Translations of

MATHEMATICAL MONOGRAPHS

Volume 221

Surfaces with Constant Mean Curvature

Katsuei Kenmotsu

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10.1090/mmono/221

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Surfaces with Constant Mean Curvature

Katsuei Kenmotsu

Translated by
Katsuhiro Moriya

American Mathematical Society
Providence, Rhode Island

This English edition has been translated from the original Japanese publication, **曲面論講義------平均曲率一定曲面入門** (A Course in Smooth Surfaces — Introduction to Constant Mean Curvature Surfaces) by **剱持勝衛** (Katsuei Kenmotsu), ©Baifukan Co. Ltd., Japan, 2000. Translated from the original Japanese by Katsuhiro Moriya.

2000 *Mathematics Subject Classification.* Primary 53-01, 53A10.

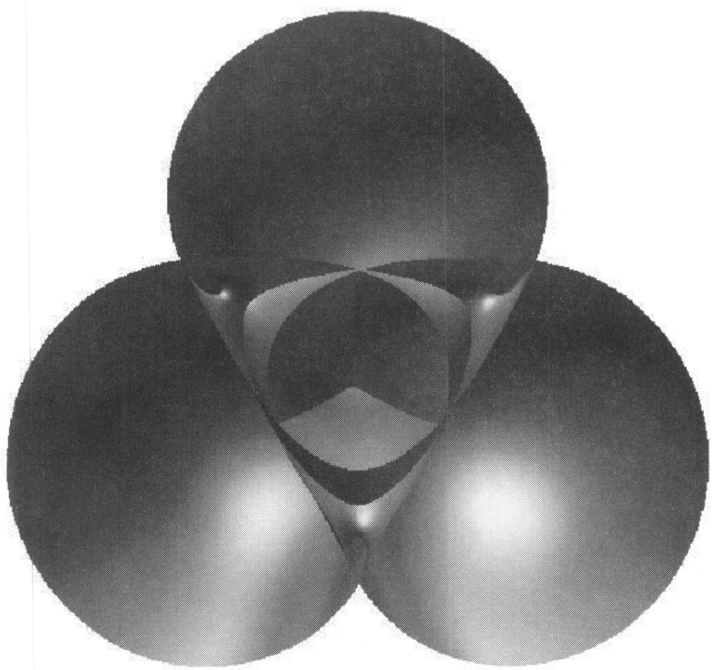
For additional information and updates on this book, visit
www.ams.org/bookpages/mmono-221

Library of Congress Cataloging-in-Publication Data

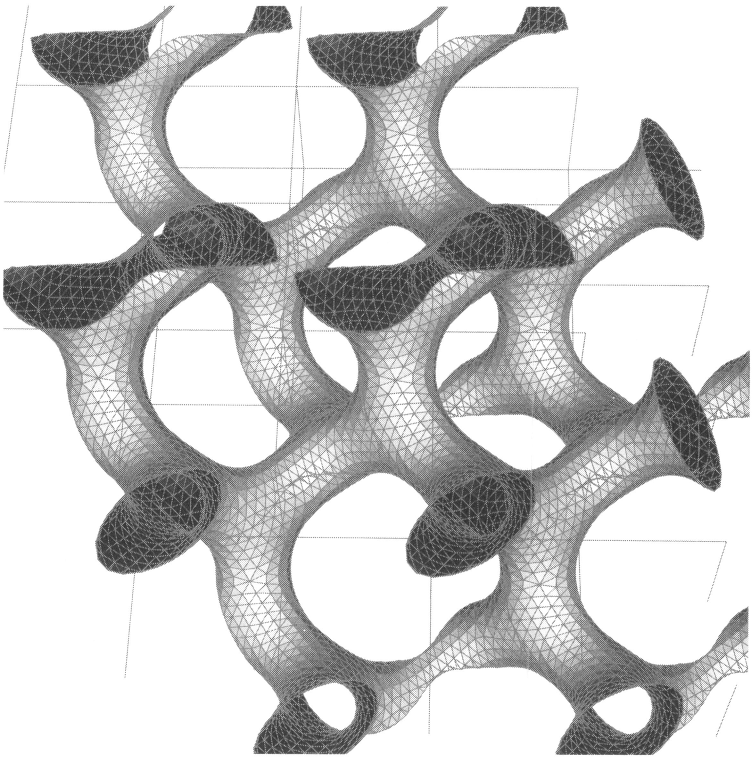
Kenmotsu, K. (Katsuei), 1942–
[Kyokumenron kogi. English]
Surfaces with constant mean curvature / Katsuei Kenmotsu ; translated by Katsuhiro Moriya.
p. cm. – (Translations of mathematical monographs ; v. 221)
Includes bibliographical references and index.
ISBN 0-8218-3479-7 (acid-free paper)
1. Minimal surfaces. 2. Curvature. 3. Geometry, Differential. I. Title. II. Series.

