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Discrete Groups

Ken'ichi Ohshika

American Mathematical Society

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Volume 207

Discrete Groups

Ken'ichi Ohshika

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ABSTRACT. This book is an introduction to discrete groups from a geometric viewpoint, which would serve as a graduate textbook. It deals mainly with three topics: hyperbolic groups in the sense of Gromov, automatic groups defined by Epstein, and Kleinian groups. Their interaction and relations to low-dimensional topology are also discussed.

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Introduction

Groups naturally appear in the study of geometric objects. The simplest and the naïvest example is the fundamental group for a path-connected topological space. In particular, when a space X is aspherical, i.e., when its higher homotopy groups vanish, it has long been known, mainly due to work of J.H.C. Whitehead, that most homotopical properties of X can be captured by studying the fundamental group of X. Also, in three-dimensional topology or knot theory, the study of 3-manifolds or knots is inseparable from combinatorial group theory, via the fundamental groups. Conversely, in combinatorial group theory, it is known that for studying a group, in many cases, it is more efficient to study a geometric object having that group as fundamental group than the group itself. The epoch-making work of Dehn and Nielsen, which is the foundation of combinatorial group theory, is based on such ideas. In such works as theirs, the correspondence between geometric objects and groups was commonly utilized, as can be seen, for instance, in the fact that subgroups correspond to covering spaces or homomorphisms correspond to continuous maps.

On the other hand, work by Cayley, Tits, Stallings and Serre among others brought in a new viewpoint by treating groups themselves as geometric objects. This trend culminated in work by Gromov in the 1980s. Groups can be regarded as geometric objects by considering their Cayley graphs. Gromov showed that groups and spaces on which they act properly discontinuously and co-compactly can be seen as nearly the same geometric object up to quasi-isometry. (The notion of quasi-isometry was first introduced and used in the proofs of Mostow's rigidity theorem and Margulis' lemma.) This can be seen as a sort of geometrization of group theory, and this viewpoint makes it possible to apply notions and tools used in manifold theory to group theory.

This book aims to serve as a concise introduction to the study of infinite groups as geometric objects, including geometric group theory à la Gromov as one of the main subjects. The main topics with which this book deals are: the theory of hyperbolic groups due to Gromov; automatic group theory, invented and developed by Epstein, whose subjects are groups that can be manipulated by computers; and Kleinian group theory, which enjoys the longest tradition and the richest content among the theory of discrete subgroups of Lie groups.

What is common among these three topics is the fact that the groups appearing there, when seen as geometric objects, have the properties of a negatively curved space rather than a positively curved space. Since Kleinian groups are groups acting on a hyperbolic space of constant negative curvature, the technique employed to study them is that of hyperbolic manifolds, typical examples of negatively curved manifolds. Although hyperbolic groups in the sense of Gromov are much more general objects than Kleinian groups, we can apply for them arguments and techniques which are quite similar to those used for Kleinian groups. Automatic groups are further general objects, including groups having properties of spaces of curvature 0. Still, when we study relationships between automatic groups and hyperbolic groups, we shall use ideas inspired by the study of hyperbolic manifolds. We can say that in all of these three topics, there is a soul of negative curvature upholding the theory, although it may not always appear on the surface. The author would be happy if readers can feel this soul through this small book.

This book is based upon the author's lectures for graduate students in Tokyo Institute of Technology, Hokkaido University, Saga University, and a series of talks at Surveys in Geometry in 1995 at Tokyo. The author expresses his gratitude to the staffs of the departments of mathematics of these three universities and the organizers of the Surveys in Geometry, Professors Sadayoshi Kojima and Kenji Fukaya. Finally, he would like to heartily thank his wife Yoshiko, who supported him spiritually throughout this work.

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