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Introduction to Prehomogeneous Vector Spaces

Tatsuo Kimura

American Mathematical Society

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Volume 215

Introduction to Prehomogeneous Vector Spaces

Tatsuo Kimura

Translated by
Makoto Nagura
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American Mathematical Society
Providence, Rhode Island

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GAIKINSHITSU BEKUTORU KUKAN (THEORY OF PREHOMOGENEOUS VECTOR BUNDLES)

by Tatsuo Kimura

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Preface to the Japanese Edition

The theory of prehomogeneous vector spaces was originated by Mikio Sato in 1961. The importance of this theory has been recognized since then, and many conferences on prehomogeneous vector spaces have been held in the U.S.A., France, and Japan. However, at the present time, there is no introductory book dealing with the basic parts of this theory, and many people seem to want one. The author has had the chance to give introductory lectures on prehomogeneous vector spaces at a few universities besides his own University of Tsukuba. In preparing for one of these lectures, at Sophia University in the spring semester in 1996, the author decided to get those lecture notes together. Later, Professor Akihiko Gyoja read the manuscript carefully and checked it in detail, and gave the author a great deal of valuable advice. Thus it was revised, and became the present book. A graduate student, Kazunari Sugiyama, was very helpful.

Takeyoshi Kogiso, Shin-ichi Otani, Tsushima Kouji, Tsuyoshi Niitani, and Makoto Nagura checked the manuscript in detail, by doing a seminar based on it. Iwao Kimura persevered heroically to make the manuscript, which was rewritten many times, into the LaTeX file. The author was given valuable advice from Professor Fumihiro Sato, besides Professor Gyoja. The author was taught some points from Professor Jun-ichi Igusa and Professor Takashi Ono. Thus this book was made up based on their cooperation, and the author expresses his gratitude to them. The author thanks also Hisao Miyauchi, Uichi Yoshida and Mamiko Hamakado, who work for Iwanami-Shoten, for their assistance.

This book is dedicated to Professor Mikio Sato, who has led the author strictly and kindly since the author first visited him at Kyoto University in the spring of 1972.

October 10, 1998
Tatsuo Kimura

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Preface to the English Edition

The original book in Japanese is written as an elementary introduction to the classical theory of prehomogeneous vector spaces originated by Mikio Sato and developed by Takuro Shintani, Fumihiro Sato, the author and others. The author tried to write the original book to be as self-contained as possible, so that it would be accessible even for undergraduate students. Even when we have to quote some theorems, we tried to show how to use them by examples, etc.

In this English edition, a new section about the residues of zeta functions (§5.5), and the example of zeta functions of indefinite quadratic forms were added at the suggestion and with the help of Professor Fumihiro Sato. For the convenience of the reader, in this English edition, we add the table of irreducible reduced prehomogeneous vector spaces as an appendix. The author would like to express his hearty thanks to Professor Fumihiro Sato. There are also some parts that are changed or added in sections of Chapters 3, 4, 6, and 7. Dr. K. Sugiyama revised the bibliography for the English edition. The author would like to express his hearty thanks to Mr. M. Nagura and Mr. T. Niitani, who translated the original book into English.

June 21, 2002
Tatsuo Kimura

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Overview of the Theory and Contents of This Book

First of all, we will roughly sketch the theory of prehomogeneous vector spaces.

Let us consider **Riemann's zeta function** $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, which is essential in number theory. This converges absolutely for $\operatorname{Re} s > 1$, and becomes a holomorphic function. Further it can be expressed as $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$ (the product is taken over all prime numbers p), which is nothing but an analytic expression of the fundamental theorem in elementary number theory that each natural number decomposes uniquely into a finite product of prime numbers. Another important property of $\zeta(s)$ is the fact that it can be extended analytically to the whole s-plane as a meromorphic function, and it satisfies the functional equation

$$\zeta(1-s) = (2\pi)^{-s}\Gamma(s)\, 2\cos\frac{\pi s}{2}\,\zeta(s).$$

This is one of basic properties of $\zeta(s)$. As the reason why $\zeta(s)$ satisfies this functional equation, we can point out the following two facts:

(I) The **Fourier transform** of a complex power of x is again a complex power of y. That is,

$$\int_0^{\infty} x^{s-1} e^{2\pi\sqrt{-1}xy}\,dx = (2\pi)^{-s}\Gamma(s)\, e^{\frac{\pi\sqrt{-1}s}{2}(\operatorname{sgn} y)}|y|^{-s} \qquad (0 < \operatorname{Re} s < 1).$$

(II) **Poisson's summation formula**; that is, for the Fourier transform $\widehat{f}(y) = \int_{-\infty}^{\infty} f(x) e^{2\pi\sqrt{-1}xy}\,dx$ of $f(x)$, we have

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)$$

(see Theorem 4.38).

Indeed, let

$$f(x) = \begin{cases} x^{s-1} & (x > 0), \\ 0 & (x \leqq 0). \end{cases}$$

Taking no account for the convergence and calculating just formally, we have

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=1}^{\infty} n^{s-1} = \zeta(1-s).$$

On the other hand, (I) implies that

$$\sum_{n=-\infty}^{\infty} \widehat{f}(n) = (2\pi)^{-s}\Gamma(s)\left(\sum_{n=-\infty}^{-1}(-n)^{-s}e^{-\frac{\pi\sqrt{-1}s}{2}} + \sum_{n=1}^{\infty} n^{-s}e^{\frac{\pi\sqrt{-1}s}{2}}\right)$$
$$= (2\pi)^{-s}\Gamma(s)\,2\cos\frac{\pi s}{2}\,\zeta(s).$$

Hence, by (II), we can see how to obtain the functional equation

$$\zeta(1-s) = (2\pi)^{-s}\Gamma(s)\,2\cos\frac{\pi s}{2}\,\zeta(s).$$

To give a rigorous proof based on this principle, we have to consider the zeta integral

$$I(s,\widehat{\varphi}) = \int_0^{\infty} t^{s-1} \sum_{x\in\mathbb{Z}-\{0\}} \widehat{\varphi}(tx)\,dt,$$

where $\widehat{\varphi}$ is the Fourier transform of a rapidly decreasing function φ on $\mathbb{R}$ (see §3.2), and the integral converges absolutely for $\operatorname{Re} s > 1$. A sketch of the proof can be given as follows: First, we have

$$I(s,\widehat{\varphi}) = \sum_{n=1}^{\infty}\int_0^{\infty} t^{s-1}\widehat{\varphi}(nt)\,dt + \sum_{n=1}^{\infty}\int_0^{\infty} t^{s-1}\widehat{\varphi}(-nt)\,dt$$
$$= \zeta(s)\int_{-\infty}^{\infty} |x|^{s-1}\widehat{\varphi}(x)\,dx.$$

On the other hand, using Poisson's summation formula (II) (that is, $\sum_{x\in\mathbb{Z}}\widehat{\varphi}(tx) = (1/t)\sum_{x\in\mathbb{Z}}\varphi(x/t)$), we can show that

$$I(s,\widehat{\varphi}) = J(s,\varphi) - \frac{\widehat{\varphi}(0)}{s} + \frac{\varphi(0)}{s-1},$$

where

$$J(s,\varphi) = \int_1^{\infty} t^{s-1}\sum_{x\in\mathbb{Z}-\{0\}}\widehat{\varphi}(tx)\,dt + \int_1^{\infty} t^{-s}\sum_{x\in\mathbb{Z}-\{0\}}\varphi(tx)\,dt.$$

Since $J(s,\varphi)$ is an entire function of s, we see that $I(s,\widehat{\varphi})$ extends analytically to a meromorphic function on the whole s-plane with poles of order at most one at $s = 0, 1$. Moreover, from this form, we see that it satisfies $I(s,\widehat{\varphi}) = I(1-s,\varphi)$. Since for each $s\in\mathbb{C}$ there exists φ satisfying $\int_{-\infty}^{\infty}|x|^{s-1}\widehat{\varphi}(x)\,dx \neq 0,\infty$, the zeta function

$$\zeta(s) = \left(\int_{-\infty}^{\infty}|x|^{s-1}\widehat{\varphi}(x)\,dx\right)^{-1}\cdot I(s,\widehat{\varphi})$$

extends analytically to a meromorphic function on the whole s-plane. On the other hand, $\int_{-\infty}^{\infty}|x|^{s-1}\widehat{\varphi}(x)\,dx$ also extends analytically to a meromorphic function on the whole s-plane. Hence it follows from (I) that

$$\int_{-\infty}^{\infty}|x|^{s-1}\widehat{\varphi}(x)\,dx = (2\pi)^{-s}\Gamma(s)\,2\cos\frac{\pi s}{2}\int_{-\infty}^{\infty}|x|^{-s}\varphi(x)\,dx.$$

(See Proposition 4.21.) By this equation and $I(s,\widehat{\varphi}) = I(1-s,\varphi)$, i.e.,

$$\zeta(s)\int_{-\infty}^{\infty}|x|^{s-1}\widehat{\varphi}(x)\,dx = \zeta(1-s)\int_{-\infty}^{\infty}|x|^{-s}\varphi(x)\,dx,$$

we obtain

$$\zeta(1-s) = (2\pi)^{-s}\Gamma(s)\,2\cos\frac{\pi s}{2}\,\zeta(s).$$

Poisson's summation formula (II) is obtained from the fact that two consecutive Fourier transformations of a function give essentially the original one (see, for example, Proposition 3.11). Its higher degree version is easily obtained (Theorem 4.38).

Thus, if we can generalize (I); that is, if we can get a systematic method of finding a polynomial $f(x)$ on a vector space V such that

(I)′ the Fourier transform of a complex power $f(x)^s$ is essentially given by a complex power of some polynomial $f^*(x)$ on the dual space V^* of V,

then it is expected that a function, for example, of the form

$$\zeta(s,f) = \sum_{x\in\mathbb{Z}^n - f^{-1}(0)} \frac{1}{|f(x)|^s}$$

satisfies a functional equation. That is to say, it is expected that **Dirichlet series** (zeta functions) with functional equations will be obtained systematically.

For example, let $P(x) = {}^t\!xAx$ ($x \in \mathbb{R}^n$) be a positive definite quadratic form, and $Q(y) = {}^t\!yA^{-1}y$ its dual. Then we have

$$\int_{\mathbb{R}^n\setminus\{0\}} |Q(y)|^{s-\frac{n}{2}}\,\widehat{\Phi}(y)\,dy = \pi^{\frac{n}{2}-2s}\cdot\sqrt{\det P}\cdot\frac{\Gamma(s)}{\Gamma\left(\frac{n}{2}-s\right)}\cdot\int_{\mathbb{R}^n\setminus\{0\}} |P(x)|^{-s}\,\Phi(x)\,dx,$$

where Φ is a rapidly decreasing function on $\mathbb{R}^n$ (Proposition 4.22), and we see that Epstein's zeta function

$$\zeta(s,P) = \sum_{x\in\mathbb{Z}^n-\{0\}} \frac{1}{|P(x)|^s}$$

satisfies a functional equation.

Then, what is the reason why the Fourier transform of x^s or $P(x)^s$ is again essentially a complex power of some polynomial?

In 1961, Mikio Sato found out that the reason is the existence of a big action of a group, and the fact that x or $P(x)$ are relative invariants with respect to that action. Thus he reached the notion of prehomogeneous vector spaces. We call a triplet (G,ρ,V) a **prehomogeneous vector space** if $\rho : G \to GL(V)$ is a rational representation of a connected algebraic group G on a finite-dimensional vector space V such that V has a dense G-orbit $\rho(G)v_0 = \{\rho(g)v_0 \in V;\ g \in G\}$ with respect to the Zariski topology, all defined over $\mathbb{C}$. In general, a G-orbit $\rho(G)v_0$ is called a homogeneous space under the action of the group G. The condition $V = \overline{\rho(G)v_0}$ implies that V is an almost homogeneous space under the action of G. Therefore we call it a prehomogeneous vector space.

A **relative invariant** of (G,ρ,V) is a nonzero rational function $f(x)$ on V satisfying $f(\rho(g)x) = \chi(g)f(x)$ with some constant $\chi(g)$ for each $g \in G$. Then $\chi : G \to GL_1$ becomes a rational character of G, and we say that f is a relative invariant corresponding to the character χ. For example, we consider the action of $G = GL_1(\mathbb{C}) = \mathbb{C}^\times$ on $V = \mathbb{C}$ by $\rho(g)x = gx$ ($g \in \mathbb{C}^\times$, $x \in \mathbb{C}$). Then, since $V - \{0\} = \rho(G)\cdot 1$ is a dense orbit, it follows that (G,ρ,V) is a prehomogeneous vector space, and $f(x) = x$ is a relative invariant corresponding to $\chi(g) = g$. Next, let $SO_n(P) = \{B \in SL_n(\mathbb{C});\ {}^t\!BAB = A\}$ be the special orthogonal group which stabilizes the quadratic form $P(x) = {}^t\!xAx$ (${}^t\!A = A \in GL_n(\mathbb{C})$) on $V = \mathbb{C}^n$. Then

$G = GL_1 \times SO_n(P)$ acts on V by $\rho(\alpha, B)x = \alpha Bx$ ($\alpha \in GL_1$, $B \in SO_n(P)$, $x \in \mathbb{C}^n$), and $\{x \in V;\ P(x) \neq 0\}$ is a dense G-orbit in V (see Example 2.15 of §2.4). Hence (G, ρ, V) is a prehomogeneous vector space, and $P(x)$ is a relative invariant corresponding to the character $\chi(\alpha, B) = \alpha^2$.

