + (Y\rho)X - g(X,Y)\operatorname{grad}\rho$$

through a simple computation for $X, Y \in \Gamma(TM)$. Consequently, we get that $\tau(\varphi) = (2-m)\operatorname{grad}\rho$, where m is the dimension of M. In particular, when $m = 2$, φ is a harmonic map.

Chapter 4

4.1 (1) Given $T \in \Gamma(TM^* \otimes u^{-1}TN)$ and $X, Y, Z \in \Gamma(TM)$, we have

$$\begin{aligned}(\nabla\nabla T)(X,Y,Z) &= (\nabla_X \nabla T)(Y,Z)\\ &= \nabla_X(\nabla T(Y,Z)) - \nabla T(\nabla_X Y, Z) - \nabla T(Y, \nabla_X Z)\\ &= \nabla_X((\nabla_Y T)(Z)) - (\nabla_{\nabla_X Y} T)(Z) - (\nabla_Y T)(\nabla_X Z)\\ &= (\nabla_X(\nabla_Y T))(Z) - (\nabla_{\nabla_X Y} T)(Z).\end{aligned}$$

(2) From the definition of the connection in $TM^* \otimes u^{-1}TN$, we have

$$'\nabla_Y(T(Z)) = (\nabla_Y)(Z) + T(\nabla_Y Z),$$

$$\begin{aligned}'\nabla_X{}'\nabla_Y(T(Z)) =&(\nabla_X\nabla_Y T)(Z) + (\nabla_Y T)(\nabla_X Z),\\ &+ (\nabla_X T)(\nabla_Y Z) + (\nabla_X T)(\nabla_Y Z).\end{aligned}$$

Similarly, computing $-'\nabla'_Y \nabla_X(T(Z))$ and $-'\nabla_{[X,Y]}(T(Z))$ and adding them together, we get

$$R'^{\nabla}(X,Y)(T(Z)) = (R^{\nabla}(X,Y)T)(Z) + T(R^M(X,Y)Z).$$

(3) From (2) and the definition of the induced connection $'\nabla$, we get

$$\begin{aligned}&\left(R^{\nabla}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)T\right)\left(\frac{\partial}{\partial x^k}\right)\\ &\quad= R'^{\nabla}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\left(T\left(\frac{\partial}{\partial x^k}\right)\right) - T\left(R^M\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k}\right)\\ &\quad= \sum_\alpha \left(\sum_{\beta,\gamma,\delta} R^N{}^{\alpha}_{\beta,\gamma,\delta} \frac{\partial u^\beta}{\partial x^i}\frac{\partial u^\gamma}{\partial x^j} T^\delta_k - \sum_l R^M{}^{l}_{ijk} T^\alpha_l\right)\frac{\partial}{\partial y^\alpha} \circ u.\end{aligned}$$

On the other hand, since we have

$$\nabla\nabla T\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) = \sum_\alpha \nabla_i \nabla_j T^\alpha_k \frac{\partial}{\partial y^\alpha} \circ u,$$

the conclusion follows from the definition of R^{∇}.

4.2 We may use an idea similar the one used in the proof of the Weitzenböck formula for $e(u_t)$. First, from the definition of $\kappa(u_t)$, we have

$$\frac{\partial \kappa(u_t)}{\partial t} = \sum_{\alpha,\beta} h_{\alpha\beta}(u_t)\nabla_t \frac{\partial u^\alpha_t}{\partial t}\frac{\partial u^\beta_t}{\partial t}.$$

On the other hand, noting that $\nabla_l \dfrac{\partial u_t^\alpha}{\partial t} = \nabla_t \nabla_l u_t^\alpha$, we get

$$\triangle \kappa(u_t) = \sum_{k,l} \sum_{\alpha,\beta} g^{kl} h_{\alpha\beta}(u_t) \nabla_k \nabla_t \nabla_l u_t^\alpha \frac{u_t^\beta}{\partial t} + \left| \nabla \frac{u_t}{\partial t} \right|^2.$$

From the Ricci identity regarding the connection ∇ in $T(M \times (0,T))^* \otimes u^{-1}TN$, we then get

$$\begin{aligned} &\nabla_k \nabla_t \nabla_l u_t^\alpha - \nabla_t \nabla_k \nabla_k u_l^\alpha \\ &\quad = -\sum_r R^{M\times(0,T)}{}^r_{ktl} \frac{\partial u_t^\alpha}{\partial x^r} + \sum_{\gamma,\delta,\epsilon} R^N{}^\alpha_{\gamma\delta\epsilon} \frac{\partial u_t^\gamma}{\partial x^k} \frac{\partial u_t^\delta}{\partial t} \frac{\partial u_t^\epsilon}{\partial x^l}. \end{aligned}$$

Noting that $M \times (0,T)$ is a product of Riemannian manifolds, we can readily verify that $R^{M\times(0,T)}{}^r_{ktl} = 0$ from the definition of the curvature tensor. Consequently, we get the desired result by substituting the Ricci identity in the above equation and by noting that u is a solution to the equation for harmonic maps.

