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Algebraic Analysis of Singular Perturbation Theory

Takahiro Kawai
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Providence, Rhode Island

特異摂動の代数解析学

ALGEBRAIC ANALYSIS OF SINGULAR PERTURBATION THEORY
by Takahiro Kawai and Yoshitsugu Takei

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Preface to the English Edition

We are glad that the English edition is now available. We should be happy if this edition could be of some help to the young people who are interested in the subject discussed in this book, that is, the exact WKB analysis. It is really exact, and not approximate, through its relevance to the Borel transformation. As this is a rapidly growing subject, we have included a Supplement to present a summary of the research done around us since the publication of the Japanese edition. For the details of the results in the Supplement we refer the reader to the papers listed in the Bibliography. (Articles referred to only in the Supplement are also included in the Bibliography.)

We are most grateful to the American Mathematical Society and Professor G. Kato for having tolerantly allowed us to make the needed linguistic corrections to the final draft prepared by Professor Kato. We believe that our corrections will save the reader from many possible misunderstandings.

Using this chance we sincerely thank Professor J. J. Duistermaat for his kind comments on Liouville's contribution to the WKB method. Our heartiest thanks also go to Professor C. M. Bender, who has advised us to include 'theory' in the title of this book, responding to our question.

August 2005 in Kyoto

Takahiro Kawai
Yoshitsugu Takei

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Preface

Perturbation theory is a natural methodology that can be phrased as follows: when the governing rule of the system in question is slightly changed by ε, an approximate solution is obtained by summing up the direct effect of the change on the object, the secondary effect produced by the (main part of) the direct effect, and all higher degree (or, at least sufficiently high degree) effects. Because of its naturality, it is no exaggeration to say that perturbation theory is constantly used in every field of the exact sciences, not only in mathematics. But nature sometimes—or we should rather say 'always'—enshrines in it seemingly complicated but actually sublimely beautiful structures, which seems so complicated that human beings cannot perceive it to be 'natural' at first. Perturbation theory is also one example of such subtleties that nature presents; most of the perturbation problems we encounter are the so-called singular perturbations whose characteristic property is that the aspect of the problem for $\varepsilon = 0$ is substantially different from that for $\varepsilon \neq 0$. As the analytic counterpart of this singular character, perturbation series almost surely diverge in singular perturbations. Probably because of this 'seeming complexity', pure mathematicians seem to have been little interested in singular perturbation theory, and we think it is not without reason. Actually, in our opinion, 'algebraic analysis of singular perturbation theory' or 'exact WKB analysis' is a field of mathematics whose core meaning can be grasped only after human beings master 'microlocal analysis'. (We hope the reader will feel so in Chapter 2, Section 2.3.)

The principal aim of this book is to sketch the recent results of our group; we have tried to describe the route to the goal, rather than the goal itself, as we understand it. Hence we confine our discussion to the case where the 'governing rule of the system' is given through a differential equation. In spite of such a restriction, we still hope this tiny monograph may trigger further development of singular perturbation theory, recalling the substantial effect on constructive field theory of the work of Bender-Wu [**14**] that discusses the eigenvalue problems in quantum mechanics, not quantum field theory.

It was Professor Mikio Sato who launched the authors' interest in the (algebraic analysis of) singular perturbation theory. The elder author (T.K.) sincerely thanks Professor Sato for having led him to the field that was seemingly alien to his own subject at that time, just at the time when one may often be tempted to choose one's subject in the 'natural' extension of one's past achievements. The younger author (Y.T.) thanks Professor Sato for having let him know of such a fruitful field at the early stage of his life as a mathematician. Both authors are truly indebted to Professor Sato for his exceptionally appropriate advice. In a word, this book (except for Chapter 4) is a report of the seminar conducted by mentor Sato, where we discussed in depth Bender-Wu [**14**], Pham [**51**], Voros [**65**], etc. The authors express their heartiest thanks to Professor Takashi Aoki for their stimulating discussions with him, from which they benefited much at every stage of their research. They express their heartiest thanks also to Professor Michio Jimbo, who gave them leads to their discussion in Chapter 4. We are thankful to the editors Kenji Ueno, Kazuhiko Aomoto, and Michio Jimbo, who gave us the opportunity to write this book, and also to Professor Orlando Neto, who arranged the Summer School supported by EU (July, 1996, Lisbon) for us to give a series of lectures on the material of this book. Without these opportunities, we could not have determined to write a book on such a rapidly changing and progressing subject as exact WKB analysis. Last but not least, we sincerely thank Professor Kazuo Murota for his careful reading of the manuscript and for providing us with invaluable advice to improve this monograph.

February 1997

Takahiro Kawai
Yoshitsugu Takei

Summary and Overview

The central theme of this book is the singular perturbation theory of differential equations, mainly the so-called WKB analysis. As Chapter 4 presents the analysis of the Painlevé transcendents whose final target is their connection formulae, the reader might wonder how it is related to the WKB analysis. But, as the reader will see, our discussion in Chapter 4 is based on the WKB analysis of a particular Schrödinger equation that underlies the Painlevé equation. In what follows, we give a summary of the theory of WKB analysis and the contents of this book, touching upon the historic background of the theory.