Then, what is the principle under which the Fourier transform of a complex power $f(x)^s$ of a relative invariant $f(x)$ on a prehomogeneous vector space is again a complex power of a polynomial $f^*(y)$?

To see this, let us consider the basic properties of relative invariants on prehomogeneous vector spaces.

First, if $f(x)$ is an absolute invariant, i.e., a relative invariant corresponding to $\chi = 1$, then it is a constant on each orbit $\rho(G)v_0$: $f(x) = c\ (= f(v_0))$. That is to say, $\{x \in V;\ f(x) = c\} \supset \rho(G)v_0$. The closure of the left-hand side is itself, since it is a closed set. The closure of the right-hand side is given by $\overline{\rho(G)v_0} = V$, so that $f(x)$ becomes a constant function on V.

Next, if f_1 and f_2 are relative invariants corresponding to the same character χ, then f_2/f_1 is an absolute invariant. Hence it is a constant, and we have $f_2(x) = cf_1(x)$ (c is a constant). Thus we see that

(III) relative invariants of a prehomogeneous vector space corresponding to the same character are identical up to a constant.

Conversely, it is known that if relative invariants of a triplet (G, ρ, V) satisfy condition (III), then (G, ρ, V) is a prehomogeneous vector space (see Proposition 2.4). In other words, a prehomogeneous vector space is characterized by condition (III).

For simplicity, we consider the case where G is a reductive algebraic group, and where the complement $S = V \setminus \rho(G)v_0$ of a dense G-orbit $\rho(G)v_0$ is the zeros $S = \{x \in V;\ f(x) = 0\}$ of an irreducible polynomial $f(x)$. Let $d = \deg f$ and $n = \dim V$. Then, it is known that $m = 2n/d$ is a natural number, and that $f(\rho(g)x) = \chi(g)f(x)$ ($g \in G$) for a character χ satisfying $\det \rho(g)^2 = \chi(g)^m$ ($g \in G$) (see Proposition 2.18). Further, the dual (G, ρ^*, V^*) of (G, ρ, V) is also a prehomogeneous vector space satisfying similar properties, and it has an irreducible relative invariant $f^*(y)$ satisfying $f^*(\rho^*(g)y) = \chi(g)^{-1}f^*(y)$.

Now, not worrying about convergence, we consider just formally the Fourier transform

$$\varphi(y) = \int_V f(x)^{s-\frac{n}{d}} \cdot e^{2\pi\sqrt{-1}\langle x,y\rangle} dx \qquad (y \in V^*)$$

of $f(x)^{s-\frac{n}{d}}$, which we also consider just formally. Since

$$\begin{aligned} \varphi(\rho^*(g)y) &= \chi(g)^{s-\frac{n}{d}} \cdot \det \rho(g) \cdot \varphi(y) \\ &= \chi(g)^s \varphi(y) \qquad (g \in G), \end{aligned}$$

the Fourier transform $\varphi(y)$ of $f(x)^{s-\frac{n}{d}}$ becomes a relative invariant on V^* corresponding to $\chi(g)^s$. On the other hand, $f^*(y)^{-s}$ is also a relative invariant on V^* corresponding to the same character $\chi(g)^s$. Since (G, ρ^*, V^*) is a prehomogeneous vector space, we expect that principal (III) will imply the coincidence of $\varphi(y)$ and $f^*(y)^{-s}$ up to a constant, although we cannot apply (III) because they are not rational functions. That is to say, we expect the existence of a constant c satisfying

$$\int_V f(x)^{s-\frac{n}{d}} \cdot e^{2\pi\sqrt{-1}\langle x,y\rangle} dx = cf^*(y)^{-s}.$$

This is the principle which says that the Fourier transform of a complex power $f(x)^s$ of a relative invariant coincides with a complex power of a polynomial $f^*(y)$. Of course, in order to make their convergence (and so on) meaningful, we need to discuss the details rigorously; for example, we have to regard $f(x)^{s-\frac{n}{d}}$ and $f^*(y)^{-s}$ as distributions. Following this principle, Mikio Sato obtained, in 1961, the fundamental theorem of prehomogeneous vector spaces concerning the Fourier transform of a complex power of a relative invariant (see Theorem 4.17). That is to say, suppose that (G, ρ, V) satisfies the conditions above, and that it is defined over $\mathbb{R}$. Then $V_{\mathbb{R}} - S_{\mathbb{R}}$ and $V_{\mathbb{R}}^* - S_{\mathbb{R}}^*$ are decomposed into the same number of connected components: $V_{\mathbb{R}} - S_{\mathbb{R}} = V_1 \cup \cdots \cup V_l$ and $V_{\mathbb{R}}^* - S_{\mathbb{R}}^* = V_1^* \cup \cdots \cup V_l^*$, where V_i, V_j^* are $G_{\mathbb{R}}^+$-orbits. Here $G_{\mathbb{R}}^+$ is the connected component containing the unit element of the subgroup $G_{\mathbb{R}}$ which consists of $\mathbb{R}$-rational points of G. Further, for rapidly decreasing functions Φ, Φ^* on $V_{\mathbb{R}}$, $V_{\mathbb{R}}^*$ respectively, the integrals

$$\int_{V_i} |f(x)|^s \Phi(x)\, dx$$

and

$$\int_{V_j^*} |f^*(y)|^s \Phi^*(y)\, dy$$

converge for $\operatorname{Re} s > 0$, and can be extended analytically to meromorphic functions on the whole s-plane, which satisfy

$$\int_{V_j} |f(x)|^{s-\frac{n}{d}} \widehat{\Phi^*}(x)\, dx = \sum_{i=1}^{l} a_{ij}(s) \int_{V_i^*} |f^*(y)|^{-s} \Phi^*(y)\, dy.$$

Now, suppose moreover that (G, ρ, V) is defined over $\mathbb{Q}$. Let L be a lattice in $V_{\mathbb{Q}}$; that is, a discrete subgroup of $V_{\mathbb{R}}$ such that $V_{\mathbb{R}}/L$ is compact. For a discrete subgroup Γ of $G_{\mathbb{R}}^+$ which stabilizes L, we would like to define the zeta function as

$$\zeta_i(L, s) = \sum_{x \in L \cap V_i} \frac{1}{|f(x)|^s}.$$

However, this does not converge for almost all cases. So taking the sum over a system of representatives of Γ-orbits in $L \cap V_i$, we define the zeta function as

$$\zeta_i(L, s) = \sum_{x \in \Gamma \backslash L \cap V_i} \frac{\mu(x)}{|f(x)|^s},$$

where $\mu(x)$ is the so-called density for $x \in V_{\mathbb{Q}}$, which depends only on the Γ-orbit $\rho(\Gamma)x$. It is known that $\mu(x)$ is finite if and only if each rational character defined over $\mathbb{Q}$ on the connected component G_x° containing the unit element of $G_x = \{g \in G;\ \rho(g)x = x\}$ $(x \in V_{\mathbb{Q}})$ is trivial. Hence, we assume here that this condition holds for any $x \in (V - S) \cap V_{\mathbb{Q}}$. For the dual space (G, ρ^*, V^*), taking the dual lattice L^* of L, we define $\zeta_j^*(L^*, s)$ similarly. Its analytic continuation and the functional equation of this zeta function are proved by an idea similar to that of Riemann's zeta function mentioned above.

It is known that $\zeta_i(L, s)$ and $\zeta_j^*(L^*, s)$ converge absolutely for sufficiently large $\operatorname{Re} s$. For example, let H be the connected component of $\{g \in G;\ \chi(g) = 1\}$ for a character χ corresponding to f. If $H_x = H \cap G_x$ is a connected semisimple algebraic

group, then $\zeta_i(L,s)$ converges absolutely for $\operatorname{Re} s > n/d$ (Theorem 6.3). There are many prehomogeneous vector spaces satisfying this condition.

The zeta integral

$$Z(\Phi,s)=\int_{G_{\mathbb{R}}^{+}/\Gamma}\chi(g)^{s}\sum_{x\in L-L\cap S}\Phi(\rho(g)x)\,dg$$

also converges absolutely for sufficiently large $\operatorname{Re} s$, where Φ is a rapidly decreasing function on $V_{\mathbb{R}}$. Moreover it satisfies

$$Z(\Phi,s)=\sum_{i=1}^{l}\zeta_i(L,s)\int_{V_i}|f(x)|^{s-\frac{n}{d}}\Phi(x)\,dx.$$

For the dual space (G,ρ^*,V^*), similarly we have

$$Z^*(\Phi^*,s)=\sum_{i=1}^{l}\zeta_i^*(L^*,s)\int_{V_i^*}|f^*(y)|^{s-\frac{n}{d}}\Phi^*(y)\,dy.$$

In the case where a rapidly decreasing function Φ^* on $V_{\mathbb{R}}^*$ satisfies $\Phi^*|_{S_{\mathbb{R}}^*}=0$ and $\widehat{\Phi^*}|_{S_{\mathbb{R}}}=0$, we see that $Z(\widehat{\Phi^*},s)$ and $Z^*(\Phi^*,s)$ can be extended analytically to entire functions of s, and that Poisson's summation formula implies the functional equation

$$Z(\widehat{\Phi^*},s)=\frac{1}{\operatorname{vol}(L)}Z^*\left(\Phi^*,\frac{n}{d}-s\right)$$

(Proposition 5.16). Here $\operatorname{vol}(L)$ denotes the volume of $V_{\mathbb{R}}/L$. The b-function $b(s)=b_0\prod_{i=1}^{d}(s+\alpha_i)$ $(d=\deg f)$ is defined by $f^*(D_x)f(x)^{s+1}=b(s)f(x)^s$. Put $\Phi=f^*(D_x)\Phi_1$ for $\Phi_1\in C_0^\infty(V_i)$. Since

$$\begin{aligned}Z(\Phi,s)&=\zeta_i(L,s)\int_{V_i}|f(x)|^{s-\frac{n}{d}}\Phi(x)\,dx\\&=\varepsilon_i\,b(-s)\,\zeta_i(L,s)\int_{V_i}|f(x)|^{s-\frac{n}{d}-1}\Phi_1(x)\,dx\end{aligned}$$

is an entire function of s (see §5.4), the function $b(-s)\,\zeta_i(L,s)$ can be extended analytically to an entire function of s, where ε_i denotes the sign of $f(x)$ on V_i. Hence, we see that $\zeta_i(L,s)$ extends analytically to a meromorphic function on the whole s-plane, and that it can have poles only at $s=\alpha_1,\cdots,\alpha_d$. Next, take $\Phi_1^*\in C_0^\infty(V_i^*)$, and put $\Phi^*(y)=f(D_y)\Phi_1^*(y)$. Then

$$Z(\widehat{\Phi^*},s)=\sum_{j=1}^{l}\zeta_j(L,s)\int_{V_j}|f(x)|^{s-\frac{n}{d}}\widehat{\Phi^*}(x)\,dx$$

is an entire function of s, and it is equal to

$$\frac{1}{\operatorname{vol}(L)}Z^*\left(\Phi^*,\frac{n}{d}-s\right)=\frac{1}{\operatorname{vol}(L)}\zeta_i^*\left(L^*,\frac{n}{d}-s\right)\int_{V_i^*}|f^*(y)|^{-s}\Phi^*(y)\,dy.$$

Since the fundamental theorem and $\operatorname{supp}\Phi^*\subset V_i^*$ imply

$$\int_{V_j}|f(x)|^{s-\frac{n}{d}}\widehat{\Phi^*}(x)\,dx=a_{ij}(s)\int_{V_i^*}|f^*(y)|^{-s}\Phi^*(y)\,dy,$$

the functional equation

$$\zeta_i^*\left(L^*, \frac{n}{d} - s\right) = \mathrm{vol}(L) \sum_{j=1}^{l} a_{ij}(s)\zeta_j(L, s)$$

is obtained from the two equations above. This is how one proves the functional equations.

Contents of this book

Now, we will explain the contents of this book.

Since this book is an introductory one, many pages are devoted to preliminaries, so that even senior undergraduate students of mathematics can understand.

Chapter 1 is devoted to algebraic preliminaries. We review basic definitions and properties of groups, rings, and fields in §1.1. Here, we give no proofs.

In §1.2, we briefly review topological spaces, in particular the notions of connectedness and irreducibility. We define Noether spaces, and we prove that each closed set of a Noether space can be uniquely expressed as a finite union of irreducible components. Further, we define constructible sets, and we establish a property of them (Proposition 1.2) that will be used in Lemma 2.1. We also prove a proposition concerning dimensions, which is needed in Chapter 7.

In §1.3, we define the Zariski topology. Since a topological space endowed with the Zariski topology becomes a Noether space, we can apply the results of §1.2. We define quasi-affine algebraic varieties, which we will simply call algebraic varieties because we consider only them. Here, it is proved that a Zariski closed set S in the vector space Ω^n over an algebraically closed field Ω is a hypersurface if and only if $\Omega^n - S$ is an affine variety.