QA644.K4613 2003
516.3′62–dc22 2003059522

10 9 8 7 6 5 4 3 2 1 08 07 06 05 04 03



FRONTISPIECE 1. One of two congruent halves of a Wente torus. It is a closed non-zero constant mean curvature surface, different from a sphere and first discovered in 1984 (see page 139, (1)).



FRONTISPIECE 2. A constant mean curvature gyroid. It is a triply periodic surface with triple junctions, drawn by K. Grosse-Brauckmann. A gyroid is used in the study of surface science and block copolymers (see page 139, (2)).

Foreword

As a measure of how a surface curves, we have the Gaussian curvature and the mean curvature. The Gaussian curvature, an intrinsic quantity, became one of the foundations for the development of Riemannian geometry. In contrast, the mean curvature is an extrinsic quantity which measures how the surface lies in space. Since the mean curvature is related to the character of the surface of a material body, it is deeply related to other sciences.

The mean curvature is defined as the arithmetic mean of the two principal curvatures at each point of a surface. A surface whose mean curvature is 0 at each point is called a minimal surface, and an area-minimizing surface is an important example of this. Minimal surfaces continue to be studied actively and have deep relations to analysis, especially the theory of functions.

A surface whose mean curvature is constant but not equal to 0 is obtained when we minimize the area of a surface while preserving its volume; the sphere is a trivial example and the constant mean curvature torus discovered by H. Wente in 1984 gave geometers a powerful incentive to study such surfaces. Subsequently, many constant mean curvature surfaces were discovered using a variety of techniques.

In this book, we aim to explain various examples of constant mean curvature surfaces and the techniques for studying them.

In Chapter 1, we define smooth surfaces and explain the basic notions of differential geometry that are necessary for the local study of surfaces. In Chapter 2, we explain the mathematical and physical meaning of the mean curvature.

In Chapter 3, we consider surfaces of revolution having constant mean curvature. Although the results in this chapter were obtained in the mid-nineteenth century by C. Delaunay, they are necessary to understand the more sophisticated modern examples. Next, we investigate constant mean curvature surfaces invariant by helicoidal motions in Chapter 4 and show that such surfaces are obtained by deforming Delaunay surfaces isometrically.

In Chapter 5, we define general surfaces. As a result, we can study global properties of surfaces, and we discuss the stability of constant mean curvature surfaces.

In Chapter 6, we introduce a closed constant mean curvature surface which is not the sphere. It is a torus, topologically, but quite different from an ordinary doughnut-shaped one.

Chapter 7 is an explanation of the basic methods used to study the general theory of complete constant mean curvature surfaces. In particular, we obtain the balancing formula, which controls behavior of infinity of complete surfaces.

In Chapter 8, we introduce the study of constant mean curvature surfaces via their Gauss maps. Harmonic maps from Riemann surfaces to the Riemann sphere appear here. We mention a representation formula for constant mean curvature surfaces in this chapter.

In Chapter 9, we explain existence theorems for constant mean curvature surfaces with or without boundary. Moreover, using material from recent studies, we explain discrete constant mean curvature surfaces and a technique used to construct constant mean curvature surfaces.

In the Appendix, we explain calculations which were too long to put in the text, some theorems used in the latter part of this book, the maximum principle for elliptic partial differential inequalities, the Alexandrov reflection technique, and so on. Moreover, we present the *Mathematica*® programs written by the author that are used for making the figures in this book.

