Translations of
MATHEMATICAL MONOGRAPHS

Volume 222

Convex Analysis: Theory and Applications

G. G. Magaril-Il'yaev
V. M. Tikhomirov

American Mathematical Society

Convex Analysis: Theory and Applications

This page intentionally left blank

10.1090/mmono/222

Translations of

MATHEMATICAL MONOGRAPHS

Volume 222

Convex Analysis: Theory and Applications

G. G. Magaril-Il'yaev
V. M. Tikhomirov

Translated by
Dmitry Chibisov

American Mathematical Society
Providence, Rhode Island

Г. Г. Магарил-Ильяев, В. М. Тихомиров
ВЫПУКЛЫЙ АНАЛИЗ: ТЕОРИЯ И ПРИЛОЖЕНИЯ
УРСС, Москва, 2000
Translated from the Russian by Dmitry Chibisov
The present translation was created under license for the American Mathematical Society and is published by permission.

2000 *Mathematics Subject Classification.* Primary 52–01, 46N10; Secondary 47N10.

For additional information and updates on this book, visit
www.ams.org/bookpages/mmono-222

Library of Congress Cataloging-in-Publication Data

Magaril-Il′yaev, G. G. (Georgii G.), 1944–
[Vypuklyi analiz teoriia i prilozheniia. English]
Convex analysis : theory and applications / G. G. Magaril-Il′yaev, V. M. Tikhomirov ; translated by Dmitry Chibisov.
p. cm. — (Translations of mathematical monographs, ISSN 0065-9282 ; v. 222)
Includes bibliographical references and index.
ISBN 0-8218-3525-4 (acid-free paper)
1. Convex geometry. 2. Discrete geometry. 3. Functional analysis. 4. Operator theory. I. Tikhomirov, V. M. (Vladimir Mikhailovich), 1934– II. Title. III. Series.

QA639.5.M3413 2003
516.3′62—dc22 2003062858

10 9 8 7 6 5 4 3 2 1   08 07 06 05 04 03

Contents

Preface vii

Introduction 1

Chapter 1. Theory 25
1. Basic definitions 25
2. Duality in convex analysis 39
3. Convex calculus 48
4. Finite-dimensional convex geometry 57
5. Convex extremal problems 69
6. Supplement: Convex analysis in vector spaces 79

Chapter 2. Applications 87
7. Convex analysis of subspaces and cones and the theory of linear equations and inequalities 87
8. Classical inequalities, problems of geometry and mechanics 92
9. Kolmogorov-type inequalities for derivatives 99
10. Convex analysis and extremal problems of approximation and recovery 118

Chapter 3. Appendix 153
11. Basic theorems of convex analysis 153
12. Supplementary topics of convex analysis 157
13. Convex analysis and the theory of extremum 163

Bibliography 177

Index 181

This page intentionally left blank

Preface

Convex analysis is a branch of mathematics that studies convex sets, convex functions, and convex extremal problems.

Historically the beginnings of convex analysis date back to antiquity, but its name was coined only in the 60s of the 20th century. Many remarkable facts of convex analysis have been obtained quite recently. So, this is at the same time an ancient and modern part of mathematics.

Convex analysis is closely related to geometry (in which it has its origins, since convexity is a geometric notion), but it is also deeply connected with analysis, and these latter connections stimulated intense interest in convex analysis in recent period. Hence the duality in the very name: *convex analysis*.

At first sight it may seem surprising that convex functions and convex sets, which are very special objects of analysis and geometry, have diverse applications in mathematics, mathematical physics, technology, and economics. In fact, there are good grounds for that (which will be exposed in this book), but anyhow, the fact that the theory of convexity has diverse and fruitful applications is now so unquestionable that basic knowledge of convex analysis is nowadays necessary for almost every mathematician (especially for those dealing with applications). Moreover, studying convex analysis can be motivated aesthetically because it comprises many beautiful phenomena and facts.

Apparently, certain elements of the convexity theory should become part of mathematical education at any level. The present book is intended to contribute to this.

Convex analysis is one of the disciplines which require little preliminary knowledge for their study. This book was also written for a broad readership. Specially for "beginners" we have written the Introduction, where the fundamentals of convex analysis are illustrated in an elementary finite-dimensional setting.

Moreover, the subject matter of Chapter 1 dealing with theoretical aspects of the convexity theory also relies only on the notion of n-dimensional space and therefore, we believe, is accessible to a fairly broad readership.

The geometrically minded reader is advised to combine reading with drawing, since many concepts and proofs have a very simple (and beautiful) geometric interpretation (which we tried to demonstrate by our figures).