In §1.4, we briefly explain the so-called algebraic group G, which is an affine variety with group structure. We prove that the irreducible component G° containing the unit element of G is uniquely determined, and that it is a normal subgroup of G of finite index. Further, we give definitions and examples of simple groups, semisimple algebraic groups, and reductive algebraic groups. The definition of a rational representation of an algebraic group G is also given. In §1.5, we define tangent spaces of algebraic varieties, and differentials of mappings. Relating to them, we also define simple points and singular points.

In §1.6, we explain the Lie algebras of algebraic groups in detail, with proofs.

First, we prove that the tangent space T_eG ($\subset M_m$) at the unit element e of an algebraic group G ($\subset GL_m$) can be equipped with the structure of a Lie algebra by $[X, Y] = XY - YX$ ($X, Y \in T_eG \subset M_m$). As examples, we calculate the Lie algebras of the algebraic groups of types A_n, B_n, C_n, and D_n.

Next, we show that the differential mapping $d\varphi : T_eG \to T_{e'}H$ induced by a homomorphism $\varphi : G \to H$ of algebraic groups becomes a homomorphism of Lie algebras, and we introduce the notion of the infinitesimal representation $d\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ of a representation $\rho : G \to GL(V)$ of an algebraic group, where $\mathfrak{g}$ denotes the Lie algebra of G. As an example, we study some properties of the adjoint representation.

Chapter 2 is devoted to the algebraic theory of relative invariants of prehomogeneous vector spaces.

In §2.1, we define prehomogeneous vector spaces. Some equivalent conditions are explained, and in particular we see that prehomogeneity is the so-called infinitesimal property, which is determined by the condition for a Lie algebra. If (G, ρ, V) is a prehomogeneous vector space, then it should be necessary that $\dim G \geqq \dim V$, which will be the first principle of our classification.

We investigate basic properties of relative invariants in §2.2. The complement $S = V \setminus \rho(G)v_0$ of a dense orbit $\rho(G)v_0$ of a prehomogeneous vector space (G, ρ, V) is a Zariski closed set which is called the singular set. If each one-codimensional irreducible component S_i is given by the zeros $S_i = \{x \in V;\ f_i(x) = 0\}$ $(1 \leqq i \leqq r)$ of an irreducible polynomial f_i, then $f_1(x), \cdots, f_r(x)$ are algebraically independent relative invariants, and further any relative invariant $f(x)$ can be uniquely expressed as $f(x) = cf_1(x)^{m_1} \cdots f_r(x)^{m_r}$, where c is a constant and $(m_1, \ldots, m_r) \in \mathbb{Z}^r$. For each relative invariant f, a mapping $\varphi_f = \operatorname{grad} \log f$ from $V - S$ to the dual V^* of V is determined, and then $\varphi_f(V - S)$ is a G-orbit in the dual triplet (G, ρ^*, V^*). If this orbit is dense in V^*, we call f nondegenerate. When there exists a nondegenerate relative invariant, we call (G, ρ, V) a regular prehomogeneous vector space. We prove that if (G, ρ, V) is a regular prehomogeneous vector space, then the dual (G, ρ^*, V^*) is also a regular prehomogeneous vector space, and moreover that there exists a relative invariant corresponding to the character $\det \rho(g)^2$ $(g \in G)$. In particular, for a regular prehomogeneous vector space which has only one irreducible relative invariant $f(x)$ up to a constant, we see that $d|2n$ and $\det \rho(g)^2 = \chi(g)^{2n/d}$ $(g \in G)$, where $n = \dim V$ and $d = \deg f$. Here, χ is the character corresponding to f. In the case where there exists a point x_1 such that $\overline{\rho(G)x_1} = \{x \in V;\ f(x) = 0\}$, using this fact, we give a proof of the degree formula that the degree $d = \deg f$ of f is determined by a calculation of a Lie algebra (i.e., by means of linear algebra).

In §2.3, we ask for a group theoretical condition ensuring that, for a reductive algebraic group G defined over $\mathbb{C}$, a prehomogeneous vector space (G, ρ, V) is regular.

Then, we introduce b-functions, which play an important role in the theory of prehomogeneous vector spaces. Using them, we show that a prehomogeneous vector space is regular if its singular set S is a hypersurface. Conversely, using D. Luna's theorem, we prove that S is a hypersurface if a prehomogeneous vector space is regular. We showed in §1.3 that S is a hypersurface if and only if $V - S \cong G/G_{x_0}$, where $G_{x_0} = \{g \in G;\ \rho(g)x_0 = x_0\}$ $(x_0 \in V - S)$, is an affine variety. Moreover, it is known, for a reductive group G, that G/G_{x_0} becomes an affine variety if and only if G_{x_0} is a reductive algebraic group. Thus, we see, for a reductive group G, that, group-theoretically, G_{x_0} $(x_0 \in V - S)$ is reductive if and only if (G, ρ, V) is regular.

We give examples of prehomogeneous vector spaces in §2.4. Examples 2.1–2.29 are all of the **reduced** irreducible regular prehomogeneous vector spaces, as was proved by M. Sato and T. Kimura [**77**]. Since the numbers of the table in that paper are frequently quoted, we use that numbering. A triplet (G, ρ, V) with an irreducible representation ρ is called an irreducible prehomogeneous vector space. In fact, each regular irreducible prehomogeneous vector space is obtained by a finite number of castling transformations (Chapter 7) from one of Examples 2.1–2.29. Example 2.30 is an example of a nonregular prehomogeneous vector space with relative invariants, and it is a basic example in the theory of prehomogeneous vector spaces without regularity condition. Through the examples, the reader will

be able to get a concrete feeling for prehomogeneous vector spaces. There are many examples; the reader may prefer to start with just a few.

Chapter 3 is devoted to analytic preliminaries to proving the fundamental theorem.

First, in §3.1, we review Lebesgue integrals, and we give the definition and examples of Haar measures. We also give a condition for a homogeneous space to have a relatively invariant measure. In §3.2, it is proved that if $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $\varphi \neq 0$, then its Fourier transform $\widehat{\varphi}$ does not belong to $C_0^\infty(\mathbb{R}^n)$. Therefore we define the space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing functions, which contains $C_0^\infty(\mathbb{R}^n)$. Then we show that $\varphi \mapsto \widehat{\varphi}$ gives a bijection from $\mathcal{S}(\mathbb{R}^n)$ to itself.

In §3.3, we consider a linear mapping $T : C_0^\infty(\mathbb{R}^n) \to \mathbb{C}$ which satisfies a certain condition of continuity. Such a mapping is called a Schwartz **distribution**. Each continuous function $f : \mathbb{R}^n \to \mathbb{C}$ can be regarded as a distribution by

$$T_f(\Phi) = \int_{\mathbb{R}^n} f(x)\Phi(x)\,dx \qquad (\Phi \in C_0^\infty(\mathbb{R}^n)).$$

If the Fourier transform $\widehat{f}$ of f is well-defined, we have $T_{\widehat{f}}(\Phi) = T_f(\widehat{\Phi})$. However, this is inconvenient, because $\widehat{\Phi} \notin C_0^\infty(\mathbb{R}^n)$. So we ask for a necessary and sufficient condition under which T can be extended to $T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$. A distribution that satisfies such a condition is called a **tempered distribution** T. We can defined its Fourier transform by $\widehat{T}(\Phi) = T(\widehat{\Phi})$ $(\Phi \in \mathcal{S}(\mathbb{R}^n))$. Since a complex power of a polynomial can be regarded as a tempered distribution, its Fourier transform is well-defined. Lastly, we prove some propositions concerning distributions, which will be used in the proof of the fundamental theorem in §4.1.

In §3.4, we prove fundamental properties of the gamma function $\Gamma(s)$. We can consider $\Gamma(s)$ to be an extension of the factorial $1 \times \cdots \times (s-1) = (s-1)!$ to a function of $s \in \mathbb{C}$.

Sections 3.5–3.7 is devoted to a proof of the theorem which says that if a reductive connected Lie group G acts transitively on an open set U in $\mathbb{R}^n$ and if there exists a G-invariant measure $\Omega(x)$ on U, then each G-invariant distribution on U can be expressed in the form

$$T(\Phi) = c\int_U \Phi(x)\Omega(x) \qquad (c \in \mathbb{C},\ \Phi \in C_0^\infty(U)).$$

We will follow Harish-Chandra's proof [**17**]. However, if we admit Theorem 3.30, induced by M. Sato's fundamental theorem of algebraic analysis, we can obtain a concise and lucid proof. Then, moreover, we need to assume neither the reductivity of G, nor the existence of a G-invariant measure $\Omega(x)$ (Remark 3.2). At a first reading, the reader can go ahead to Chapter 4 by admitting Theorem 3.31.

The aim of §3.5 is to prove the **de Rham theorem** concerning differential forms with compact supports. The definition of C^∞-manifold, the notion of orientability, and the integral of a differential form are explained. In §3.6, we prove Koszul's theorem which says that the cohomologies induced by left invariant differential forms under an action of a reductive connected Lie group coincide with those of a subcomplex of two-sided invariant differential forms. From this, we prove a lemma which will be used in §3.7.

Following Harish-Chandra's paper [**17**], under the preparations in §§3.5 and 3.6, we prove a certain theorem concerning invariant distributions in §3.7. Although the theorem is often quoted, it seems that references are rarely given. So, for

the reader's convenience, we give an explanation of some propositions in Harish-Chandra's paper in §§3.5–3.7. Proposition 3.27 in §3.7 is also used in Example 2 in §5.6. Lastly, we sketch another proof, using algebraic analysis.

The goal of Chapter 4 is to prove the fundamental theorem, which says that the Fourier transform of a complex power of a relative invariant on a space is again a complex power of some relative invariant on its dual space.

A proof of the fundamental theorem is given in §4.1. Here, for simplicity, we assume that G is a reductive algebraic group, that the singular set S is an irreducible hypersurface, and that (G, ρ, V) is defined over $\mathbb{R}$. All spaces explained in Examples 2.1–2.29 of §2.4 satisfy these assumptions. Using the propositions proved in §§3.3 and 3.7, we prove the fundamental theorem.

In §4.2, as a typical example of the fundamental theorem, we calculate the Fourier transform of x^s for the case of $(GL_1, \Lambda_1, V(1))$ in Example 1. Next, in Example 2, we calculate the Fourier transform of a complex power of a positive definite quadratic form $P(x)$ by using the fundamental theorem. Here we use the properties of the gamma function proved in §3.4. In Example 3, we prove Witt's theorem on quadratic forms, and using the theorem we give the orbital decomposition of $(GL_1(k) \times SO(f;k),\ \Lambda_1 \otimes \Lambda_1,\ V(1) \otimes V(n))$ with a quadratic form f on an arbitrary field k of characteristic different from two. In particular, the orbital decomposition can be obtained in the case of an indefinite quadratic form $P(x)$ on $\mathbb{R}$. Then we calculate the Fourier transform of a complex power of $P(x)$. In §4.3, we prove a theorem due to Takuro Shintani, which determines the form of $c_{ij}(s)$ appearing in the fundamental theorem. We will give a detailed explanation of what he described briefly in his paper [**79**], since the present book is an introductory one.

In §4.4, we explain Poisson's summation formula. In §4.5, we define the zeta distribution, and we prove its functional equation. The reader may go ahead to Chapter 5, skipping this section.

Chapter 5 is devoted to a general theory of zeta functions associated with prehomogeneous vector spaces. In §5.1, we define arithmetic subgroups, and consider the condition that $G_{\mathbb{R}}/G_{\mathbb{Z}}$ has a finite measure. We define zeta functions in §5.2, and we investigate zeta integrals in §5.3. The functional equation of zeta integrals is obtained from Poisson's summation formula.

In §5.4, using zeta integrals, we give the analytic continuation of a zeta function on the whole s-plane. We also prove the functional equation of zeta functions by that of zeta integrals, i.e., Poisson's summation formula, and the fundamental theorem.

In §5.5, we discuss how to calculate the residues of zeta functions under stronger conditions (Assumptions A and B) on convergence. Here we mention a criterion that tells us whether Assumption A holds for a given concrete case. Although we do not give the proof of the criterion, the reader is expected to understand its usage through an example. In §5.6, we investigate three examples corresponding to those in §4.2. The zeta function of Example 1 is the well-known Riemann zeta function, and that of Example 2 coincides with Epstein's zeta function up to a constant. Example 3 deals with Siegel's zeta function of indefinite quadratic forms. In this case Assumption A in §5.5 is satisfied, so we calculate its residues by using the results in §5.5.

Chapter 6 deals with the convergence of zeta functions.

In §6.1, we mention, without proof, Hiroshi Saito's theorem which gives an extremely general result on convergence. Its proof is beyond the level of this book, so we prove another theorem on convergence under stronger conditions in §§6.2–6.7.

Sections 6.2–6.8 are devoted to an explanation of a part of Fumihiro Sato's paper [**66**]. In §6.2, we reduce the convergence of our zeta function to that of another Dirichlet series. In §6.3, we review p-adic fields briefly. It is important that these fields are locally compact fields.

In §6.4, we give a brief review on adeles, in the case of $\mathbb{Q}$ which is needed later. In §§6.5 and 6.6, using the notion of adeles, we estimate a certain infinite sum and an integral, so that we reduce the convergence of zeta functions associated with a prehomogeneous vector space to that of the multiplicative adelic zeta functions. In §6.7, we prove a result, due to Takashi Ono, concerning the convergence of the multiplicative adelic zeta functions. Thus, in the case where $H_x = \{g \in (\ker \chi)^\circ;\ \rho(g)x = x\}$ is a connected semisimple algebraic group, the convergence theorem of zeta functions is proved.

