

Translations of
**MATHEMATICAL
MONOGRAPHS**

Volume 228

**Principal Structures
and Methods of
Representation Theory**

D. Zhelobenko



American Mathematical Society

Principal Structures
and Methods of
Representation Theory

This page intentionally left blank

Translations of
**MATHEMATICAL
MONOGRAPHS**

Volume 228

Principal Structures
and Methods of
Representation Theory

D. Zhelobenko

Translated by Alex Martsinkovsky



American Mathematical Society
Providence, Rhode Island

EDITORIAL COMMITTEE

AMS Subcommittee

Robert D. MacPherson Grigorii A. Margulis James D. Stasheff (Chair)

ASL Subcommittee Steffen Lempp (Chair)

IMS Subcommittee Mark I. Freidlin (Chair)

Д. П. Желобенко

ОСНОВНЫЕ СТРУКТУРЫ И МЕТОДЫ ТЕОРИИ ПРЕДСТАВЛЕНИЙ

МЦНМО, МОСКВА, 2004

This work was originally published in Russian by МЦНМО under the title “Основные структуры теории представлений” ©2004. The present translation was created under license for the American Mathematical Society and is published by permission.

Translated from the Russian by Alex Martsinkovsky

2000 *Mathematics Subject Classification*. Primary 20–01, 20Cxx;
Secondary 17B10, 20G05, 20G42.

For additional information and updates on this book, visit
www.ams.org/bookpages/mmono-228

Library of Congress Cataloging-in-Publication Data

Zhelobenko, D. P. (Dmitrii Petrovich)

[Osnovnye struktury i metody teorii predstavlenii. English]

Principal structures and methods of representation theory / D. Zhelobenko ; translated by Alex Martsinkovsky.

p. cm. — (Translations of mathematical monographs ; v. 228)

“Originally published in Russian by MTSNMO under the title ‘Osnovnye struktury i metody teorii predstavlenii’ c2004”—T.p. verso.

Includes bibliographical references and index.

ISBN 0-8218-3731-1 (alk. paper)

1. Representations of groups. 2. Representations of algebras. I. Title. II. Series.

QA176.Z5413 2004

512'.22—dc22

2005052352

Copying and reprinting. Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy a chapter for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication is permitted only under license from the American Mathematical Society. Requests for such permission should be addressed to the Acquisitions Department, American Mathematical Society, 201 Charles Street, Providence, Rhode Island 02904-2294, USA. Requests can also be made by e-mail to reprint-permission@ams.org.

© 2006 by the American Mathematical Society. All rights reserved.

The American Mathematical Society retains all rights
except those granted to the United States Government.

Printed in the United States of America.

∞ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.

Visit the AMS home page at <http://www.ams.org/>

10 9 8 7 6 5 4 3 2 1 11 10 09 08 07 06

Contents

Preface	ix
Part 1. Introduction	1
Chapter 1. Basic Notions	3
1. Algebraic structures	3
2. Vector spaces	8
3. Elements of linear algebra	14
4. Functional calculus	20
5. Unitary spaces	26
6. Tensor products	35
7. S -modules	40
Comments to Chapter 1	46
Part 2. General Theory	49
Chapter 2. Associative Algebras	51
8. Algebras and modules	51
9. Semisimple modules	58
10. Group algebras	64
11. Systems of generators	70
12. Tensor algebras	75
13. Formal series	80
14. Weyl algebras	86
15. Elements of ring theory	93
Comments to Chapter 2	98
Chapter 3. Lie Algebras	99
16. General questions	99
17. Solvable Lie algebras	105
18. Bilinear forms	109
19. The algebra $U(\mathfrak{g})$	115
20. Semisimple Lie algebras	120
21. Free Lie algebras	125
22. Examples of Lie algebras	130
Comments to Chapter 3	137
Chapter 4. Topological Groups	139
23. Topological groups	139
24. Topological vector spaces	145

