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Volume 217

D-modules and Microlocal Calculus

Masaki Kashiwara

Translated by
Mutsumi Saito

American Mathematical Society

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Preface

This book provides an introduction to the general theory of D-modules, whose framework was built up by my teacher Mikio Sato, Takahiro Kawai, and me. As an application of D-module theory, I planned to write how the geometry of flag manifolds and the representation theory of semisimple Lie algebras are related to each other through the theory. Since an excellent Japanese book ([HT]) by Ryoshi Hotta and Toshiyuki Tanisaki covering this topic has been already published, I have decided to write, instead, a book on the algorithm for b-functions using microlocal analysis. For this reason, the title of this book has been changed from the one announced.* I apologize for any confusion this may have caused.

It was around 1969 when I started to study D-modules and microlocal analysis with Sato and Kawai. Among the results I have obtained through over 30 years' study, the microlocal algorithm for b-functions is one of the most beautiful ones (its only defect is a limited range of applications).

I would like to thank Kiyoshi Takeuchi, Toshiyuki Tanisaki, and Kenji Iohara for pointing out many mistakes in preliminary versions of the book, and the staff from Iwanami Shoten for their cooperation.

Masaki Kashiwara
Paris, March 2000

*The original Japanese book had been announced to be published under the title of "Algebraic Analysis and Representation Theory."

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Preface to the English Edition

This book was originally written in Japanese, and was published by Iwanami Shoten on March 28, 2000. The Japanese title is "Foundation of Algebraic Analysis". In this edition, I changed the title in order to express the contents more exactly. The phrase "microlocal calculus" in the present title is not a popular terminology. As explained in the introduction, I intend by this to send the message that the microlocal point of view helps concrete calculations, as powerfully as the Cauchy integral formula provides the values of many definite integrals.

Finally, but not the least, special thanks go to the translator, Mutsumi Saito.

Masaki Kashiwara
Kyoto, August 2002

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Introduction

The study of D-modules was launched when Mikio Sato gave a colloquium talk on them at the Department of Mathematics, University of Tokyo, in 1960. Let us briefly explain his motivation for studying D-modules.

The general form of a system of linear partial differential equations with unknown functions $u_1, \ldots, u_p$ in $x = (x_1, \ldots, x_n)$ is

$$\sum_{j=1}^{p} P_{ij}(x, \partial) u_j = 0 \quad (i = 1, \ldots, q), \tag{0.1}$$

where $P_{ij}(x, \partial)$ $(i = 1, \ldots, q,\ j = 1, \ldots, p)$ are linear partial differential operators, and each of them is written as

$$\sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} a_\alpha(x) \partial^\alpha, \quad \partial^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}, \quad \alpha = (\alpha_1, \ldots, \alpha_n).$$

The unknown functions $u_1, \ldots, u_p$ are not intrinsic in the system of equations; they are just dummies for the purpose of writing the system in an explicit form. This point of view is a starting point for the introduction of D-modules. As a simple example, let us consider the equation

$$\left(x \frac{d}{dx} - \lambda\right) u = 0, \tag{0.2}$$

where λ is a complex number. Let $v(x) = x u(x)$; then this is transformed into the equation

$$\left(x \frac{d}{dx} - \lambda - 1\right) v(x) = 0, \tag{0.3}$$

because

$$\left(x \frac{d}{dx} - \lambda - 1\right) x = x \left(x \frac{d}{dx} - \lambda\right).$$

Conversely, if v is a solution to equation (0.3), then, assuming that $\lambda \neq -1$, we see that $u = \frac{1}{\lambda+1}\frac{d}{dx}v$ satisfies equation (0.2), since

$$\left(x\frac{d}{dx} - \lambda\right)\frac{d}{dx} = \frac{d}{dx}\left(x\frac{d}{dx} - \lambda - 1\right).$$

Moreover, by letting $v = xu(x)$ for a solution u to equation (0.2), we obtain $u = \frac{1}{\lambda+1}\frac{d}{dx}v$, since

$$\frac{1}{\lambda+1}\frac{d}{dx}v - u = \frac{1}{\lambda+1}\left(\frac{d}{dx}x - \lambda - 1\right)u = 0.$$

Conversely, by letting $u = \frac{1}{\lambda+1}\frac{d}{dx}v$ for a solution v to equation (0.3), we obtain $v = xu$, since

$$xu - v = \frac{1}{\lambda+1}\left(x\frac{d}{dx} - \lambda - 1\right)v = 0.$$

By the transformations $v = xu$ and $u = \frac{1}{\lambda+1}\frac{d}{dx}v$, we thus see that equations (0.2) and (0.3) are equivalent to each other. Hence they mean the same, although they look different. In other words, taking u or v as an unknown function is quite artificial. We can formulate this idea by using D-modules.