In §6.8, we prove that the convergence of zeta functions associated with a prehomogeneous vector space is equivalent to that of additive adelic zeta functions. The author learned this part from Professor Fumihiro Sato.

Chapter 7 is devoted to a classification theory.

In §7.1, we explain the **castling transformation**, which is indispensable in our classification theory of irreducible prehomogeneous vector spaces. In §7.2, we review a general theory of irreducible representations. In particular, we prove E. Cartan's theorem which says that $d\rho(\mathfrak{g})$, where $d\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is an irreducible representation of a Lie algebra $\mathfrak{g}$, is a semisimple Lie algebra or a composite of a semisimple Lie algebra and $\{cI_V;\ c \in \mathbb{C}\}$. We also show that an arbitrary irreducible representation $\rho : G_1 \times G_2 \to GL(V)$ over $\mathbb{C}$ can be expressed as a tensor representation $\rho = \rho_1 \otimes \rho_2$, $V \cong V_1 \otimes V_2$ with irreducible representations $\rho_i : G_i \to GL(V_i)$ $(i = 1, 2)$.

In §7.3, first, we review the representation theory and the classification theory of simple Lie algebras over $\mathbb{C}$. Using these, we show that the generic isotropy subalgebra of Example 2.6 $(GL_7, \Lambda_3, V(35))$ of §2.4 is a simple Lie algebra of type G_2, and we introduce the formula to determine the degrees of irreducible representations of G_2. We also show that the composition of the adjoint representation of a simple algebraic group and a scalar multiplication induces a prehomogeneous vector space if and only if the rank of its group is equal to 1.

In §7.4, we describe a classification of irreducible prehomogeneous vector spaces. It follows from the results in §7.2 that if (G, ρ, V) is a prehomogeneous vector space, then we may assume that $G = GL_1 \times G_1 \times \cdots \times G_k$, $\rho = \Lambda_1 \otimes \rho_1 \otimes \cdots \otimes \rho_k$, and $V = V(d_1) \otimes \cdots \otimes V(d_k)$ $(d_1 \geqq \cdots \geqq d_k \geqq 2)$. Here, each G_i is a simple algebraic group, ρ_i is an irreducible representation of G_i on a d_i-dimensional vector space $V(d_i)$, and Λ_1 means the scalar multiplication by GL_1. The principle of our classification is as follows: First, we take out reduced ones satisfying $\dim G \geqq \dim V$ in each case where G_1 is of type A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7, or E_8. Next we check whether each one is a prehomogeneous vector space. Although in this book we carry out it completely only in the case where G_1 is a simple algebraic group of type G_2, the reader can get a feeling for the method of our classification.

Then, we give a table of b-functions associated with irreducible prehomogeneous vector spaces.

In §7.5, we give a table of prehomogeneous vector spaces induced by the composition of a representation $\rho = \rho_1 \oplus \cdots \oplus \rho_l$ of a simple algebraic group G and scalar multiplications ${GL_1}^l$ on each irreducible component. In §7.6, we introduce the notion of weakly spherical spaces, which is a generalization of that of prehomogeneous vector spaces. Using this, we prove the generalized castling transformation. This also gives another proof of the castling transformation.

APPENDIX

Table of Irreducible Reduced Prehomogeneous Vector Spaces

Let (G, ρ, V) be an irreducible reduced prehomogeneous vector space with a relatively invariant irreducible polynomial $f(x)$ which is unique up to a constant multiple.

Let $\chi : G \to GL_1$ be the corresponding character of $f(x)$. Then the dual space (G, ρ^*, V^*) has a relatively invariant irreducible polynomial $f^*(y)$ corresponding to χ^{-1}.

The b-function $b(s)$ of $f(x)$ (or in this case, of (G, ρ, V)) is defined by

$$f^*(\mathrm{grad}_x) f(x)^{s+1} = b(s) f(x)^s.$$

The isotropy subgroup $G_v = \{g \in G;\ \rho(g)v = v\}$ for $v \in V - S$ is called a generic isotropy subgroup. Let G_v° be the connected component of G_v.

In the following table (I), we give (i) $f(x)$, or $\deg f$, (ii) the corresponding character χ, (iii) the b-function $b(s)$, (iv) a generic isotropy subgroup G_v, or its connected component G_v°, and (v) $\ker \rho$, or $G_v/(G_v^\circ \cdot \ker \rho)$.

We denote by S_n the symmetric group of degree n.

(v) is due to A. Sasada, *Generic isotropy subgroup of irreducible prehomogeneous vector spaces with relative invariants*, Kyoto-Math 98-06, Kyoto University, 1998. The reader will notice that item (v) is missing for some entries in table (I). But since the open orbits and the relative invariant of these spaces are the same as those of various other spaces whose item (v) is not missing, we are not really interested in the information of the missing part (v)s from the viewpoint of relative invariants.

(I) Irreducible reduced prehomogeneous vector spaces with relative invariants

(1) $(\boldsymbol{H \times GL_m,\ \sigma \otimes \Lambda_1,\ M(m)})$ $(m \geqq 1)$, where $\sigma : H \to GL_m$ is an irreducible representation of a connected semisimple algebraic group H, and $\rho = \sigma \otimes \Lambda_1$ is given by $x \mapsto \sigma(A) x^t B$ for $g = (A, B) \in H \times GL_m$ and $x \in M(m)$.

(i) $f(x) = \det x$, $\deg f = m$.
(ii) $\chi(g) = \det B$ for $g = (A, B) \in H \times GL_m$.
(iii) $b(s) = \prod_{\nu=1}^{m} (s + \nu) = (s+1)(s+2)\cdots(s+m)$.
(iv) $G_v \cong H$ for $v = I_m$.
(v) $G_v/(G_v^\circ \cdot \ker \rho) \cong S_1$.

(2) $(\boldsymbol{GL_n,\ 2\Lambda_1,\ V(n(n+1)/2)})$ ($n \geqq 2$), where the action $\rho = 2\Lambda_1$ on $\{x \in M(n);\ {}^t x = x\} = V(n(n+1)/2)$ is given by $x \mapsto gx^t g$ for $g \in GL_n$ and $x = {}^t x \in M(n)$.
(i) $f(x) = \det x$, $\deg f = n$.
(ii) $\chi(g) = (\det g)^2$ for $g \in GL_n$.
(iii) $b(s) = \prod_{\nu=1}^{n} \left(s + \frac{\nu+1}{2}\right) = (s+1)\left(s+\frac{3}{2}\right)\cdots\left(s+\frac{n+1}{2}\right)$.
(iv) $G_v = O_n$ for $v = I_n$.
(v) $n = 2m$; even, $G_v/(G_v^\circ \cdot \ker\rho) \cong S_2$ ($\ker\rho = \{\pm I_n\} \subset G_v^\circ = SO_n$).
$n = 2m+1$; odd, $G_v/(G_v^\circ \cdot \ker\rho) \cong S_1$ ($\ker\rho \cap G_v^\circ = \{I_n\}$).

(3) $(\boldsymbol{GL_{2m},\ \Lambda_2,\ V(m(2m-1))})$ ($m \geqq 3$), where the action $\rho = \Lambda_2$ on $\{x \in M(2m);\ {}^t x = -x\} = V(m(2m-1))$ is given by $x \mapsto gx^t g$ for $g \in GL_{2m}$ and $x = -{}^t x \in M(2m)$.
(i) $f(x) = \mathrm{Pf}(x)$, $\deg f = m$.
(ii) $\chi(g) = \det g$ for $g \in GL_{2m}$.
(iii) $b(s) = \prod_{k=1}^{m}(s+2k-1) = (s+1)(s+3)\cdots(s+2m-1)$.
(iv) $G_v = Sp_m$ for $v = J_m = \left(\begin{smallmatrix} O & I_m \\ -I_m & O \end{smallmatrix}\right)$.
(v) $G_v/(G_v^\circ \cdot \ker\rho) \cong S_1$.

(4) $(\boldsymbol{GL_2,\ 3\Lambda_1,\ V(4)})$, where the action $\rho = 3\Lambda_1$ on the space of the binary cubic forms $V(4) = \{F_x(u,v) = x_1u^3 + x_2u^2v + x_3uv^2 + x_4v^3;\ x = {}^t(x_1,x_2,x_3,x_4) \in \mathbb{C}^4\}$ is given by $F_{\rho(g)x}(u,v) = F_x((u,v)g)$ for $g \in GL_2$.
(i) $f(x)$ is "the discriminant of $F_x(u,v)$"
$= {x_2}^2{x_3}^2 + 18x_1x_2x_3x_4 - 4x_1{x_3}^3 - 4{x_2}^3x_4 - 27{x_1}^2{x_4}^2$, $\deg f = 4$.
(ii) $\chi(g) = (\det g)^6$ for $g \in GL_2$.
(iii) $b(s) = (s+1)^2\left(s+\frac{5}{6}\right)\left(s+\frac{7}{6}\right)$.
(iv) $G_{v_0} = \left\{\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right);\ a^3 = d^3 = 1\right\} \cup \left\{\left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right);\ b^3 = c^3 = 1\right\}$
for $v_0 = u^3 + v^3 = {}^t(1,0,0,1)$.
(v) $\ker\rho = \{\omega I_2;\ \omega^3 = 1\}$ and $G_{v_0}/(G_{v_0}^\circ \cdot \ker\rho) \cong S_3$.

(5) $(\boldsymbol{GL_6,\ \Lambda_3,\ V(20)})$, where the action $\rho = \Lambda_3$ on $V(20) = \sum \mathbb{C}u_i \wedge u_j \wedge u_k$ ($1 \leqq i < j < k \leqq 6$) is given by $u_i \wedge u_j \wedge u_k \mapsto \rho_1(g)u_i \wedge \rho_1(g)u_j \wedge \rho_1(g)u_k$ with $\rho_1(g)(u_1,\ldots,u_6) = (u_1,\ldots,u_6)g$ for $g \in GL_6$.
(i) $\deg f = 4$.
(ii) $\chi(g) = (\det g)^2$ for $g \in GL_6$.
(iii) $b(s) = (s+1)\left(s+\frac{5}{2}\right)\left(s+\frac{7}{2}\right)(s+5)$.
(iv) $G_v = \left\{\left(\begin{smallmatrix} A & O \\ O & B \end{smallmatrix}\right);\ A, B \in SL_3\right\} \times \left\{I_6, \left(\begin{smallmatrix} O & I_3 \\ I_3 & O \end{smallmatrix}\right)\right\}$
for $v = u_1 \wedge u_2 \wedge u_3 + u_4 \wedge u_5 \wedge u_6$.
(v) $\ker\rho = \{\omega I_6;\ \omega^3 = 1\} \subset G_v^\circ$ and $G_v/(G_v^\circ \cdot \ker\rho) \cong S_2$.

(6) $(\boldsymbol{GL_7,\ \Lambda_3,\ V(35)})$, where the action $\rho = \Lambda_3$ on $V(35) = \sum \mathbb{C}u_i \wedge u_j \wedge u_k$ ($1 \leqq i < j < k \leqq 7$) is given by $u_i \wedge u_j \wedge u_k \mapsto \rho_1(g)u_i \wedge \rho_1(g)u_j \wedge \rho_1(g)u_k$ with $\rho_1(g)(u_1,\ldots,u_7) = (u_1,\ldots,u_7)g$ for $g \in GL_7$.
(i) $\deg f = 7$.
(ii) $\chi(g) = (\det g)^3$ for $g \in GL_7$.
(iii) $b(s) = (s+1)(s+2)\left(s+\frac{5}{2}\right)\left(s+\frac{7}{2}\right)(s+3)(s+4)(s+5)$.

(iv) $G_v = (G_2) \times \{\omega I_7;\ \omega^3 = 1\}$, where (G_2) is the exceptional algebraic groups of rank 2 (see [**77**], p. 85), and
$v = u_2 \wedge u_3 \wedge u_4 + u_5 \wedge u_6 \wedge u_7 + u_1 \wedge (u_2 \wedge u_5 + u_3 \wedge u_6 + u_4 \wedge u_7)$.
(v) $\ker \rho = \{\omega I_7;\ \omega^3 = 1\}$ and $G_v/(G_v^\circ \cdot \ker \rho) \cong S_1$.