25. Topological modules	152
26. Invariant measures	157
27. Group algebras	164
28. Compact groups	170
29. Solvable groups	175
30. Algebraic groups	181
Comments to Chapter 4	185
Chapter 5. Lie Groups	187
31. Manifolds	187
32. Lie groups	192
33. Formal groups	198
34. Local Lie groups	203
35. Connected Lie groups	209
36. Representations of Lie groups	214
37. Examples and exercises	219
Comments to Chapter 5	224
Part 3. Special Topics	225
Chapter 6. Semisimple Lie Algebras	227
38. Cartan subalgebras	227
39. Classification	233
40. Verma modules	238
41. Finite-dimensional \mathfrak{g} -modules	244
42. The algebra $Z(\mathfrak{g})$	250
43. The algebra $F_{\text{ext}}(\mathfrak{g})$	256
Comments to Chapter 6	262
Chapter 7. Semisimple Lie Groups	263
44. Reductive Lie groups	263
45. Compact Lie groups	268
46. Maximal tori	272
47. Semisimple Lie groups	277
48. The algebra $\mathcal{A}(G)$	283
49. The classical groups	289
50. Reduction problems	294
Comments to Chapter 7	300
Chapter 8. Banach Algebras	301
51. Banach algebras	301
52. The commutative case	307
53. Spectral theory	312
54. C^* -algebras	317
55. Representations of C^* -algebras	323
56. Von Neumann algebras	329
57. The algebra $C^*(G)$	335
58. Abelian groups	340
Comments to Chapter 8	346

Chapter 9. Quantum Groups	347
59. Hopf algebras	347
60. Weyl algebras	353
61. The algebra $U_q(\mathfrak{g})$	359
62. The category \mathcal{O}_{int}	365
63. The algebra $\mathcal{A}_q(\mathfrak{g})$	370
64. Gaussian algebras	377
65. Projective limits	383
Comments to Chapter 9	388
Appendix A. Root Systems	391
Comments to Appendix A	402
Appendix B. Banach Spaces	403
Appendix C. Convex Sets	407
Appendix D. The Algebra $B(H)$	413
Bibliography	421
Index	425

This page intentionally left blank

Preface

The title of this book admits two interpretations, with emphasis on either the “principal structures” or the “representation theory”. The latter is more preferable, as it is difficult to identify what the basic structures of modern mathematics are. Nevertheless, in a sense, the two interpretations agree.

Indeed, representation theory deals with fundamental aspects of mathematics, beginning with algebraic structures like semigroups, groups, rings, associative algebras, Lie algebras, etc. Eventually topology enters the play by way of algebro-topological and algebro-analytical structures like topological groups, manifolds, Lie groups, etc. Formally speaking, the subject of representation theory is the study of homomorphisms (representations) of abstract structures into linear structures consisting, as a rule, of linear operators on vector spaces. But in fact representation theory is tied up with structure theory. Very early the students of mathematics learn that “ring theory is inseparably linked with module theory”. An important feature of this setting is that the above structures are either linear or have suitable linearizations (linear hulls of semigroups, tangent Lie algebras of Lie groups, etc.).

Here we come to the question of the role representation theory plays in modern mathematics. Originally (in the beginning of the 20th century) that role was rather modest and was confined to the representation theory of finite groups and, eventually, finite-dimensional (associative) algebras. We should mention the connections of that theory with problems of symmetry in algebra and geometry, including Galois theory (the symmetries of algebraic equations), and with problems of crystallography. Eventually the subject of representation theory significantly expanded in response to general questions from analysis, geometry, and physics. Fundamental discoveries in theoretical physics, such as the theory of relativity and quantum mechanics, played a significant role in that process. For example, it turned out that logical foundations of quantum mechanics can be adequately expressed in terms of automorphisms of certain algebras (the algebras of observables). The process of describing observables reduces to representation theory of certain Lie groups and algebras. Among classical results of that period we specifically mention the works of E. Cartan and H. Weyl on the general aspects of the theory of Lie groups and on harmonic analysis on compact groups.