As explained in Chapter 2, WKB analysis (or the WKB method) is a method of obtaining a formal solution $\psi(x, \eta)$ of a (1-dimensional) Schrödinger equation

$$\left(-\frac{d^2}{dx^2} + \eta^2 Q(x)\right) \psi(x, \eta) = 0 \tag{1}$$

in the form

$$\exp\left(\int_{x_0}^{x} S(x, \eta) dx\right), \tag{2}$$

where

$$S(x, \eta) = S_{-1}(x)\eta + S_0(x) + S_1(x)\eta^{-1} + \cdots, \tag{3}$$

$$x_0 \text{ is a properly chosen constant.} \tag{4}$$

(Here $Q(x)$ is a holomorphic function or a rational function, and $\eta = 1/\hbar$, where $\hbar$ is the Planck constant. Hence, η is interpreted as a large parameter.) Such a formal solution is said to be a *WKB solution*. WKB is named for the three physicists Wentzel, Kramers, and Brillouin, who used this method efficiently for the study of quantum physics. As is usually the case with singular perturbations, this approach is a very natural one and the expansion of this sort had been used in analysis before them; as examples, we may count Jeffreys [**31**] and the Debye expansion for the Bessel function with a large order (cf. [**46**, p. 156] for example).* Actually the Debye formula, which may appear to be really a mysterious one, is understood to be a natural one if regarded as an example of the application of this method (cf. Chapter 2, Remark 2.3). Parenthetically, we note that we discuss in this book not a 'differential equation with a small parameter' but a 'differential equation with a large parameter (to be always denoted by η)', as we used $\eta = 1/\hbar$, not $\hbar$ itself, as a parameter in (1). This is a matter of convenience, but we call the reader's attention to this point in view of its importance; we use a large parameter mainly because the Borel transformation, which plays the central role in this book, can be more neatly described with the use of a large parameter, and the reason why we stick to the symbol η is just because the resulting notations are most well-balanced if the corresponding variable is denoted by y in the Borel transformation. (See Chapter 2, (2.20) for example.)

As will be explained in Chapter 2, Section 2.2, the formal solution $S(x, \eta)$ as given in (3) is uniquely determined recursively by

$$S_{-1}^2 = Q, \tag{5}$$

$$2S_{-1}S_j + \sum_{\substack{k+l=j-1 \\ k,l \geq 0}} S_k S_l + \frac{dS_{j-1}}{dx} = 0 \qquad (j \geq 0) \tag{6}$$

once the sign of $S_{-1} = \pm\sqrt{Q(x)}$ is fixed. From this construction, each S_j is holomorphic except at a zero point (which is called a *turning point* of (1)) and a singular point of $Q(x)$. Thus, the algebraic structure is clear. However, unfortunately, $S(x, \eta)$ almost always diverges as a series in η^{-1} (Chapter 2, Sections 2.2 and 2.3), reflecting its singular perturbative character. And, as is always the case with divergent series, disputes over the legitimacy and the scope of applicability of WKB analysis continued until quite recently; such foggy conditions were cleared up only in the 1980's by the key-word 'Borel sum' (Chapter 1). To be more concrete, by considering the Borel

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Note added in proof* (August, 2005): Concerning this point Professor J. J. Duistermaat has kindly informed one of the authors (T.K.) that WKB expansions follow directly from the techniques of Liouville [44**] and that we should not forget the original form of the Sturm-Liouville theory. We share the same viewpoint of the Sturm-Liouville theory with Professor Duistermaat, and we are most grateful to him for his kind and informative letter dated September 4, 2000.

sum of WKB solutions, Voros [**65**] established in 1983 the connection formulae for WKB solutions (cf. Chapter 2, Section 2.3) when all the zero points of $Q(x)$ are simple, and, in 1985, Silverstone [**57**] gave a clear-cut argument to show that the doubts over the legitimacy of the WKB analysis are due to the arguments applied—without any logical grounds—to the points where the Borel sum of a WKB solution is not well defined. We note that, behind these epoch-making papers, there had been, on the mathematics side, the suggestion of Dingle [**20**] to the effect that correct connection formulae for WKB solutions could be obtained only through the information on all terms even though they are divergent, and, on the physics side, several affirmative results on the Borel summability of perturbative expansions that followed the remarkable paper of Bender-Wu [**14**]. (Cf. e.g. Magnen-Sénéor [**45**], 't Hooft [**64**] and Eckmann-Epstein [**24**].) Further, just around that time Ecalle ([**21**], [**22**], etc.) was developing a new analysis 'resurgent theory', which is also based on the Borel sum, and located at the crossing point of these two trends are the works of the Nice group led by Pham (Pham [**51**], Candelpergher-Nosmas-Pham [**16**], Delabaere-Dillinger-Pham [**18**], [**19**], etc.). This kind of WKB analysis based on Borel sums has been recently called the exact WKB analysis. In this book, in most cases, we simply say 'the WKB analysis' for the exact WKB analysis. (Ignoring whether a 'proof' is given or not (actually the argument on the Borel summability in Chapter 4 is not yet perfect; see Future Directions and Problems), as an idea, we are always considering the exact WKB analysis.)