(7) $(\boldsymbol{GL_8,\ \Lambda_3,\ V(56)})$, where the action $\rho = \Lambda_3$ on $V(56) = \sum \mathbb{C} u_i \wedge u_j \wedge u_k$ $(1 \leqq i < j < k \leqq 8)$ is given by $u_i \wedge u_j \wedge u_k \mapsto \rho_1(g)u_i \wedge \rho_1(g)u_j \wedge \rho_1(g)u_k$ with $\rho_1(g)(u_1, \ldots, u_8) = (u_1, \ldots, u_8)g$ for $g \in GL_8$.
(i) $\deg f = 16$.
(ii) $\chi(g) = (\det g)^6$ for $g \in GL_8$.
(iii) $b(s) = (s+1)\left(s+\frac{3}{2}\right)^2\left(s+\frac{11}{6}\right)(s+2)^3\left(s+\frac{13}{6}\right)\left(s+\frac{7}{3}\right)\left(s+\frac{5}{2}\right)^3 \times \left(s+\frac{8}{3}\right)(s+3)^2\left(s+\frac{7}{2}\right)$.
(iv) $G_v^\circ = \mathrm{Ad}(SL_3)$ $(\subset SL_8$; the image of the adjoint representation of $SL_3)$, where $v = u_1 \wedge u_2 \wedge u_3 + u_4 \wedge u_5 \wedge u_6 + u_7 \wedge (u_1 \wedge u_4 - u_2 \wedge u_5) + u_8 \wedge (u_1 \wedge u_4 - u_3 \wedge u_6)$, and $g_1 \in GL_8$ defined by $u_i \mapsto -u_i$ $(i = 1, 4)$, $u_j \mapsto u_{j+1}$ $(j = 2, 5, 7)$ and $u_k \mapsto u_{k-1}$ $(k = 3, 6, 8)$ has $\det g_1 = -1$ and $g_1 \in G_v$ but $g_1 \notin G_v^\circ$.
(v) $G_v/(G_v^\circ \cdot \ker \rho) \cong S_2$ (see [**12**]).

(8) $(\boldsymbol{SL_3 \times GL_2,\ 2\Lambda_1 \otimes \Lambda_1,\ V(6) \otimes V(2)})$, where the action $\rho = 2\Lambda_1 \otimes \Lambda_1$ on $V(6) \otimes V(2) = \{x = (x_1, x_2) \in M(3) \oplus M(3);\ {}^t x_1 = x_1,\ {}^t x_2 = x_2\}$ is given by $\rho(A, B)x = (Ax_1{}^tA, Ax_2{}^tA)^tB$ for $(A, B) \in SL_3 \times GL_2$.
(i) $f(x)$ is the discriminant of the binary cubic form $\det(ux_1 + vx_2)$ for $x = (x_1, x_2)$, $\deg f = 12$.
(ii) $\chi(g) = (\det B)^6$ for $g = (A, B) \in SL_3 \times GL_2$.
(iii) $b(s) = \left\{(s+1)^2\left(s+\frac{5}{6}\right)\left(s+\frac{7}{6}\right)\left(s+\frac{3}{4}\right)\left(s+\frac{5}{4}\right)\right\}^2$.
(iv) G_v is a finite group of order 72 for $v = \left\{\left(I_3, \begin{pmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{pmatrix}\right)\right\}$ with $\omega^3 = 1$, $\omega \neq 1$ (see p. 54).
(v) $\ker \rho = \{(\omega I_3, \omega I_2);\ \omega^3 = 1\}$ and $G_v/(G_v^\circ \cdot \ker \rho) \cong S_4$.

(9) $(\boldsymbol{SL_6 \times GL_2,\ \Lambda_2 \otimes \Lambda_1,\ V(15) \otimes V(2)})$, where the action $\rho = \Lambda_2 \otimes \Lambda_1$ on $V(15) \otimes V(2) = \{x = (x_1, x_2) \in M(6) \oplus M(6);\ {}^t x_1 = -x_1,\ {}^t x_2 = -x_2\}$ is given by $\rho(g)x = (Ax_1{}^tA, Ax_2{}^tA)^tB$ for $g = (A, B) \in SL_6 \times GL_2$ and $x = (x_1, x_2)$.
(i) $f(x)$ is the discriminant of the binary cubic form $\mathrm{Pf}(ux_1 + vx_2)$ for $x = (x_1, x_2)$ and $\deg f = 12$.
(ii) $\chi(g) = (\det B)^6$ for $g = (A, B) \in SL_6 \times GL_2$.
(iii) $b(s) = (s+1)^2\left(s+\frac{5}{6}\right)\left(s+\frac{7}{6}\right)\left(s+\frac{3}{2}\right)^2(s+2)^2\left(s+\frac{5}{2}\right)^2\left(s+\frac{7}{3}\right) \times \left(s+\frac{8}{3}\right)$.
(iv) $G_v^\circ = \left\{\left(\begin{pmatrix} \mathrm{diag}(a_1,a_2,a_3) & \mathrm{diag}(b_1,b_2,b_3) \\ \mathrm{diag}(c_1,c_2,c_3) & \mathrm{diag}(d_1,d_2,d_3) \end{pmatrix}, I_2\right);\ \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in SL_2\right\} \cong SL_2 \times SL_2 \times SL_2$ for $v = \left\{\begin{pmatrix} O & I_3 \\ -I_3 & O \end{pmatrix}, \begin{pmatrix} O & \Lambda \\ -\Lambda & O \end{pmatrix}\right\}$ with $\Lambda = \begin{pmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{pmatrix}$ $(\omega^3 = 1$, $\omega \neq 1)$.
(v) $\ker \rho = \{(\omega I_6, \omega^{-2} I_2);\ \omega^6 = 1\}$ and $G_v/(G_v^\circ \cdot \ker \rho) \cong S_3$.

(10) $(\boldsymbol{SL_5 \times GL_3,\ \Lambda_2 \otimes \Lambda_1,\ V(10) \otimes V(3)})$, where the action $\rho = \Lambda_2 \otimes \Lambda_1$ on $V(10) \otimes V(3) = V(10) \oplus V(10) \oplus V(10)$ is given by $\rho(g)x = (\Lambda_2(A)x_1, \Lambda_2(A)x_2, \Lambda_2(A)x_3)^t B$ for $g = (A, B) \in SL_5 \times GL_3$ and the action Λ_2 on $V(10) = \sum \mathbb{C}u_i \wedge u_j$ $(1 \leqq i < j \leqq 5)$ is given by $\Lambda_2(A)(u_i \wedge u_j) = \rho_1(A)u_i \wedge \rho_1(A)u_j$ with $\rho_1(A)(u_1, \ldots, u_5) = (u_1, \ldots, u_5)A$ for $A \in SL_5$.

(i) $\deg f = 15$.

(ii) $\chi(g) = (\det B)^5$ for $g = (A, B) \in SL_5 \times GL_3$.

(iii) $b(s) = \left\{(s+1)\left(s+\frac{3}{2}\right)(s+2)\right\}^3 \cdot \left\{\left(s+\frac{4}{3}\right)\left(s+\frac{5}{3}\right)\right\}^2 \left(s+\frac{5}{4}\right)\left(s+\frac{7}{4}\right)$.

(iv) $\mathrm{Lie}(G_v) = \{(4\Lambda_1(C), 2\Lambda_1(C)) \in \mathfrak{sl}_5 \oplus \mathfrak{sl}_3;\ C \in \mathfrak{sl}_2\} \cong \mathfrak{sl}_2$ and $G_v^\circ \cong SL_2/\{\pm I_2\}$ for $v = (u_1 \wedge u_2 + u_3 \wedge u_4,\ u_2 \wedge u_3 + u_4 \wedge u_5,\ u_1 \wedge u_3 + u_2 \wedge u_5)$.

(v) $\ker \rho = \{(\omega I_5, \omega^{-2} I_3);\ \omega^5 = 1\}$ and $G_v/(G_v^\circ \cdot \ker \rho) \cong S_1$.

(11) $(\boldsymbol{SL_5 \times GL_4,\ \Lambda_2 \otimes \Lambda_1,\ V(10) \otimes V(4)})$, where the action $\rho = \Lambda_2 \otimes \Lambda_1$ on $V(10) \otimes V(4) = V(10) \oplus V(10) \oplus V(10) \oplus V(10)$ is given by $\rho(g)x = (\Lambda_2(A)x_1, \ldots, \Lambda_2(A)x_4)^t B$ for $g = (A, B) \in SL_5 \times GL_4$ and the action Λ_2 on $V(10) = \sum \mathbb{C}u_i \wedge u_j$ $(1 \leqq i < j \leqq 5)$ is given by $\Lambda_2(A)(u_i \wedge u_j) = \rho_1(A)u_i \wedge \rho_1(A)u_j$ with $\rho_1(A)(u_1, \ldots, u_5) = (u_1, \ldots, u_5)A$ for $A \in SL_5$.

(i) $\deg f = 40$. (See Corollary 2.20.)

(ii) $\chi(g) = (\det B)^{10}$ for $g = (A, B) \in SL_5 \times GL_4$.

(iii) $b(s) = \prod_{i=1}^3 \left(s + \frac{1}{2} + \frac{i}{4}\right)^4 \cdot \prod_{j=1}^4 \left(s + \frac{1}{2} + \frac{j}{5}\right)^2 \cdot \prod_{k=1}^5 \left(s + \frac{1}{2} + \frac{k}{6}\right)^4 = (s+1)^8 \cdot \left\{\left(s+\frac{2}{3}\right)\left(s+\frac{4}{3}\right)\left(s+\frac{3}{4}\right)\left(s+\frac{5}{4}\right)\left(s+\frac{5}{6}\right)\left(s+\frac{7}{6}\right)\right\}^4 \times \left\{\left(s+\frac{7}{10}\right)\left(s+\frac{9}{10}\right)\left(s+\frac{11}{10}\right)\left(s+\frac{13}{10}\right)\right\}^2$.
The microlocal structure of this prehomogeneous vector space is very complicated. The b-function $b(s)$ was conjectured by Ikuzo Ozeki and was solved affirmatively in T. Yano and I. Ozeki, *Microlocal structure of the regular prehomogeneous vector space associated with $SL(5) \times GL(4)$, II*, Sūrikaisekikenkyūsho Kōkyūroku **999** (1997), 92–115 (in English).

(iv) G_v is a finite group of order 600 for $v = (u_1 \wedge u_2 + u_3 \wedge u_4,\ u_2 \wedge u_3 + u_4 \wedge u_5,\ u_1 \wedge u_3 + u_2 \wedge u_5,\ u_2 \wedge u_4 + u_3 \wedge u_5)$.

(v) $\ker \rho = \{(\omega I_5, \omega^{-2} I_4);\ \omega^5 = 1\}$, and $G_v/(G_v^\circ \cdot \ker \rho) \cong G_v/\ker \rho \cong S_5$.

(12) $(\boldsymbol{SL_3 \times SL_3 \times GL_2,\ \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1,\ V(3) \otimes V(3) \otimes V(2)})$, where the action $\rho = \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1$ on $V(3) \otimes V(3) \otimes V(2) = M(3) \oplus M(3)$ is given by $\rho(g)x = (Ax_1{}^t B, Ax_2{}^t B)^t C$ for $g = (A, B, C) \in SL_3 \times SL_3 \times GL_2$ and $x = (x_1, x_2) \in M(3) \oplus M(3)$.

(i) $f(x)$ is the discriminant of the binary cubic form $\det(ux_1 + vx_2)$ for $x = (x_1, x_2)$, $\deg f = 12$.

(ii) $\chi(g) = (\det C)^6$ for $g = (A, B, C) \in SL_3 \times SL_3 \times GL_2$.

(iii) $b(s) = (s+1)^4 \left(s+\frac{3}{2}\right)^4 \left(s+\frac{4}{3}\right)\left(s+\frac{5}{3}\right)\left(s+\frac{5}{6}\right)\left(s+\frac{7}{6}\right)$.

(iv) $G_v^\circ = \{(\mathrm{diag}(a, b, a^{-1}b^{-1}), \mathrm{diag}(a^{-1}, b^{-1}, ab), I_2);\ a, b \in GL_1\} \cong GL_1 \times GL_1$ for $v = \left(I_3, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$.

(v) $\ker \rho = \{(\omega I_3, \omega' I_3, \omega^{-1}\omega'^{-1} I_2);\ \omega^3 = \omega'^3 = 1\}$ and $G_v/(G_v^\circ \cdot \ker \rho) \cong S_3$.

(13) $(\boldsymbol{Sp_n \times GL_{2m},\ \Lambda_1 \otimes \Lambda_1,\ V(2n) \otimes V(2m)})$ $(n \geqq 2m \geqq 2)$, where the action $\rho = \Lambda_1 \otimes \Lambda_1$ on $V(2n) \otimes V(2m) = M(2n, 2m)$ is given by $\rho(g)x = Ax^tB$, for $g = (A, B) \in Sp_n \times GL_{2m}$ and $x \in M(2n, 2m)$.

(i) $f(x) = \mathrm{Pf}({}^txJx)$ for $x \in M(2n, 2m)$ and $J = \left(\begin{smallmatrix} O & I_n \\ -I_n & O \end{smallmatrix}\right)$, $\deg f = 2m$.

(ii) $\chi(g) = \det B$ for $g = (A, B) \in Sp_n \times GL_{2m}$.

(iii) $b(s) = \prod_{k=1}^{m}(s + 2k - 1) \cdot \prod_{l=0}^{m-1}(s + 2n - 2l) = (s+1)(s+3)\cdots(s+2m-1)(s+2n)(s+2n-2)\cdots(s+2n-2m+2)$.

(iv) $G_v = \left\{ \left(\begin{pmatrix} A_1 & O & B_1 & O \\ O & A_2 & O & B_2 \\ C_1 & O & D_1 & O \\ O & C_2 & O & D_2 \end{pmatrix}, {}^t\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}^{-1} \right);\ \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in Sp_m,\ \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \in Sp_{n-m} \right\}$
$\cong Sp_m \times Sp_{n-m}$ for $v = {}^t\begin{pmatrix} I_m & O_{m,n-m} & O_m & O_{m,n-m} \\ O_m & O_{m,n-m} & I_m & O_{m,n-m} \end{pmatrix} \in M(2n, 2m)$.