This book was originally conceived as the translation of the Russian publication by Magaril-Il'yaev and Tikhomirov (2000), but in the course of preparation of the English edition many parts were essentially revised, so that it became actually a different book. For this reason we somewhat changed its title.

We express our deep gratitude to our colleagues, J. Brinkhuis, A. D. Ioffe, C. V. Konyagin, V. L. Levin, K. Yu. Osipenko, and V. Yu. Protasov for the material kindly provided to us, for useful discussions, and for assistance in preparation of the Russian edition of this book. We thank the student S. S. Chudova for drawing the figures. We are grateful to the translator, Professor D. M. Chibisov, for competent advice and comments which led to improvement of the book.

Bibliography

N. I. Akhiezer (1965), *Lectures on the theory of approximation*, Nauka, Moscow; English transl., Frederik Ungar, New York, 1956.

V. M. Alekseev, E. M. Galeev, and V. M. Tikhomirov (1984), *Optimization problem book*, Nauka, Moscow (Russian); French transl., *Recueil de problèmes d'optimisation*, Mir, Moscow, 1987.

V. M. Alekseev, V. M. Tikhomirov, and S. V. Fomin (1979), *Optimal control*, Nauka, Moscow; English transl., Consultants Bureau, New York, 1987.

V. V. Arestov (1989), *Best recovery of operators and related problems*, Trudy Mat. Inst. Steklov **189**, 3–20; English transl. in Proc. Steklov Inst. Math. **189** (1991).

——— (1996), *Approximation of unbounded operators by bounded operators and related extremal problems*, Uspekhi Mat. Nauk **51**, no. 6, 89–124; English transl. in Russian Math. Surveys **51**.

T. Bonnesen and W. Fenchel (1934), *Theorie der konvexen Körper*, Springer-Verlag, Berlin.

Yu. D. Burago and V. A. Zalgaller (1988), *Geometric inequalities*, Springer-Verlag, Berlin.

G. B. Dantzig (1963), *Linear programming and extensions*, Princeton Univ. Press, Princeton.

A. Ya. Dubovitskii and A. A. Milyutin (1965), *Problems for extremum under constraints*, Zh. Vychisl. Mat. i Mat. Fiz. **5**, no. 3, 395–453; English transl. in USSR Comput. Math. and Math. Phys. **5** (1965).

N. Dunford and J. T. Schwartz (1958), *Linear operators*. Vol. I, Interscience, New York.

I. Ekeland and R. Teman (1976), *Convex analysis and variational problems*, SIAM, New York.

J. Farkas (1901), *Über die Theorie der einfachen Ungleichungen*, J. Reine und Angew. Math. **124**, 1–24.

W. Fenchel (1951), *Convex cones, sets and functions*, Princeton Univ. Press, Princeton.

A. T. Fuller (1960), *Relay control systems optimized by various performance criteria*, Proc. First World Congress IFAC, Moscow, vol. 1, London, pp. 510–519.

V. N. Gabushin (1969), *On the best approximation of the differentiable operator on the half-line*, Mat. Zametki **6**, no. 5, 573–582; English transl. in Math. Notes **6**.

D. Gale (1960), *The theory of linear economic models*, McGraw-Hill, New York.

J. B. Giles (1982), *Convex analysis with applications in differentiation of convex functions*, Pitman and Hall, London.

J. Hadamard (1914), *Sur le module maximum d'une fonction et de ses dérivées*, Bull. Soc. Math. France, C. R. de Séances **42**, 68–72.

G. G. Hardy, J. E. Littlewood, and G. Pólya (1934), *Inequalities*, Cambridge Univ. Press, Cambridge.

E. Helly (1923), *Über Mengen konvexer Körper mit gemeinschatlichen Punkten*, Über. Deutsch. Math. Verein. **32**, 175–176.

R. B. Holmes (1975), *Geometric functional analysis and its applications*, Springer-Verlag, New York–Heidelberg–Berlin.

A. D. Ioffe and V. M. Tihomirov [Tikhomirov] (1974), *Theory of extremal problems*, Nauka, Moscow; English transl., North-Holland, Amsterdam, 1979.

F. John (1948), *Extreme problems with inequalities as subsidary conditions*, Studies and essays presented to R. Courant on his 60th birthday, January 8, 1948, Intersociety, New York, pp. 187–204.