The underlying idea of harmonic analysis on groups is based on the connection between a group G and the “dual object” \hat{G} consisting, roughly speaking, of irreducible representations of G . Usually G can be recovered, up to isomorphism, from its dual object \hat{G} . A remarkable feature of harmonic analysis is that numerical functions on G can be recovered from their (operator) “Fourier images”, where the role of elementary harmonics is played by irreducible representations of G . A meaningful definition of Fourier images on locally compact groups is possible because of the fundamental results of A. Haar, J. von Neumann, and A. Weil on the existence

(and uniqueness) of invariant measures on such groups. In that sense, the classical Fourier analysis (Fourier series and integrals) is subsumed into an impressive development program of harmonic analysis on topological groups.

Logical foundations of Fourier analysis can be significantly clarified within the framework of “abstract harmonic analysis”, where the group G is replaced by a C^* -algebra. Fundamental results in that direction are due to I. M. Gelfand and M. A. Naimark (in the 1940s). Beginning with the 1950s, the theory of C^* -algebras develops very rapidly and, to a large extent, characterizes the functional analysis of the 20th century. It is important to observe that that theory has fundamental applications to operator algebras, Hopf algebras, dynamical systems, statistical mechanics, quantum field theory, etc.

Modern representation theory deals with a wide variety of associative algebras, including structure algebras of manifolds and Lie groups, universal enveloping algebras of Lie algebras, group (convolution) algebras, Hopf algebras, quantum groups, etc. Notice that the theory of Lie groups, born within the context of differential geometry, is now included in the framework of functional analysis by way of bialgebras and formal groups associated with Lie groups.

One may also expand the definition of representation theory to include, if desired, such neighboring disciplines as abstract theory of differential equations, theory of sheaves on homogenous spaces, microanalysis, quantum field theory, etc.

There is a known thesis according to which “mathematics is representation theory”. The corresponding antithesis can be stated as “mathematics does not reduce to representation theory”. It is worthwhile to note the nature of the question. Whatever is true, it appears that the scope of representation theory is already comparable with that of the entire mathematics.

It may be that the desire to systematize mathematics in the spirit of representation theory made N. Bourbaki write the multi-volume set “Elements of mathematics”. Despite certain shortcomings of that titanic work (excessive formalism, unfinished parts) one finds original treatment of several fundamental issues, including general aspects of algebra, topology, the theory of integration, the theory of Lie groups and Lie algebras, etc.

At present, there is a large number of monographs dealing with various aspects of representation theory, including Lie groups and Lie algebras ([4], [10], [14], [31], [35], [61]). Banach algebras ([6], [8], [13], [22], [49], [58]), algebraic groups ([3], [29], [64], [73]), infinite-dimensional groups ([53]), general representation theory ([40]). The author’s monograph [75] can be used as an easily accessible source of information on representations of Lie groups, especially suitable for physicists. However, there is still no monograph which would put together all of those aspects of representation theory.

To fill the gap, this book was conceived as a compilation of canonical texts on representation theory. It provides a systematic description of a wide spectrum of algebro-topological structures. On one hand, the concept of such a book is appealing because it allows us to compare ideas and methods from different parts of representation theory. On the other hand, it is also risky just because of the sheer volume of the material to be covered. Nevertheless, the author thinks that a partial resolution of this dilemma is possible because the offered texts have been carefully worked upon and refined.

The contents of the book split into three parts. Part I (Introduction) contains general facts for beginners, including linear algebra and functional analysis. The survey-type sections on topology, theory of integration, etc. (see [23], [24], [26], [31]) as well as Appendices A, B, C, and D are written in the same spirit. In the main Part II (General theory) we consider associative algebras, Lie algebras, topological groups, and Lie groups. We also mention some aspects of ring theory and the theory of algebraic groups. We provide a detailed account of classical results in those branches of mathematics, including invariant integration and Lie's theory of connections between Lie groups and Lie algebras. In Part III (Special topics) we consider semisimple Lie algebra and Lie groups, Banach algebras, and quantum groups.

The book brings the reader close to the modern aspects of “noncommutative analysis”, including harmonic analysis on locally compact groups. The author regards the contents of this book as a prerequisite for those who want to seriously study representation theory.

The style of the book allows the author to choose the depth of the exposition to his taste. For example, we prove the theorem on the conjugacy of Cartan subalgebras (in complex Lie algebras) but omit a similar result for Borel subalgebras (in semisimple Lie algebras). Yet the author hopes that the reader will see a detailed enough panoramic description of representation theory.