(v) $\ker\rho = \{\pm(I_{2n}, I_{2m})\} \subset G_v^\circ = G_v$ and $G_v/(G_v^\circ \cdot \ker\rho) \cong S_1$.

(14) $(\boldsymbol{GL_1 \times Sp_3,\ \Lambda_1 \otimes \Lambda_3,\ V(1) \otimes V(14)})$, where the action $\rho = \Lambda_1 \otimes \Lambda_3$ on $V(1) \otimes V(14) = \{x = \sum x_{ijk}u_i \wedge u_j \wedge u_k;\ 1 \leqq i < j < k \leqq 6,\ x_{l14} + x_{l25} + x_{l36} = 0\ (1 \leqq l \leqq 6)\}$ is given by $u_i \wedge u_j \wedge u_k \mapsto t\rho_1(A)u_i \wedge \rho_1(A)u_j \wedge \rho_1(A)u_k$ with $\rho_1(A)(u_1, \ldots, u_6) = (u_1, \ldots, u_6)A$ for $(t, A) \in GL_1 \times Sp_3$.

(i) $\deg f = 4$.

(ii) $\chi(g) = t^4$ for $g = (t, A) \in GL_1 \times Sp_3$.

(iii) $b(s) = (s+1)\left(s+2\right)\left(s+\frac{5}{2}\right)\left(s+\frac{7}{2}\right)$.

(iv) $G_v^\circ = \left\{\left(1, \left(\begin{smallmatrix} B & O \\ O & {}^tB^{-1} \end{smallmatrix}\right)\right);\ B \in SL_3\right\} \cong SL_3$ for $v = u_1 \wedge u_2 \wedge u_3 + u_4 \wedge u_5 \wedge u_6$, and $g_1 = \left(\sqrt{-1}, \left(\begin{smallmatrix} O & \sqrt{-1}I_3 \\ \sqrt{-1}I_3 & O \end{smallmatrix}\right)\right) \in G_v$ but $g_1 \notin G_v^\circ$.

(v) $\ker\rho = \{\pm(1, I_6)\}$ and $G_v/(G_v^\circ \cdot \ker\rho) \cong S_2$.

(15) $(\boldsymbol{SO_n \times GL_m,\ \Lambda_1 \otimes \Lambda_1,\ V(n) \otimes V(m)})$ $(n \geqq 2m \geqq 2)$, where the action $\rho = \Lambda_1 \otimes \Lambda_1$ on $V(n) \otimes V(m) = M(n, m)$ is given by $\rho(g)x = Ax^tB$ for $g = (A, B) \in SO_n \times GL_m$ and $x \in M(n, m)$.

(i) $f(x) = \det({}^txx)$, $\deg f = 2m$.

(ii) $\chi(g) = (\det B)^2$ for $g = (A, B) \in SO_n \times GL_m$.

(iii) $b(s) = \prod_{k=1}^{m}\left(s + \frac{k+1}{2}\right) \cdot \prod_{l=1}^{m}\left(s + \frac{n-l+1}{2}\right)$
$= (s+1)\left(s+\frac{3}{2}\right)\cdots\left(s+\frac{m+1}{2}\right)\left(s+\frac{n}{2}\right)\left(s+\frac{n-1}{2}\right)\cdots\left(s+\frac{n-m+1}{2}\right)$.

(iv) $G_v = \left\{\left(\left(\begin{smallmatrix} A_1 & O \\ O & A_2 \end{smallmatrix}\right), {}^tA_1{}^{-1}\right);\ A_1 \in O_m,\ A_2 \in O_{n-m},\ \det A_1 \cdot \det A_2 = 1\right\}$ and $G_v^\circ \cong SO_m \times SO_{n-m}$ for $v = {}^t(I_m, O) \in M(n, m)$.

(v) $\ker\rho = \begin{cases} (I_n, I_m) & (n = \text{odd}), \\ \pm(I_n, I_m) & (n = \text{even}), \end{cases}$
and $G_v/(G_v^\circ \cdot \ker\rho) = \begin{cases} S_1 & (n = \text{even},\ m = \text{odd}), \\ S_2 & (\text{otherwise}). \end{cases}$

(16) $(\boldsymbol{GL_1 \times Spin_7,\ \Lambda_1 \otimes \sigma,\ V(1) \otimes V(8)})$, where σ is the spin representation of $Spin_7$. Since $\Lambda_1 \otimes \sigma(GL_1 \times Spin_7) \subset \Lambda_1 \otimes \Lambda_1(GL_1 \times SO_8)$, this space reduces to (15) with $n = 8$ and $m = 1$.

(i) $\deg f = 2$.

(ii) $\chi(g) = t^2$ for $g = (t, A) \in GL_1 \times Spin_7$.

(iii) $b(s) = (s+1)(s+4)$.

(iv) $G_v^\circ = (G_2)$ for $v = 1 + e_1e_2e_3e_4$.

(17) $(\boldsymbol{Spin_7 \times GL_2,\ \sigma \otimes \Lambda_1,\ V(8) \otimes V(2)})$, where σ is the spin representation of $Spin_7$. Since $\sigma \otimes \Lambda_1(Spin_7 \times GL_2) \subset \Lambda_1 \otimes \Lambda_1(SO_8 \times GL_2)$, this space reduces to (15) with $n = 8$ and $m = 2$.
(i) $\deg f = 4$.
(ii) $\chi(g) = (\det B)^2$ for $g = (A, B) \in Spin_7 \times GL_2$.
(iii) $b(s) = (s+1)\left(s+\frac{3}{2}\right)(s+4)\left(s+\frac{7}{2}\right)$.
(iv) $G_v^\circ \cong SL_3 \times SO_2$ for $v = (1, e_1e_2e_3e_4)$.

(18) $(\boldsymbol{Spin_7 \times GL_3,\ \sigma \otimes \Lambda_1,\ V(8) \otimes V(3)})$, where σ is the spin representation of $Spin_7$. Since $\sigma \otimes \Lambda_1(Spin_7 \times GL_3) \subset \Lambda_1 \otimes \Lambda_1(SO_8 \times GL_3)$, this space reduces to (15) with $n = 8$ and $m = 3$.
(i) $\deg f = 6$.
(ii) $\chi(g) = (\det B)^2$ for $g = (A, B) \in Spin_7 \times GL_3$.
(iii) $b(s) = (s+1)\left(s+\frac{3}{2}\right)(s+2)(s+4)\left(s+\frac{7}{2}\right)(s+3)$.
(iv) $G_v^\circ \cong SL_2 \times SO_3$ for $v = (1,\ e_1e_2e_3e_4,\ e_1e_2 - e_3e_4)$.

(19) $(\boldsymbol{GL_1 \times Spin_9,\ \Lambda_1 \otimes \sigma,\ V(1) \otimes V(16)})$, where σ is the spin representation of $Spin_9$. Since $\Lambda_1 \otimes \sigma(GL_1 \times Spin_9) \subset \Lambda_1 \otimes \Lambda_1(GL_1 \times SO_{16})$, this space reduces to (15) with $n = 16$ and $m = 1$.
(i) $\deg f = 2$.
(ii) $\chi(g) = t^2$ for $g = (t, A) \in GL_1 \times Spin_9$.
(iii) $b(s) = (s+1)(s+8)$.
(iv) $G_v^\circ \cong Spin_7$ for $v = 1 + e_1e_2e_3e_4$.

(20) $(\boldsymbol{Spin_{10} \times GL_2,\ \sigma \otimes \Lambda_1,\ V(16) \otimes V(2)})$, where σ is the even half-spin representation of $Spin_{10}$.
(i) $\deg f = 4$.
(ii) $\chi(g) = (\det B)^2$ for $g = (A, B) \in Spin_{10} \times GL_2$.
(iii) $b(s) = (s+1)(s+4)(s+5)(s+8)$.
(iv) $G_v^\circ = (G_2) \times SL_2$ for $v = (1 + e_1e_2e_3e_4,\ e_1e_5 + e_2e_3e_4e_5)$.
(v) $\ker \rho = \{\pm(1, I_2), \pm(z_0, \sqrt{-1}I_2)\}$ with $z_0 = (\sqrt{-1})^5 \cdot \prod_{i=1}^5 (e_if_i - f_ie_i)$, and $G_v/(G_v^\circ \cdot \ker \rho) \cong S_1$.

(21) $(\boldsymbol{Spin_{10} \times GL_3,\ \sigma \otimes \Lambda_1,\ V(16) \otimes V(3)})$, where σ is the even half-spin representation of $Spin_{10}$.
(i) $\deg f = 12$.
(ii) $\chi(A, B) = (\det B)^4$ for $g = (A, B) \in Spin_{10} \times GL_3$.
(iii) $b(s) = (s+1)\left(s+\frac{3}{2}\right)(s+2)^2(s+3)^2\left(s+\frac{7}{2}\right)(s+4)\left(s+\frac{5}{3}\right)\left(s+\frac{7}{3}\right) \times \left(s+\frac{8}{3}\right)\left(s+\frac{10}{3}\right)$.
(iv) $G_v^\circ \cong SL_2 \times SO_3$ for $v = (1 + e_1e_2e_3e_4,\ e_1e_5 + e_2e_3e_4e_5,\ e_1e_2 + e_1e_3e_4e_5)$.
(v) $\ker \rho = \{\pm(1, I_3), \pm(z_0, \sqrt{-1}I_3)\}$ with $z_0 = (\sqrt{-1})^5 \cdot \prod_{i=1}^5 (e_if_i - f_ie_i)$ and $G_v/(G_v^\circ \cdot \ker \rho) \cong S_1$.

(22) $(\boldsymbol{GL_1 \times Spin_{11},\ \Lambda_1 \otimes \sigma,\ V(1) \otimes V(32)})$, where σ is the spin representation of $Spin_{11}$. Since $\sigma(Spin_{11}) \subset$ the even half-spin representation of $Spin_{12}$, this space reduces to (23).
(i) $\deg f = 4$.

(ii) $\chi(g) = t^4$ for $g = (t, A) \in GL_1 \times Spin_{11}$.
(iii) $b(s) = (s+1)\left(s+\frac{7}{2}\right)\left(s+\frac{11}{2}\right)(s+8)$.
(iv) $G_v^\circ \cong SL_5$ for $v = 1 + e_1e_2e_3e_4e_5e_6$.

(23) $(\boldsymbol{GL_1 \times Spin_{12}}, \boldsymbol{\Lambda_1 \otimes \sigma}, \boldsymbol{V(1) \otimes V(32)})$, where σ is the even half-spin representation.
(i) $\deg f = 4$.
(ii) $\chi(g) = t^4$ for $g = (t, A) \in GL_1 \times Spin_{12}$.
(iii) $b(s) = (s+1)\left(s+\frac{7}{2}\right)\left(s+\frac{11}{2}\right)(s+8)$.
(iv) $G_v^\circ \cong SL_6$ for $v = 1 + e_1e_2e_3e_4e_5e_6$.
(v) $\ker\rho = \{\pm(1,1), \pm(-1, z_0)\}$ with $z_0 = (\sqrt{-1})^6 \cdot \prod_{i=1}^6 (e_if_i - f_ie_i)$ and $G_v/(G_v^\circ \cdot \ker\rho) \cong S_2$.

(24) $(\boldsymbol{GL_1 \times Spin_{14}}, \boldsymbol{\Lambda_1 \otimes \sigma}, \boldsymbol{V(1) \otimes V(64)})$, where σ is the even half-spin representation.
(i) $\deg f = 8$.
(ii) $\chi(g) = t^8$ for $g = (t, A) \in GL_1 \times Spin_{14}$.
(iii) $b(s) = (s+1)\left(s+\frac{5}{2}\right)\left(s+\frac{7}{2}\right)(s+4)(s+5)\left(s+\frac{11}{2}\right)\left(s+\frac{13}{2}\right)(s+8)$.
(iv) $G_v^\circ \cong (G_2) \times (G_2)$ for $v = 1 + e_1e_2e_3e_7 + e_4e_5e_6e_7 + e_1e_2e_3e_4e_5e_6$.
(v) $\ker\rho = \{\pm(1,1), \pm(-\sqrt{-1}, z_0)\}$ with $z_0 = (\sqrt{-1})^7 \cdot \prod_{i=1}^7 (e_if_i - f_ie_i)$, and $G_v/(G_v^\circ \cdot \ker\rho) \cong S_2$.

(25) $(\boldsymbol{GL_1 \times (G_2)}, \boldsymbol{\Lambda_1 \otimes \Lambda_2}, \boldsymbol{V(1) \otimes V(7)})$. Since $\Lambda_2(G_2) \subset \Lambda_1(SO_7)$, this space reduces to (15) with $n = 7$ and $m = 1$.
(i) $\deg f = 2$.
(ii) $\chi(g) = t^2$ for $g = (t, A) \in GL_1 \times (G_2)$.
(iii) $b(s) = (s+1)\left(s+\frac{7}{2}\right)$.
(iv) $G_v^\circ \cong SL_3$ for $v = {}^t(1,0,0,0,0,0,0)$.