L. V. Kantorovich (1939), *Mathematical methods of production management and planning*, Leningrad Univ., Leningrad. (Russian)

——— (1942), *On transportation of masses*, Dokl. Akad. Nauk SSSR **37**, 227–229. (Russian)

L. V. Kantorovich and G. Sh. Rubinstein (1957), *On a functional space and some extremal problems*, Dokl. Akad. Nauk SSSR **115**, no. 6, 1058–1061. (Russian)

A. Khovansky, D. Bernstein, and A. Kushnirenko (1976), *Mixed volumes of Newton polynomials, Milnor's numbers and numbers of solutions of algebraic equations*, Uspekhi Mat. Nauk **32**, no. 3, 196. (Russian)

A. N. Kolmogorov (1939), *Inequalities between the suprema of consecutive derivatives of an arbitrary function on an infinite interval*, Uchen. Zap. Moskov. Univ. Mat. **3**, no. 30, 3–16; English transl. in: Selected works of A. N. Kolmogorov, Vol. 1: Mathematics and Mechanics, Kluwer, Dordrecht, 1991, pp. 277–290.

A. Koushnirenko (1976), *Polyhedres de Newton et nombre de Milnor*, Inv. Math. **32**, 1–31.

N. P. Kuptsov (1975), *Kolmogorov estimates for derivatives in $L_2(0,\infty)$*, Trudy Mat. Inst. Steklov **138**, 94–117; English transl. in Proc. Steklov Inst. Math. **138**.

S. S. Kutateladze (1979), *Convex operators*, Uspekhi Mat. Nauk **34**, no. 1, 167–196. (Russian)

Ky Fan (1956), *On systems of linear inequalities*, Linear Inequalities and Related Systems, Princeton Univ. Press, Princeton, NJ.

E. Landau (1913), *Ungleichungen für zweimal differenzierbar Funktionen*, Proc. London Math. Soc. **2**, no. 13, 43–49.

V. L. Levin (1978), *The Monge–Kantorovich problem on transportation of masses*, Matematicheskaya Ekonomika i Funkcional'nyi Analis (Mathematical Economics and Functional Analysis), Nauka, Moscow, pp. 94–108. (Russian)

——— (1985), *Convex analysis in spaces of measurable functions and its applications in mathematics and economics*, Nauka, Moscow. (Russian)

V. L. Levin and A. A. Milyutin (1979), *The problem on transportation of masses with discontinuous cost function and the duality of convex extremal problems in terms of masses*, Uspekhi Mat. Nauk **34**, no. 3, 3–68. (Russian)

Yu. I. Lyubich (1960), *Inequalities between powers of a differential operator*, Izv. Akad. Nauk SSSR Ser. Mat. **24**, 825–864. (Russian)

L. A. Lyusternik (1934), *On conditional extrema of functionals*, Mat. Sbornik **41**, no. 3, 390–401. (Russian)

——— (1956), *Convex figures and polyhedra*, Gostekhizdat, Moscow. (Russian)

G. G. Magaril-Il'yaev (1983), *Inequalities for derivatives and duality*, Trudy Math. Inst. Steklov **161**, 183–194. (Russian)

G. G. Magaril-Il'yaev and K. Yu. Osipenko (1991), *Optimal recovery of functionals based on inaccurate data*, Mat. Zametki **50**, no. 1, 85–93; English transl. in Math. Notes **50**, 1274–1279.

——— (2002), *Optimal recovery of functions and their derivatives from Fourier coefficients prescribed with an error*, Mat. Sbornik **193**, no. 3, 79–100; English transl. in Sbornik: Mathematics **193**, no. 3, 387–407.

——— (2003), *Optimal recovery of functions and their derivatives from approximate information on the spectrum and inequalities for derivatives*, Funktsional Anal. i Prilozhen. **37**, no. 3. (Russian)

G. G. Magaril-Il'yaev and V. M. Tikhomirov (1981), *Kolmogorov's inequality for fractional derivatives on a half-line*, Analysis Math. **7**, no. 1, 37–47. (Russian)

——— (1997), *Kolmogorov-type inequalities for derivatives*, Mat. Sbornik **188**, no. 12, 73–106; English transl. in Sbornik: Mathematics **188**, no. 12, 1799–1832.

——— (2000), *Convex analysis and its applications* (2nd ed. 2003), Editorial URSS, Moscow. (Russian)

C. A. Micchelli and T. J. Rivlin (1977), *A survey of optimal recovery*, Optimal Estimation in Approximation Theory, Plenum, New York, pp. 1–54.

——— (1985), *Lectures on optimal recovery*, Lecture Notes in Math., vol. 1129, Springer-Verlag, Berlin, pp. 21–93.

H. Minkowski (1896), *Geometrie der Zahlen*, Teubner, Leipzig.

G. Monge (1784), *Mémoire sur la théorie des déblais et des remblais*, Histoire de l'Académie Royale des Science, 666–704.