We assume that the reader has some knowledge of calculus (at least as far as Green's theorem), linear algebra, and elemental differential geometry. In addition, some acquaintance with manifold theory, such as differential forms and elementary topology, is desirable for reading Chapter 5.

Part of this book is based on the author's lectures at Tohoku University, Kanazawa University, and the Federal University of Ceará. The author is grateful to Professor Haruo Kitahara of Kanazawa University, Professor L. Barbosa of the Federal University of Ceará, and all the audiences of his lectures. He expresses his gratitude to Masaaki Umehara of Hiroshima University, Miyuki Koiso of Kyoto University of Education, and Kotaro Yamada of Kyushu University, for reading through the manuscript and giving him much valuable advice. Figures

3.8 and 3.9 were made by Yusuke Sakane of Osaka University using numerical data supplied by the author. The author is deeply grateful to him for offering these beautiful figures. Moreover, he thanks Seiji Iwata at Baifu-kan for his patience and warm encouragement, and for helpful advice.

The author was urged to begin writing this book by Syukichi Tanno, honorary professor of Tokyo Institute of Technology. The author dedicates this book to Professor Tanno, who died in September last year at the age of 62.

September, 2000

Sendai

Katsuei Kenmotsu

Regarding the second edition: We corrected some mistakes in the first edition and replaced Figure 3.8. Moreover, we improved and revised the programs in Appendix B (Figure 3.4, Figure 3.7, Figure 3.8, and Figure 3.9) so that the figures can be drawn more quickly. The author expresses his appreciation to Professor Takashi Ogata of Yamagata University, Professor Yusuke Sakane of Osaka University, and Shinya Hirakawa, graduate student of Tohoku University, for their assistance in making the revisions.

October 2001

The author

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List of Sources for the Figures

The author gratefully acknowledges the individuals, universities, and publishers that allowed him to use the following figures in this book, and especially Professor K. Grosse-Brauckmann of Bonn University in Germany for offering his postscript files.

(1) Frontispiece 1, Figures 2.2, 2.3, 5.3, 5.4, 6.1, 6.2, 6.3 and 6.4:
`http://www.gang.umass.edu/gallery/cmc/`
Nicholas Schmitt, Center for Geometry, Analysis, Numerics and Graphics (GANG), University of Massachusetts at Amherst

(2) Frontispiece 2:
`http://www.math.uni-bonn.de/kgb/Research/research.html`
Karsten Grosse-Brauckmann, Mathematiches Institut, Universität Bonn
Software used: Surface Evolver by Ken Brakke, Exp. Math. **6** (1997), 33–50

(3) Figure 2.4:
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(6) Figures 9.4 and 9.5: Karsten Grosse-Brauckmann and Konrad Polthier

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(7) Figure 9.6:

`http://www-sfb256.iam.uni-bonn.de/grape/EXAMPLES/AMANDUS/knoid.html`

Martin Steffens, Arts–Unframed, Münchener Str.24 85774 Unterföhring, Germany and Christian Teitzel, Hofäckeralle 3, 85774 Unterföhring, Germany

Software used: Grape

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The mean curvature of a surface is an extrinsic parameter measuring how the surface is curved in the three-dimensional space. A surface whose mean curvature is zero at each point is a minimal surface, and it is known that such surfaces are models for soap film. A surface whose mean curvature is constant but nonzero is obtained when we try to minimize the area of a closed surface without changing the volume it encloses. A trivial example of a surface of constant mean curvature is the sphere. A nontrivial example is provided by the constant curvature torus, whose discovery in 1984 gave a powerful incentive for studying such surfaces. Later, many examples of constant mean curvature surfaces were discovered using various methods of analysis, differential geometry, and differential equations.

In this book, the author presents the numerous examples of constant mean curvature surfaces and the techniques for studying them. Many figures illustrate the presented results and allow the reader to visualize and better understand these beautiful objects.

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ISBN 0-8218-3479-7

9 780821 834794

MMONO/221