(26) $(\boldsymbol{(G_2) \times GL_2}, \boldsymbol{\Lambda_2 \otimes \Lambda_1}, \boldsymbol{V(7) \otimes V(2)})$. Since $\Lambda_2(G_2) \subset \Lambda_1(SO_7)$, this space reduces to (15) with $n = 7$ and $m = 2$.
(i) $\deg f = 4$.
(ii) $\chi(g) = (\det B)^2$ for $g = (A, B) \in (G_2) \times GL_2$.
(iii) $b(s) = (s+1)\left(s+\frac{3}{2}\right)\left(s+\frac{7}{2}\right)(s+3)$.
(iv) $G_v^\circ \cong GL_2$ for $v = {}^t\left(\begin{smallmatrix} 0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0 \end{smallmatrix}\right)$.

(27) $(\boldsymbol{GL_1 \times E_6}, \boldsymbol{\Lambda_1 \otimes \Lambda_1}, \boldsymbol{V(1) \otimes V(27)})$. The space $V(1) \otimes V(27)$ is the exceptional simple Jordan algebra

$$\mathcal{J} = \left\{ x = \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix}; \quad \begin{array}{l} \xi_1, \xi_2, \xi_3 \in \mathbb{C}, \\ x_1, x_2, x_3 \in \mathfrak{C} \end{array} \right\},$$

where $\mathfrak{C}$ denotes the Cayley algebra over $\mathbb{C}$.
(i) $f(x) = \det x = \xi_1\xi_2\xi_3 + \operatorname{tr} x_1x_2x_3 - \xi_1x_1\overline{x_1} - \xi_2x_2\overline{x_2} - \xi_3x_3\overline{x_3}$, $\deg f = 3$.
(ii) $\chi(g) = t^3$ for $g = (t, A) \in GL_1 \times E_6$.
(iii) $b(s) = (s+1)(s+5)(s+9)$.
(iv) $G_v^\circ \cong F_4$ for $v = I_3 \in \mathcal{J}$.
(v) $\sharp \ker\rho = 3$, and $G_v/(G_v^\circ \cdot \ker\rho) \cong S_1$.

(28) $(\boldsymbol{E_6 \times GL_2, \Lambda_1 \otimes \Lambda_1, V(27) \otimes V(2)})$. The space $V(27) \otimes V(2)$ is identified with $\mathcal{J} \oplus \mathcal{J}$, where $\mathcal{J}$ is the exceptional simple Jordan algebra.
(i) $f(x)$ is the discriminant of the binary cubic form $\det(ux_1 + vx_2)$ for $x = (x_1, x_2) \in \mathcal{J} \oplus \mathcal{J}$, $\deg f = 12$.
(ii) $\chi(g) = (\det B)^6$ for $g = (A, B) \in E_6 \times GL_2$.
(iii) $b(s) = (s+1)^2 \left(s + \frac{5}{6}\right) \left(s + \frac{7}{6}\right) \left(s + \frac{5}{2}\right)^2 (s+3)^2 \left(s + \frac{9}{2}\right)^2 \left(s + \frac{13}{3}\right) \times \left(s + \frac{14}{3}\right)$.
(iv) $G_v^\circ \cong SO_8$ with $v = \left(I_3, \begin{pmatrix} 1 & & \\ 0 & & \\ & -1 & \end{pmatrix}\right) \in \mathcal{J} \oplus \mathcal{J}$.
(v) $\sharp \ker \rho = 3$ and $G_v/(G_v^\circ \cdot \ker \rho) \cong S_3$.

(29) $(\boldsymbol{GL_1 \times E_7, \Lambda_1 \otimes \Lambda_6, V(1) \otimes V(56)})$.
(i) $\deg f = 4$.
(ii) $\chi(g) = t^4$ for $g = (t, A) \in GL_1 \times E_7$.
(iii) $b(s) = (s+1)\left(s + \frac{11}{2}\right)\left(s + \frac{19}{2}\right)(s+14)$.
(iv) $G_v^\circ \cong E_6$.
(v) $\sharp \ker \rho = 4$, and $G_v/(G_v^\circ \cdot \ker \rho) \cong S_2$.

(30) $(\boldsymbol{GL_1 \times Sp_n \times SO_3, \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1, V(1) \otimes V(2n) \otimes V(3)})$ $(n \geqq 2)$. The action $\rho = \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1$ on $V(1) \otimes V(2n) \otimes V(3) = M(2n, 3)$ is given by $\rho(g)x = tAx^tB$ for $g = (t, A, B) \in GL_1 \times Sp_n \times SO_3$ and $x \in M(2n, 3)$.
(i) $f(x) = \mathrm{tr}({}^txJx)^2$ for $x \in M(2n, 3)$, $\deg f = 4$.
(ii) $\chi(g) = t^4$ for $g = (t, A, B) \in GL_1 \times Sp_n \times SO_3$.
(iii) $b(s) = (s+1)\left(s + \frac{3}{2}\right)\left(s + \frac{2n}{2}\right)\left(s + \frac{2n+1}{2}\right)$.
(iv) $G_v^\circ \cong (Sp_{n-2} \times SO_2) \cdot U(2n-3)$, where $U(2n-3)$ denotes the $(2n-3)$-dimensional unipotent group for $v = {}^t\left(\begin{smallmatrix} 1 & 0 & \\ 0 & 1 & O_{3,n-2} \\ 0 & 0 & \end{smallmatrix}\middle|\begin{smallmatrix} 0 & 0 & \\ 0 & 0 & O_{3,n-2} \\ 1 & 0 & \end{smallmatrix}\right) \in M(2n, 3)$.
(v) $\ker \rho = \{(tk, tI_{2n}, kI_3);\ t^2 = k^2 = 1\}$ and $G_v/(G_v^\circ \cdot \ker \rho) \cong S_2$.

(II) Irreducible reduced prehomogeneous vector spaces without relative invariants

(1) $(\boldsymbol{H \times GL_m, \sigma \otimes \Lambda_1, M(n, m)})$ $(n < m)$,
(1)′ $(\boldsymbol{H \times SL_m, \sigma \otimes \Lambda_1, M(n, m)})$ $(n < m)$, where $\sigma : H \to GL_m$ is an irreducible representation of a connected semisimple algebraic group H, and $\rho = \sigma \otimes \Lambda_1$ is given by $x \mapsto \sigma(A)x^tB$ for $g = (A, B) \in H \times GL_m$ (resp. $H \times SL_m$) and $x \in M(n, m)$.

When $(H, \sigma) = (SL_n, \Lambda_1)$, we need $m \geqq 2n$ to be reduced.

(2) $(\boldsymbol{GL_{2m+1}, \Lambda_2, V(m(2m+1))})$,
(2)′ $(\boldsymbol{SL_{2m+1}, \Lambda_2, V(m(2m+1))})$. The action Λ_2 on $V(m(2m+1)) = \{x \in M(2m+1);\ {}^tx = -x\}$ is given by $x \mapsto gx^tg$ for $g \in GL_{2m+1}$ (resp. SL_{2m+1}) and $x = -{}^tx \in M(2m+1)$.

(3) $(\boldsymbol{SL_{2m+1} \times GL_2, \Lambda_2 \otimes \Lambda_1, V(m(2m+1)) \otimes V(2)})$ $(m \geqq 2)$,
(3)′ $(\boldsymbol{SL_{2m+1} \times SL_2, \Lambda_2 \otimes \Lambda_1, V(m(2m+1)) \otimes V(2)})$ $(m \geqq 2)$. The action $\rho = \Lambda_2 \otimes \Lambda_1$ on $V(m(2m+1)) \otimes V(2) = \{x = (x_1, x_2) \in M(2m+1) \oplus M(2m+1);\ {}^tx_1 = -x_1,\ {}^tx_2 = -x_2\}$ is given by $\rho(g)x = (Ax_1{}^tA, Ax_2{}^tA)^tB$

for $g = (A, B) \in SL_{2m+1} \times GL_2$ (resp. $SL_{2m+1} \times SL_2$) and $x = (x_1, x_2)$.

(4) $\boldsymbol{(Sp_n \times GL_{2m+1},\ \Lambda_1 \otimes \Lambda_1,\ V(2n) \otimes V(2m+1))}$ $(n > 2m+1 \geqq 1)$,

(4)′ $\boldsymbol{(Sp_n \times SL_{2m+1},\ \Lambda_1 \otimes \Lambda_1,\ V(2n) \otimes V(2m+1))}$ $(n > 2m+1 \geqq 1)$. The action $\rho = \Lambda_1 \otimes \Lambda_1$ on $V(2n) \otimes V(2m+1) = M(2n, 2m+1)$ is given by $\rho(g)x = Ax^tB$ for $g = (A, B) \in Sp_n \times GL_{2m+1}$ (resp. $Sp_n \times SL_{2m+1}$) and $x \in M(2n, 2m+1)$.

(5) $\boldsymbol{(GL_1 \times Spin_{10},\ \Lambda_1 \otimes \sigma,\ V(1) \otimes V(16))}$,

(5)′ $\boldsymbol{(Spin_{10},\ \sigma,\ V(16))}$, where σ is the even half-spin representation of $Spin_{10}$.

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Bibliography

In fact, for prehomogeneous vector spaces we have a useful bibliography [**70**] made by F. Sato. However, since it is written in Japanese, we give a short bibliography and some comments for the convenience of the readers. It should be emphasized that this bibliography is not complete and important references might not be quoted.

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Comments on the bibliography

Prehistory. The theory of prehomogeneous vector spaces was originated by M. Sato around 1960. The motivation that led him to prehomogeneous vector spaces is stated in [**74**]. Let us quote a part of his words (translated by A. Gyoja in [**15**]):

> Now, as for the linear partial differential equations, what is particular for the classical Laplace equations and the wave equations is that their fundamental solutions can be explicitly described. Therefore, as a testing ground for a general theory of linear differential equations, I tried to find differential operators of higher order other than those of quadratic type. If possible, I wanted to find the general scheme of the differential operators with constant coefficients which are given by some homogeneous polynomials and whose fundamental solutions can be expressed as rational powers or log of some homogeneous polynomials. I remember it was late 1960 or early 61.

As originally expected, this theory worked as the "testing ground" for the general theory of linear differential equations, which is now called algebraic analysis. For example, M. Sato, M. Kashiwara, T. Kimura and T. Oshima [**76**] is the first application of algebraic analysis to prehomogeneous vector spaces.

The papers of M. Sato and T. Shintani. The first article on prehomogeneous vector spaces is M. Sato [**74**] (lecture notes by T. Shintani in Japanese). In this article the fundamental theorem and its supplement by T. Shintani are proved under the assumption that the group G is reductive and the singular set S is a hypersurface. However, he does not assume the irreducibility of S here. In this case, the b-functions and so on are functions in several variables. This paper discusses the zeta distributions, but not the zeta functions as analytic functions. Chapter I, "Algebraic theory of prehomogeneous vector spaces", of [**74**] is translated into English by M. Muro in [**75**]. At any rate, this article has been a fundamental reference for a long time.

Secondly, we will pick up T. Shintani [**79**]. In the first part, he introduces, as Sato's theory, the fundamental theorem and its supplement in the case that the group G is reductive and the singular set S is an irreducible hypersurface. The proof here imitates that of M. Sato [**74**], and is the same as that of this book. In the later part, Shintani investigates the Dirichlet series whose coefficients involve class numbers of integral binary cubic forms. These are nothing but the zeta functions of the space of binary cubic forms, which is Example 2.4 of §2.4. He also gives an application of the zeta functions to asymptotic estimation of class numbers.

Through the above work on particular zeta functions, M. Sato and T. Shintani [**78**] constructed the general theory of the zeta functions in one variable. In this paper, they prove the fundamental theorem in the case that the group G is reductive and the singular set S is an irreducible hypersurface by using a different method from M. Sato [**74**] or T. Shintani [**79**]. The functional equations and the analytic continuations of zeta functions were first established in this article [**78**]. They also discuss the poles of zeta functions, under Assumptions A and B in §5.5.

T. Shintani [**80**] investigates the zeta functions of the space of symmetric matrices, namely, Example 2.2 $(GL_n, 2\Lambda_1)$ of §2.4. As mentioned in §5.2, the space $(GL_2, 2\Lambda_1)$ with the split form does not satisfy Assumption 4, and we cannot appeal to the general theory. The first part of this article is devoted to proving the functional equations of the zeta function of $(GL_2, 2\Lambda_1)$. These zeta functions can be regarded as Siegel's zeta functions of the ternary zero form $xz - y^2$, and these are also studied in F. Sato [**64**]. In the latter part of [**80**], T. Shintani discusses the zeta functions of $(GL_n, 2\Lambda_1)$ with $n \geq 3$, and shows that the special values of them are related to certain integrals, which appear in the dimension formula for Siegel's modular cusp forms.

Algebraic theory. The algebraic theory of prehomogeneous vector spaces is explained in M. Sato [**74**], M. Sato and T. Kimura [**77**] and A. Gyoja [**13**]. The explicit constructions of the relative invariants of irreducible prehomogeneous vector spaces are given by M. Sato, H. Kawahara, A. Gyoja and the author. H. Ochiai [**54**] reformulated the construction in A. Gyoja [**11**] by introducing the notion of quotients. By using this idea, T. Kogiso, G. Miyabe, M. Kobayashi and T. Kimura gave the explicit constructions of 2-simple prehomogeneous vector spaces of type I. For example, see [**48**].