J. J. Moreau (1962), *Fonctions convexes en dualité*, Fac. des Sciences de Montpellier. Sem. de Mathématiques.

K. Yu. Osipenko (1976), *Best approximation of analytic functions from their values at a finite number of points*, Mat. Zametki **19**, no. 1, 29–40; English transl. in Math. Notes **19**, 17–23.

——— (2000), *Optimal recovery of analytic functions*, Nova Science Publ., Huntington, NY.

R. R. Phelps (1978), *Differentiability of convex functions on Banach spaces*, Saunders, London.

L. Plaskota (1996), *Noisy information and computational complexity*, Cambridge Univ. Press, Cambridge.

J. Radon (1921), *Körper, die einen gemeinsamen Punkt enthalten*, Math. Ann. **83**, 113–115.

R. T. Rockafellar (1970), *Complex analysis*, Princeton Univ. Press, Princeton, NJ.

A. Sard (1949), *Best approximate integration formulae; best approximation formulae*, Amer. J. Math. **71**, 80–91.

S. A. Smolyak (1965), *Optimal recovery of functions and functionals thereof*, Kandidat Thesis, Moscow. (Russian)

B. Sz.-Nagy (1941), *Über Integralungleichungen zwischen einer Function und ihrer Ableitung*, Acta Sci. Math. **10**, 64–74.

L. N. Taikov (1968), *Kolmogorov-type inequalities and the best formulae for numerical differentiation*, Mat. Zametki **4**, no. 2, 233–238; English transl. in Math. Notes **4**.

V. M. Tikhomirov and G. G. Magaril-Il'yaev (1991), *Inequalities for derivatives*, Selected works of A. N. Kolmogorov, Vol. 1: Mathematics and Mechanics, Kluwer, Dordrecht, pp. 442–446.

H. Weyl (1935), *Elementare Theorie der konvexen Polyheder*, Comm. Math. Helv. **7**, 290–306.

M. I. Zelikin and V. F. Borisov (1994), *Theory of chattering control*, Birkhäuser, Boston.

Index

accumulation point, 28
admissible point (in an extremal problem), 70
affine combination (of vectors), 31
affine hull (of a set), 31
affine independent points, 31
affine manifold, 13, 26, 31
affine subspace, 13, 26, 31, 80
algebraic sum, 30
analog of Fermat's theorem, 49
annihilator, 42, 81

Banach–Alaoglu theorem, 153
barycentric coordinates, 32
basis, 25
Bernstein's inequality, 138
biannihilator, 42, 81
biconjugate cone, 81
bipolar, 81
Blaschke's compactness theorem, 68
Brunn–Minkowski inequality, 67
Brunn–Minkowski symmetrization, 67

Carathéodory's theorem, 8, 62
Cauchy theorem on rigidity, 58
Cauchy's formula, 65
Cauchy–Steiner lemma, 60
clean-up theorem, 156
  for cones, 62
closed set, 4, 27
closure, 28
compact set, 28
complementary slackness condition, 71
cone, 13, 30
conic combination, 31
conic hull (of a set), 31
conjugate cone, 42, 81
conjugate function, 15, 39
conjugation operator, 39
  for cones, 42
constraint, 70
continuous function, 28
convex analysis, 1
convex closure, 7
convex combination, 31
convex cone, 13, 31, 80
convex curve, 1
convex function, 1, 34
convex homogeneous function, 13
convex hull, 7
  of minimum (of convex functions), 54
  of a set, 31
  of a union (of convex sets), 53
convex polyhedron, 88
convex problem, 19, 69
convex programming problem, 19, 70
convex set, 1, 30, 31, 80
convex surface, 1
convolution (of convex functions), 54

directional derivative, 36
distance between vectors, 27

dual problem, 21
dual problem (to an extremal problem), 73
dual space, 26
Dubovitskii–Milyutin formula, 17
Dubovitskii–Milyutin theorem, 52

epigraph, 1, 34
Euclidean norm, 27
extremal subset, 154
extreme point, 154

Farkas theorem, 90
Fenchel–Moreau theorem, 11, 16, 44
Fermat–Torricelli–Steiner problem, 96
finitely generated cone, 88
first separability theorem, 38
Fredholm theorem, 92
Fuller, A. T., 111

Gabushin, V. N., 111
Gale theorem, 90
generalized Gabushin's inequality, 110
generalized Hölder's inequality, 93
Goddard problem, 97
Grünbaum–Hammer theorem, 68