The diverse nature of the compiled material unavoidably leads to discrepancies in traditions, which sometimes cause certain redundancy in the definitions and notation. For example, the notation $\text{End } X$ in the category of vector spaces is sometimes replaced by $L(X)$ where $\dim X < \infty$.

The exercises included in the book, as a rule, are designed as tests for beginners. Sometimes (in moderation) the results of the exercises are used to shorten certain proofs. Only the exercises marked with an asterisk can be viewed as more or less serious problems.

While working on the book the author felt himself a chronicler. Indeed, the book covers a century in the development of mathematics, a period which is probably not yet fully appreciated.

The contents of the book are, to a large extent, based on two elective courses the author gave at the Independent University of Moscow in 1996–1998. The lecture notes of one of those courses were published in 2001 ([78]). The work on this book was partially supported by the RFFI Grant 01-01-00490 and NWO 047-008-009.

The author is grateful to V. R. Nigmatullin for his help during the proofreading of the text.

D. Zhelobenko

This page intentionally left blank

This page intentionally left blank

Bibliography

- [1] Yu. A. Bakhturin, *Basic structures of modern algebra*. Nauka, Moscow, 1990; English transl., Kluwer, Dordrecht, 1993.
- [2] I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand, *Structure of representations that are generated by vectors of higher weight*. Funktsional. Anal. i Pril. **5** (1971), no. 1, 1–9. (Russian)
- [3] A. Borel, *Linear algebraic groups*. Second edition. Graduate Texts in Mathematics, vol. 126. Springer-Verlag, New York, 1991.
- [4] N. Bourbaki, *Lie groups and Lie algebras*. Chapters 1–3, 4–6, 7–9. Springer-Verlag, Berlin, 1998, 2002, 2005.
- [5] ———, *Integration*. Chapters 1–6, 7–9. Springer-Verlag, Berlin, 2004.
- [6] ———, *Théories spectrales*. Chapitre I: Algèbres normées. Chapitre II: Groupes localement compacts commutatifs. Actualités Scientifiques et Industrielles, No. 1332, Hermann, Paris 1967.
- [7] ———, *Topological vector spaces*. Springer-Verlag, Berlin, 1987.
- [8] O. Bratteli and D. W. Robinson, *Operator algebras and quantum statistical mechanics*. Vol. I. C^* - and W^* -algebras, algebras, symmetry groups, decomposition of states. Springer-Verlag, New York–Heidelberg, 1979.
- [9] V. Chari and A. Pressley, *A guide to quantum groups*. Cambridge Univ. Press, Cambridge, 1995.
- [10] C. Chevalley, *Théorie des groupes de Lie I, II, III*. Vol. I is available in English: Princeton University Press, Princeton, NJ, 1999. Vols. II and III: Hermann, Paris, 1951, 1955.
- [11] A. Connes, *Noncommutative geometry*. Academic Press, San Diego, CA, 1994.
- [12] E. Demidov, *Quantum groups*. Factorial, Moscow, 1998. (Russian)
- [13] J. Dixmier, *Enveloping algebras*. Amer. Math. Soc., Providence, RI, 1996.
- [14] ———, *C^* -algebras*. North-Holland, Amsterdam–New York–Oxford, 1977.
- [15] V. G. Drinfeld, *Quantum groups*. Proc. Intern. Congress Math., Berkeley, 1986. Amer. Math. Soc., Providence, RI, 1987.
- [16] M. Enock and J.-M. Schwartz, *Kac algebras and duality of locally compact groups*. Springer-Verlag, 1992.
- [17] L. D. Faddeev and A. O. Yakubovskii, *Lectures on quantum mechanics for students of mathematics*. Leningrad State University, 1980. (Russian)
- [18] G. Gasper and M. Rahman, *Basic hypergeometric series*. With a foreword by Richard Askey. Encyclopedia of Mathematics and its Applications, vol. 35. Cambridge Univ. Press, Cambridge, 1990.
- [19] L. Gårding, *Vecteurs analytiques dans les représentations des groupes de Lie*. Bull. Soc. Math. France **88** (1960), 73–93.
- [20] I. M. Gelfand and A. A. Kirillov, *Structure of the Lie sfield connected with a semisimple decomposable Lie algebra*. Funktsional. Anal. i Pril. **3** (1969), no. 1, 7–26. (Russian)
- [21] I. M. Gelfand and M. A. Naimark, *Unitary representations of the classical groups*. Trudy Mat. Inst. Steklov., vol. 36, Izdat. Akad. Nauk SSSR, Moscow–Leningrad, 1950. (Russian)
- [22] I. Gelfand, D. Raikov, and G. Shilov, *Commutative normed rings*. Chelsea, New York, 1964.
- [23] M. Goto and F. D. Grosshans, *Semisimple Lie algebras*. Lecture Notes Pure Appl. Math., vol. 38, Marcel Dekker, New York–Basel, 1978.
- [24] P. R. Halmos, *Measure theory*. D. Van Nostrand, New York, 1950.
- [25] ———, *A Hilbert space problem book*. 2nd ed., Springer-Verlag, New York–Berlin, 1982.
- [26] S. Helgason *Groups and geometric analysis*. Integral geometry, invariant differential operators, and spherical functions. Mathematical Surveys and Monographs, vol. 83, Amer. Math. Soc. Society, Providence, RI, 2000.