4.3 If we consider the derivative $d\exp_{(p,0)}$ of the map $\exp : \mathcal{U} \to N$ at $(p,0) \in TM^\perp$ following the line along the proofs of Theorems 1.24 and 1.25, we see that its matrix representation with respect to the canonical coordinate system gives rise to

$$\begin{pmatrix} I0 \\ 0I \end{pmatrix}.$$

Hence, noting that M is compact, the existence of the desired ϵ follows from the inverse function theorem.

4.4 Noting Lemma 3.3 and the definition of the induced connection, we get, for $X, Y \in \Gamma(TM_1)$,

$$\begin{aligned} \nabla d(f_2 \circ f_1)(X,Y) &= \nabla_X (df_2 \circ df_1(Y)) - d(f_2 \circ df_1)(\nabla_X Y) \\ &= (\nabla_{df_1(X)} df_2)(df_1(Y)) + df_2(\nabla_X (df_1(Y))) - df_2 \circ df_1(\nabla_X Y) \\ &= \nabla df_2(df_1(X), df_1(Y)) + df_2(\nabla df_1(X,Y)). \end{aligned}$$

From this follows the first equation. The second equation readily follows from the first equation.

4.5 If $\nabla du = 0$, the formula for the second fundamental form of composition maps implies that

$$\nabla_{d/dt} \frac{d(u \circ c)}{dt} = du \left(\nabla_{d/dt} \frac{dc}{dt} \right) + \nabla du \left(\frac{dc}{dt}, \frac{dc}{dt} \right) = 0$$

for any geodesic $c : I \to M$. From this follows (i) $\Rightarrow$ (ii). Conversely, if (ii) holds, $\nabla du(dc/dt, dc/dt) = 0$ holds for the tangent vector dc/dt of the geodesic. $\nabla du = 0$ holds, since any tangent vector $v \in TM$ at each point $x \in M$ can be given in the form of dc/dt for some geodesic.

4.6 Set $Q = M \times [0, T]$. For example, regarding $C^{2+\alpha,1+\alpha/2}(Q, \mathbb{R}^q)$ being a Banach space, a proof goes as follows. Let $\{u_k\}$ be a Cauchy sequence in $C^{2+\alpha,1+\alpha/2}(Q, \mathbb{R}^q)$. Since $\{u_k\}$ is a uniformly bounded and equicontinuous sequence in $C^{2,1}(Q, \mathbb{R}^q)$, the Ascoli-Arzelà theorem implies that there is a subsequence $\{u_{k'}\}$ of $\{u_k\}$ such that it converges to some u in $C^{2,1}(Q, \mathbb{R}^q)$. Then it suffices to show that u is an element of $C^{2+\alpha,1+\alpha/2}(Q, \mathbb{R}^q)$ and that $\{u_k\}$ converges to u in $C^{2+\alpha,1+\alpha/2}(Q, \mathbb{R}^q)$. These can be directly verified from the definition of the norm $|u|_Q^{(2+\alpha,1+\alpha/2)}$.

4.7 Since $|\zeta w|_Q^{(\alpha',\alpha'/2)} = |\zeta w|_Q + \langle \zeta w \rangle_x^{(\alpha')} + \langle \zeta w \rangle_x^{(\alpha'/2)}$ from the definition of the norm, it suffices to estimate individually $|\zeta w|_Q$, $\langle \zeta w \rangle_x^{(\alpha')}$ and $\langle \zeta w \rangle_x^{(\alpha'/2)}$. From the assumption, $w \in C^{2,1}(Q, \mathbb{R}^q)$ and $w(x, 0) = 0$, we see that $|\zeta w|_Q \le C_1 \epsilon^{\alpha/2} |\zeta w|_Q^{(\alpha,\alpha/2)}$. Similarly, we see that $\langle \zeta w \rangle_x^{(\alpha')} \le C_2 \epsilon^{(\alpha-\alpha')/2} |\zeta w|_Q^{(\alpha,\alpha/2)}$. (For example, we may treat the two cases where $d(x, x') \ge \epsilon^{1/2}$ and $d(x, x') \le \epsilon^{1/2}$.)

On the other hand, regarding $\langle \zeta w \rangle_x^{(\alpha'/2)}$, for $0 \le t < t' \le 2\epsilon$, we have

$$\begin{aligned}
&|\zeta(t)w(x,t) - \zeta(t')w(x,t')| \\
&\qquad \le |\zeta(t)w(x,t) - w(x,t')| + |(\zeta(t) - \zeta(t'))(w(x,t) - w(x,0))| \\
&\qquad \le |\zeta(t)||w|_Q^{(\alpha,\alpha/2)} |t - t'|^{\alpha/2} + 2\epsilon^{-1} |t - t'||w|^{(\alpha,\alpha/2)} |t'|^{\alpha/2}.
\end{aligned}$$

Dividing both sides of the above inequalities by $|t - t'|^{\alpha/2}$, we get

$$\begin{aligned}
&|\zeta(t)w(x,t) - \zeta(t')w(x,t')||t - t'|^{-\alpha/2} \\
&\qquad \le |w|_Q^{(\alpha,\alpha/2)} (2\epsilon)^{(\alpha-\alpha')/2} + 2\epsilon^{-1} (2\epsilon)^{1-\alpha'/2} |w|^{(\alpha,\alpha/2)} (2\epsilon)^{\alpha/2} |t'|^{\alpha/2} \\
&\qquad \le C_3 \epsilon^{(\alpha-\alpha')/2} |w|_Q^{(\alpha,\alpha/2)}.
\end{aligned}$$

From this follows $\langle \zeta w \rangle_x^{(\alpha'/2)} \le C_3 \epsilon^{(\alpha-\alpha')/2} |w|_Q^{(\alpha,\alpha/2)}$.