Hahn–Banach–Kantorovich theorem, 161
half-interval, 29
half-space, 37
Helly's theorem, 9, 63
Hölder's inequality, 93
homogeneous function, 13
hyperplane, 26, 37

image of a function under a mapping, 35, 54
image of a set under a mapping, 30, 53
indicator function, 42
  of a convex set, 34
indicator operator, 42
interior point of a set, 27
interior of a set, 27
intersection, 30
  of convex sets, 53
interval, 29
inverse image
  of a function under a mapping, 35, 54
  of a set under a mapping, 30, 53

Jensen inequality, 34

Karush–Kuhn–Tucker theorem, 20, 70
Kelley's sum
  of convex functions, 54
  of convex sets, 53
Kolmogorov-type inequalities, 99
Ky Fan theorem, 90

Lagrange function, 70, 163
Lagrange multipliers, 70, 163
Lagrange principle, 20
Landau–Arestov theorem, 118
least possible constant, 100
Legendre transform, 39
Legendre–Young–Fenchel transformation, 15
lemma on the annihilator of the kernel of a regular operator, 165
lemma on nontriviality of the annihilator, 168
Levin's theorem, 52, 156
line segment, 27, 29
linear combination (of vectors), 30
linear programming problem, 74
linear span (of a set), 31
locally convex space, 13
Lyapunov, A. A., theorem, 155, 169
Lyusternik Theorem, 165

Magaril-Il'yaev, G. G., 111
Magaril-Il'yaev theorem, 116, 118
Masur theorem, 160
maximum (of convex functions), 35, 54
method of central sections, 75
method of circumscribed ellipsoids, 76
Michael's selection theorem, 161
Minkowski–M. Krein–D. Milman theorem, 7, 154
Minkowski's function, 36, 43

Minkowski's operator, 43
Minkowski's theorem, 6, 64, 90
Monge–Kantorovich problem, 158
Moreau–Rockafellar formula, 17
Moreau–Rockafellar theorem, 51

neighborhood, 27
nonatomic measure, 155

open ball, 27
open set, 4, 27

perturbation of problem, 21, 72
polar, 41, 81
polyhedral cone, 88
problem without constraints, 20, 49, 70
proper function, 34

Rademacher theorem, 160
Radon theorem, 9, 62
ray, 27
relative interior (of a set), 32
relatively interior point (of a set), 32
Riesz, F., theorem, 92
Riesz, M., identity, 138

second conjugate, 15, 40
second separability theorem, 37
second separation theorem, 4, 83
separable sets, 37
separation theorems, 16, 37
set of constraints, 70
S-function (of the family of problems), 72
sharp constant, 100
simplex, 31
simplex method, 77
Slater's condition, 70
solution (to an extremal problem), 70
standard basis, 25
standard perturbation, 73
stationarity condition, 101
Steiner–Minkowski formula, 66
straight line, 27
strictly differentiable mapping, 164
strictly separable sets, 37
subdifferential, 17, 40, 48
subdifferential form of the clean-up theorem, 52
subgradient, 17, 48
sublinear function, 13, 35
subspace, 13, 26, 30, 80
sum of convex functions, 54
sum of convex sets, 53
summation (of convex functions), 35
support function, 18, 36, 41
support operator, 41
Sz.-Nagy, B., 106, 107
  theorem, 108

tangent vector, 164
theorem, Lagrange principle for smooth-convex problems, 164
theorem: the main formulas of convex calculus, 55
theorem on convex duality, 43
theorem on duality of sets, 83
theorem on existence and structure of subdifferential, 50
theorem on isoperimetric property of circle, 94
theorem on polyhedral sets, 69
theorem on strict separation, 4

value (of an extremal problem), 70
vertex, 31

weak topology, 82
Weierstrass theorem, 29
Weyl theorem, 88

Young–Fenchel transform, 39
Young's inequality, 94

zero-convex set, 13, 80

Convex analysis is a branch of mathematics that studies convex sets, convex functions, and convex extremal problems. It has surprisingly diverse and fruitful applications in mathematics, mathematical physics, technology, and economics.

This book is an introduction to convex analysis and some of its applications. It starts with basic theory, which is explained within the framework of finite-dimensional spaces. The only prerequisites are basic analysis and simple geometry. The second chapter presents some applications of convex analysis, including problems of linear programming, geometry, and approximation. Special attention is paid to applications of convex analysis to Kolmogorov-type inequalities for derivatives of functions in one variable. Chapter 3 collects some results on geometry and convex analysis in infinite-dimensional spaces. A comprehensive introduction written "for beginners" illustrates the fundamentals of convex analysis in finite-dimensional spaces.

The book can be used for an advanced undergraduate or graduate-level course on convex analysis and its applications. It is also suitable for independent study of this extremely important area of mathematics.

ISBN 0-8218-3525-4

9 780821 835258

MMONO/222