The discussion given so far might have given the reader an impression that the exact WKB analysis was developed just to avoid divergence problems. However, the consideration of the Borel sum of a WKB solution has much more positive merit; as will be explained in Chapter 1, the notion of a Borel sum is based on the analytic continuation of the *Borel transform* of a divergent series, giving the exact description of the divergent series. The above fact is the reason why the exact WKB analysis is effective for the treatment of exponentially small terms in eigenvalue problems, but the argument of Voros in [**65**] is effective for general problems of differential equations in the large, beyond the framework of eigenvalue problems ([**2**], [**54**]). As one of the most outstanding examples, we will show the fact (see Chapter 3, Section 3.1) that the monodromy group of a second order Fuchsian type differential equation (in a generic situation) can be described in terms of 'the contour integral of the logarithmic derivative of the

WKB solution' (precisely speaking, of its odd part) (see Chapter 2, Section 2.1). The reader will get an impression 'Yes, indeed exact'.

Now that the WKB analysis is shown to be effective for describing the monodromy group, the following question can be naturally raised: How is the WKB analysis related to the monodromy preserving deformation? (See Jimbo [**32**].) (Note that 'naturally' does not mean 'trivially'. As we mentioned in the Preface, it was Professor Michio Jimbo who led us to this question.) We initially considered this problem to be easy to answer. However, when we began our investigation by introducing a large parameter η (Chapter 4, Section 4.1), to our surprise, the monodromy preserving deformation is always associated with a double turning point (Chapter 4, Section 4.3). This kind of an inevitable degeneration often indicates something interesting behind it; in fact, we found an analysis centered around this 'double turning point'. To be more explicit, we first construct a formal solution (with the double turning point as the principal part) of the Painlevé equation relevant to the monodromy preserving deformation in question. Then we look for the connection formula for general Painlevé transcendents by using the following properties (7) and (8). (For the history of the Painlevé transcendent, see [**32**] by Professor Jimbo, who has made a substantial contribution to it. We like to note, at least, that it is an interesting example to be seriously considered when we discuss the problem: what is 'useful mathematics'?, and that about one hundred years after the work of Painlevé, the Painlevé transcendent (a special function of the twentieth century) is still an attractive subject to study, which has not yet been fully understood.)

(7) For Painlevé I, the analytic continuation of the Borel sum of the formal solution can be explicitly described (Chapter 4, Section 4.5).

(8) With an appropriate correspondence of the parameters, their formal solutions are locally 'equivalent' (see Chapter 4, Section 4.6 for the precise meaning).

Although we do not describe the details of our discussion here (see Chapter 4 for them), we should emphasize the following fact.

"The 'transformation' to be used to show the 'equivalence' asserted in (8) should be, logically speaking, found by studying the Painlevé equation only; but, in our actual construction of the 'transformation', the transformation of the Schrödinger equation (SL_J) that underlies the Painlevé equation (P_J) appears naturally."

Here we should recall the fact that the Painlevé equation was originally found by Painlevé (and his student Gambier) while looking for a second order differential equation whose solutions do not admit movable branch points. This viewpoint is completely different from what is employed in this book, i.e., '(P_J) is a condition that lets (SL_J) be deformed isomonodromically'. (See R. Fuchs [**28**] for (P_{VI}), and Okamoto [**48**] for other (P_J)'s. See also Jimbo-Miwa-Ueno [**33**] and Jimbo [**32**].) Even the formulation should be difficult, if we were to study connection formulae for solutions of an equation which admits solutions with movable branch points, i.e., branch points that depend on parameters contained in the solution (not the equation) [note that the singular points of solutions of linear differential equations (cf. Chapters 2 and 3) are confined to the singular points of the equation and that such movable branch points do not appear for linear equations]. Honestly speaking, we 'naturally' arrived at the connection formulae for the Painlevé transcendents, without being seriously aware of the above characterization (the so-called Painlevé property) of the Painlevé transcendents, at least at the first stage of our study. In retrospect, we ourselves are really impressed by the miraculous harmony that the Painlevé transcendents enjoy. We are, however, still far away from the level of 'applicable mathematics'; e.g., we do not know how our constructed formal solution corresponds to the true solution. We hope that some of the readers will join us in this quest. Shall we make a collection of formulas where the Painlevé transcendent is a twentieth century special function?

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**Note*: RIMS Koukyuuroku is a series of the proceedings of a conference or a workshop held at the Research Institute for Mathematical Sciences, Kyoto

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The topic of this book is the study of singular perturbations of ordinary differential equations, i.e., perturbations that represent solutions as asymptotic series rather than as analytic functions in a perturbation parameter. The main approach used by the authors is the so-called WKB (Wentzel–Kramers–Brillouin) method, originally invented for the study of quantum-mechanical systems. The authors describe in detail the WKB method and its applications to the study of monodromy problems for Fuchsian differential equations and to the analysis of Painlevé functions. The volume is suitable for graduate students and researchers interested in differential equations and special functions.

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