4.8 Given a constant C and an $\epsilon > 0$, set

$$\hat{u} = e^{-(C+1)t} u, \quad Q = M \times [0, T - \epsilon].$$

From the definition, the signs of $\hat{u}$ and u are the same. They satisfy, in $M \times (0.T)$, the inequality

$$\partial_t \hat{u} \leq \triangle \hat{u} - \hat{u}.$$

Since $\hat{u}$ is a continuous function in Q, there is a point $(x^\circ, t^\circ) \in Q$ where it assumes the maximum value. It suffices to derive a contradiction assuming $\hat{u}(x^\circ, t^\circ) > 0$. Since $\hat{u}(\cdot, 0) \leq 0$ from the assumption of u, we must have $t^\circ > 0$. We readily see that this contradicts the above inequality regarding $\hat{u}$, applying a similar argument to the proof of Lemma 4.11. Since ϵ is arbitrary, we get the desired conclusion.

4.9 Consider M to be a Riemannian manifold with any Riemannian metric. Since M is compact, we can choose a finite cover of M consisting of geodesic spheres $B_r(x_1), \ldots, B_r(x_k)$ such that $f(B_{3r}(x_i))$ is contained in a coordinate neighborhood of N for each $i = 1, \ldots, k$. In $B_{3r}(x_1)$ and for sufficiently small ϵ, define

$$f_1(x) = \int_0^s \int_{B_{2r}(x_1)} (4\pi t)^{-m/2} \exp(-d(x,y)^2/4t) f(y) d\mu_g(y) dt.$$

Choose a C^∞ function $\varphi_i : M \to \mathbb{R}$ such that

$$0 \leq \varphi_i \leq 1, \quad \varphi_i(x) = 1 \ (x \in B_r(x_i)), \quad \varphi_i(x) = 0 \ (x \notin B_{3r/2}(x_i)).$$

Set $\tilde{f}_1 = \varphi_1 f_1 + (1 - \varphi_1) f$. Then $\tilde{f}_1$ becomes a function defined in the entire M. We inductively define f_i by

$$f_i(x) = \int_0^s \int_{B_{2r}(x_1)} (4\pi t)^{-m/2} \exp(-d(x,y)^2/4t) \tilde{f_{i-1}}(y) d\mu_g(y) dt,$$

and set $\tilde{f}_i = \varphi_i f_i + (1 - \varphi_i) \overline{f}_{i-1}$. $\overline{f}_k$ gives the desired C^∞ map $\tilde{f}$. It is readily verified that $\tilde{f}$ is free-homotopic to f as $s \to 0$.

4.10 An element α in $\pi_k(M)$ is nothing but the homotopy class of a continuous map $f : S^k \to M$ from the k-dimensional sphere S^k into M. Since $K_M \leq 0$, f is free-homotopic to a harmonic map $u : S^k \to M$ by Corollary 4.18. Proposition 4.24, then, implies that u is a constant map for $k \geq 2$; hence, the desired conclusion is obtained.

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7. T. Ochiai, Editor, *Nonlinear problems in differential geometry*, Reports on Global Analysis I, Seminar Kankokai, 1979; S. Nishikawa and T. Ochiai, *Existence and applications of harmonic maps*, Reports on Global Analysis II, Seminar Kankokai, 1980.

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This consists of the differential geometric aspects of minimal surfaces (Volume 1), applications of minimal surfaces (Volume 2) and the analytics aspects of minimal surfaces (Volume 3). Volume 3 contains a detailed discussion of the solution of Douglas-Radó-Morrey regarding the existence of minimal surfaces and its properties.

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These reports contain papers by J. L. Kazdan, Some applications of partial differential equations to problems in geometry; S. Nishikawa, On continuity of weak solutions to non-linear elliptic partial differential equations, I, II; A. Tachikawa, On differentiability of the solutions to variational problems, etc. The lecture notes of Kazdan are best suited as an introduction to these topics.

10. T. Sakai, *Riemannian Geometry*, Shokabo, 1992.

This book contains detailed accounts on the relationship between the curvature and topology of Riemannian manifolds. Special attention is paid to the method called the comparison theorem and its application.

11. H. Urakawa, *The variational method and harmonic maps*, Shokabo, 1990.

This well written book explains the general theory of the variational method and the theory of harmonic maps starting from a very basic level. In particular, it contains Uhlenbeck's proof of the theorem of Eells and Sampson from the point of view of Morse theory on infinite dimensional manifolds.

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