Classification. A fundamental article on classification of prehomogeneous vector spaces is M. Sato and T. Kimura [**77**], in which the irreducible prehomogeneous vector spaces are classified. Since then, many attempts at the classification of non-irreducible prehomogeneous vector spaces have been made. For example, a triplet

(G, ρ, V) such that V decomposes into a finite number of G-orbits must be a prehomogeneous vector space. Such triplets (G, ρ, V), where G is reductive with full scaler multiplications, are classified in T. Kimura, S. Kasai and O. Yasukura [**44**].

T. Kimura's paper [**41**] is a survey of classification theory, including some unpublished results of M. Sato (with his permission, of course). For example, let A be a finite dimensional associative $\mathbb{C}$-algebra with unit element and $A^\times$ the multiplicative group of A. We consider the representation ρ of $A^\times$ on A defined by $\rho(a)b = ab$ ($a \in A^\times$, $b \in A$). Clearly the triplet $(A^\times, \rho, A)$ is a prehomogeneous vector spaces. Then M. Sato proved the following theorem: (i) *The triplet $(A^\times, \rho, A)$ is a regular prehomogeneous vector space if and only if A is a semisimple algebra.* (ii) *The dual triplet of $(A^\times, \rho, A)$ again becomes a prehomogeneous vector space if and only if A is a Frobenius algebra.* For the details, see §10 of T. Kimura [**41**].

While the above articles concern classification over an algebraically closed field of characteristic zero, H. Rubenthaler [**57**] classifies reduced irreducible regular prehomogeneous vector spaces of parabolic type over the real number field. H. Saito [**59**] classifies reduced irreducible prehomogeneous vector spaces over non-archimedean local fields of characteristic zero and algebraic number fields. Z. Chen studies the classification of irreducible prehomogeneous prehomogeneous vector spaces over a field of positive characteristic. For example, see Z. Chen [**6**].

***b*-Functions.** For the b-functions of prehomogeneous vector spaces, the paper by M. Sato, M. Kashiwara, T. Kimura and T. Oshima [**76**] is fundamental. In it the microlocal calculus for b-functions is explained. By using the microlocal calculus, all the b-functions of irreducible regular prehomogeneous vector spaces are determined. This was done by M. Sato, M. Muro, I. Ozeki, T. Yano and the author. See T. Kimura [**40**] and its references. J. Bernstein [**2**] defines the b-functions for arbitrary polynomials. Rationality of roots of b-functions is proved by M. Kashiwara [**37**].

Local zeta functions. The tempered distribution $F_i(s, \Phi)$ that appears in the fundamental theorem (Theorem 4.17) is a kind of local zeta function. Now we shall give the general definition of local zeta functions. Let K be a local field of characteristic zero, and $X = K^n$. If f is a nonconstant polynomial over K in n variables, Φ is a Schwartz–Bruhat function on X and s is a complex variable with $\operatorname{Re} s > 0$, then the local zeta function $Z_f(s, \Phi)$ is defined to be

$$Z_f(s, \Phi) = \int_X |f(x)|_K^s \, \Phi(x) \, dx,$$

where dx is the normalized Haar measure on X and $|\cdot|_K$ is the absolute value in K. If Φ is the standard function, namely $\exp(-\pi^t x x)$ for $K = \mathbb{R}$, $\exp(-2\pi^t x \bar{x})$ for $K = \mathbb{C}$, and the characteristic function of O_K^n for a p-adic field K with maximal compact subring O_K, then we write simply $Z_f(s) = Z_f(s, \Phi)$.

J. Igusa [**34**] is an excellent introduction to local zeta functions, and also discusses the case of prehomogeneous vector spaces. If f is an irreducible relative invariant of an irreducible regular prehomogeneous vector space and $K = \mathbb{C}$, then $Z_f(s)$ can be described explicitly in terms of the b-function $b(s)$. By using this fact, one can determine the explicit form of the functional equation of $Z_f(s, \Phi)$ in the above case. See J. Igusa [**28**] for the details. This result is also obtained by M. Sato [**74**], although there was an ambiguity up to signatures in M. Sato's formula.

In the p-adic case, $Z_f(s)$ is often called the Igusa local zeta function (this is Serre's terminology). J. Igusa [**26**] showed that $Z_f(s)$ for an arbitrary polynomial f is a rational function of q^{-s}, where q is the cardinality of the residue class field of O_K, and as an application, he settled the Borevich–Shafarevich conjecture. J. Igusa [**30**], [**32**] made extensive calculations of $Z_f(s)$ for the irreducible relative invariants f of irreducible regular prehomogeneous vector spaces, except for five cases. By such explicit computations, J. Igusa conjectured that the real parts of the poles of $Z_f(s)$ are zeros of the b-function $b(s)$, more precisely, the Bernstein polynomial of f. In the case of irreducible regular prehomogeneous vector spaces, this conjecture was settled affirmatively by T. Kimura, F. Sato and X.-W. Zhu [**47**]. Nowadays, Igusa local zeta functions are important in many areas of mathematics. J. Igusa [**33**] and J. Denef [**9**] are well-organized reports on this subject with complete bibliographies.

Fundamental theorems. F. Sato [**65**] generalizes the fundamental theorem to the partial Fourier transforms with respect to regular subspaces without assuming the reductivity of the group G. Here the partial Fourier transform is defined as follows: Let (G, ρ, V) be a prehomogeneous vector space of the form $(G, \rho, V) = (G, \rho_1 \oplus \rho_2, E \oplus F)$. We call F a regular subspace if there exists a relative invariant $P(x, y)$ $(x \in E,\ y \in F)$ such that $\det\left(\frac{\partial^2 P}{\partial y_i \partial y_j}(x, y)\right)$ is not identically zero with respect to $y \in F$. We assume that the singular set S is a hypersurface. For $V = E \oplus F$, we put $V^* = E \oplus F^*$. For $\varphi \in \mathcal{S}(V_{\mathbb{R}})$, we define the partial Fourier transform $\widehat{\varphi}$ of φ with respect to the regular subspace F by

$$\widehat{\varphi}(x, y^*) = \int_{F_{\mathbb{R}}} \varphi(x, y) e^{2\pi\sqrt{-1}\langle y, y^*\rangle} dy.$$

F. Sato [**65**] proves the fundamental theorem for such partial Fourier transforms. When G is reductive, $E = \{0\}$, and $F = V$, it is nothing but a result from M. Sato [**74**].

A. Gyoja [**13**] proves the fundamental theorem for reductive prehomogeneous vector spaces without assuming the regularity.

In general, it is difficult to determine the explicit forms of functional equations of zeta functions. In other words, the calculation of $c_{ij}(s)$ in Theorem 4.17 would be difficult. Using M. Sato's idea, M. Kashiwara applied the theory of simple holonomic systems to prehomogeneous vector spaces, and developed an algorithm to calculate $c_{ij}(s)$, which is called microlocal calculus. Although the whole theory of Kashiwara's algorithm is not published yet, M. Kashiwara [**36**] (note by T. Miwa in Japanese) gives an outline of this method. By using this algorithm, M. Muro and T. Suzuki calculated the explicit forms of $c_{ij}(s)$ for some irreducible regular prehomogeneous vector spaces. For example, see M. Muro [**53**].

The fundamental theorem over p-adic fields is proved by J. Igusa [**27**] in the case that the group G is reductive and the singular set S is an irreducible hypersurface with finite orbits condition. F. Sato [**68**] generalizes this result without assuming the reductivity of G nor the irreducibility of S. However, he also assumes some finiteness condition for singular orbits. Examples of the functional equations over p-adic fields can be found in J. Igusa [**28**], [**30**] and F. Sato [**68**]. Y. Hironaka and F. Sato applied the fundamental theorem over p-adic fields to calculate local densities. For example, see [**20**].

T. Kimura [**42**] discusses the arithmetic calculus of Fourier transforms. T. Kimura, T. Kogiso and M. Fujinaga [**46**] proved the fundamental theorem over local fields of positive characteristic under certain conditions.

Zeta functions. F. Sato [**65**] constructed the theory of zeta functions in several variables. By using this theory, F. Sato [**67**] proves the functional equations of Eisenstein series for indefinite quadratic forms, which generalizes the result of A. Selberg. Another application is given in F. Sato [**64**].

As we pointed out in §5.2, C. L. Siegel introduced the volume $\mu(x)$ which appears in the definition of zeta functions to measure the size of sets of integral solutions of some indeterminate equations. Y. Hironaka and F. Sato introduce certain completions of the sets of rational points of homogeneous spaces, and give a rigorous measure-theoretic interpretation of the volume $\mu(x)$. See [**69**], [**21**]

In general, the zeta functions of prehomogeneous vector spaces do not have the Euler product. However, as mentioned in §6.7, J. Igusa [**30**] shows that if an irreducible regular prehomogeneous vector space satisfies the universal transitivity, the additive adelic zeta function has the Euler product. Further, J. Igusa [**29**] classifies all the irreducible regular prehomogeneous vector spaces of split type satisfying universal transitivity. T. Kimura, S. Kasai and H. Hosokawa [**43**], and also T. Kimura and T. Kogiso [**45**], study the same problems for non-irreducible prehomogeneous vector spaces.

When Assumption A in §5.5 is not satisfied, the calculation of the residue of zeta functions is quite difficult. A. Yukie [**89**] generalizes the method in T. Shintani [**79**] and determines the principal parts of some zeta functions.

Several attempts to generalize the zeta functions have been made. For example, A. Murase and T. Sugano [**52**] defined the zeta functions of prehomogeneous affine spaces. As stated in §7.6, the viewpoint of weakly spherical homogeneous spaces gives some insight into the study of prehomogeneous vector spaces. In the case of reductive symmetric spaces, functional equations of Eisenstein series are discussed in Y. Hironaka and F. Sato [**21**]. Recently, F. Sato [**71**], [**72**] constructed the theory of Eisenstein series of weakly spherical homogeneous spaces of general linear groups, by using the theory of zeta functions of prehomogeneous vector spaces.

Castling transformations. Castling transformations, defined in §7.1, play an important role in the theory of prehomogeneous vector spaces. As stated there, the behavior of b-functions and functional equations under castling transformations were studied by T. Shintani. Later, F. Sato and H. Ochiai [**73**] generalized Shintani's result to the case of several variables. The behavior of Igusa local zeta functions is studied by J. Igusa [**31**], and the behavior of zeta functions is studied by F. Sato [**71**].

As stated in §7.6, Y. Teranishi [**82**] gives an example of a generalization of castling transformations. A. Gyoja [**14**] defines some contractions of prehomogeneous vector spaces.

Applications to representation theory. By the Dynkin–Kostant theory, the study of nilpotent orbits of semisimple Lie algebras is reduced to that of prehomogeneous vector spaces. See A. Gyoja [**12**] for the prehomogeneous vector spaces of Dynkin–Kostant type.

N. Kawanaka [**39**] began to study the Gaussian sums of prehomogeneous vector spaces in relation to the representation theory of finite reductive groups. A. Gyoja

and N. Kawanaka conjectured that the Gaussian sums satisfy certain functional equations. Finally, J. Denef and A. Gyoja [**10**] settled this conjecture. See also the exposition [**15**] by A. Gyoja. As an application of the functional equations of the Gaussian sums, one can prove the functional equations of L-functions of prehomogeneous vector spaces. For example, see F. Sato [**68**].

Prehomogeneous vector spaces of parabolic type and their relation to representation theory have been studied mainly in the French school. For this subject, see H. Rubenthaler [**58**], N. Bopp and H. Rubenthaler [**3**], and I. Muller, H. Rubenthaler and G. Schiffmann [**50**].

Applications to number theory. It is expected that the study of zeta functions of prehomogeneous vector spaces will have many applications in number theory, such as density theorems of arithmetic quantities. For example, B. Datskovsky and D. J. Wright generalized the result of T. Shintani [**79**] by using adelic language, and as an application, they proved the density theorems of discriminants of cubic extensions. See [**7**], [**8**], and [**86**]. In order to consider applications to number theory, rational orbit decompositions are important. D. J. Wright and A. Yukie [**87**] solved this problem in some cases.

T. Shintani [**80**] showed that special values of zeta functions of prehomogeneous vector spaces of symmetric matrices are closely related to the explicit forms of dimension formulas of automorphic forms. Later, T. Shintani [**81**] developed a method of evaluating special values of zeta functions by using contour integrals. By generalizing Shintani's method, I. Satake calculated special values of zeta functions of self-dual homogeneous cones, which are special cases of zeta functions of prehomogeneous vector spaces. See I. Satake and S. Ogata [**63**] and I. Satake and J. Faraut [**62**]. T. Arakawa [**1**] also evaluated special values of L-functions of prehomogeneous vector spaces of symmetric matrices. Recently, T. Ibukiyama and H. Saito [**22**] gave the explicit description of the zeta functions of the prehomogeneous vector spaces of symmetric matrices, in terms of Riemann's zeta functions and Dirichlet series attached to some Eisenstein series of half integral weight. Also T. Ibukiyama and H. Katsurada calculated the explicit forms of certain Dirichlet series attached to the prehomogeneous vector spaces given in Example 2.15 of §2.4. Through the above works, H. Saito [**60**] found a formula expressing the additive adelic zeta functions in terms of orbital local zeta functions. By using this result, H. Saito [**61**] proved convergence of the zeta functions as stated in §6.1. See the exposition [**23**] by T. Ibukiyama and H. Saito.

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