Let us consider equation (0.1). Let D be the (noncommutative) ring of linear partial differential operators, and $P : D^{\oplus q} \to D^{\oplus p}$ the map given by

$$(Q_1, \ldots, Q_q) \mapsto \left(\sum_{i=1}^{q} Q_i P_{i1}, \ldots, \sum_{i=1}^{q} Q_i P_{ip}\right).$$

This map P is left D-linear, and its cokernel M is the D-module corresponding to equation (0.1). We can derive the solutions to (0.1) from M as follows: Let F be a space of functions in which we want to find the solutions. Various spaces can be taken as F, such as the space of C^∞-functions, the space of distributions, or the space of holomorphic functions, depending on the problem being considered. Since we can apply differential operators to the functions belonging to F, we regard F as a left D-module. Then

$$\mathrm{Hom}_D(D^{\oplus p}, F) = F^{\oplus p} = \{(u_1, \ldots, u_p); u_1, \ldots, u_p \in F\}.$$

By a well known property (left exactness) of Hom_D,

$$\begin{aligned}\mathrm{Hom}_D(M, F) &= \mathrm{Ker}(\mathrm{Hom}_D(D^{\oplus p}, F) \xrightarrow{P} \mathrm{Hom}_D(D^{\oplus q}, F)) \\ &= \{(u_1, \ldots, u_p) \in F^{\oplus p}; u_1, \ldots, u_p \,\text{satisfy (0.1)}\}.\end{aligned}$$

Hence the space of solutions to (0.1) in F equals $\mathrm{Hom}_D(M, F)$, which depends only on M. Conversely, given a D-module M, for each isomorphism

$$M \simeq \mathrm{Coker}(D^{\oplus q} \xrightarrow{P} D^{\oplus p}), \tag{0.4}$$

we obtain equation (0.1). Then (0.1) is considered an explicit presentation of M corresponding to the isomorphism (0.4). There are many such isomorphisms for a given D-module. Each isomorphism gives a different explicit presentation (0.1). In the previous example, one D-module has two distinct isomorphisms; it is isomorphic to two cokernels:

$$\mathrm{Coker}(D \xrightarrow{x\frac{d}{dx}-\lambda} D) \simeq \mathrm{Coker}(D \xrightarrow{x\frac{d}{dx}-\lambda-1} D),$$

which lead to (0.2) and (0.3) respectively.

From this point of view, D-modules were introduced, and the theory has grown into a rich one with many subtheories such as characteristic varieties, microlocal analysis, holonomic systems (maximally overdetermined systems), etc. We give an outline of the theory in this book.

Today, besides the theory of linear partial differential equations, which is a root of D-module theory, the theory is also applied to representation theory, conformal field theory, etc. For applications to representation theory, [HT] has a detailed exposition. In this book, we describe applications of the theory to b-functions; in particular, we give an algorithm to compute b-functions using microlocal analysis. Sato seems to have expected that the D-module theory (in particular the theory of holonomic systems) would be useful for explicit computations. This is analogous to Cauchy's integration theorem, which has not only theoretical beauty and importance, but also an application to explicit computations of definite integrals through Cauchy's residue formula. The theory of b-functions described in this book is a good example of applicability of microlocal analysis of holonomic systems to explicit computations.

Conventions and Notation. In this book, a manifold means a nonsingular complex analytic manifold. A morphism of manifolds $f : X \to Y$ is said to be smooth if the corresponding maps of tangent spaces $T_xX \to T_{f(x)}Y$ are surjective at all $x \in X$.

We abbreviate the dimension $\dim X$ of a manifold X to d_X, and write $d_{X/Y} = d_X - d_Y$.

We simply call a sheaf of rings a ring, and a sheaf of modules a module.

$\rightarrowtail$ indicates injectivity, and $\twoheadrightarrow$ surjectivity.

For an analytic set Z, we say that a property (P) holds at a generic point p on Z if there exists a nowhere dense closed analytic subset Y such that (P) holds at all $p \in Z \backslash Y$.

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[HT] R. Hotta, and T. Tanisaki, *D-modules and Algebraic Groups*, Springer Modern Mathematics Series, Springer-Verlag Tokyo, Tokyo, 1995 (in Japanese)

is recommended, which contains a detailed exposition particularly of their applications to representation theory. Other books on D-modules are the following:

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A book in the same series as the present one,

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Masaki Kashiwara is undoubtedly one of the masters of the theory of D-modules, and he has created a good, accessible entry point to the subject. The theory of D-modules is a very powerful point of view, bringing ideas from algebra and algebraic geometry to the analysis of systems of differential equations. It is often used in conjunction with microlocal analysis, as some of the important theorems are best stated or proved using these techniques. The theory has been used very successfully in applications to representation theory.

Here, there is an emphasis on b-functions. These show up in various contexts: number theory, analysis, representation theory, and the geometry and invariants of prehomogeneous vector spaces. Some of the most important results on b-functions were obtained by Kashiwara.

A hot topic from the mid '70s to mid '80s, it has now moved a bit more into the mainstream. Graduate students and research mathematicians will find that working on the subject in the two-decade interval has given Kashiwara a very good perspective for presenting the topic to the general mathematical public.

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