- [27] E. Hewitt and K. A. Ross, *Abstract harmonic analysis*. I, II. Springer-Verlag, Berlin–New York, 1979, 1970.
- [28] A. S. Kholevo, *Introduction to quantum information theory*, MCCME, Moscow, 2002. (Russian)
- [29] J. E. Humphreys, *Linear algebraic groups*. Graduate Texts in Mathematics, No. 21. Springer-Verlag, New York–Heidelberg, 1975.
- [30] ———, *Introduction to Lie algebras and representation theory*. Springer-Verlag, New York–Berlin, 1978.
- [31] N. Jacobson, *Lie algebras*. Dover, New York, 1979.
- [32] A. Joseph, *Quantum groups and their primitive ideals*. Springer-Verlag, Berlin, 1995.
- [33] V. G. Kac, *Infinite-dimensional Lie algebras*. Third edition, Cambridge Univ. Press, Cambridge, 1990.
- [34] V. G. Kac and D. A. Kazhdan, *Structure of representations with highest weight of infinite-dimensional Lie algebras*, Adv. Math. **24** (1979), 97–108.
- [35] I. Kaplansky, *Lie algebras and locally compact groups*. Univ. Chicago Press, Chicago, IL, 1995.
- [36] M. Kashiwara, *The universal Verma module and the b-function*. In: Algebraic groups and related topics (Kyoto–Nagoya 1983), North-Holland, 1985, pp. 69–81.
- [37] ———, *Crystallizing the q-analogue of the universal enveloping algebras*. Comm. Math. Phys. **122** (1990), 249–260.
- [38] Ch. Kassel, *Quantum groups*. Springer-Verlag, Berlin–New York–Heidelberg, 1995.
- [39] J. Kelley, *General topology*. Springer-Verlag, New York–Berlin, 1975.
- [40] A. A. Kirillov, *Elements of the theory of representations*. Springer-Verlag, Berlin–New York, 1976.
- [41] A. I. Kostrikin and Yu. I. Manin, *Linear algebra and geometry*. Gordon and Breach, Amsterdam, 1997.
- [42] A. W. Knap, *Representation theory of semisimple groups*. Princeton Univ. Press, Princeton, NJ, 2001.
- [43] A. N. Kolmogorov and S. V. Fomin, *Elements of representation theory*. Nauka, Moscow, 1978. (Russian)
- [44] S. Lang, *Algebra*. Third Edition, Springer-Verlag, New York, 2002.
- [45] L. H. Loomis, *An introduction to abstract harmonic analysis*. Van Nostrand, Toronto–New York–London, 1953.
- [46] Yu. I. Manin, *Quantum groups and noncommutative geometry*. Montreal, 1988.
- [47] A. Molev, *A basis for representations of symplectic Lie algebras*. Comm. Math. Phys. **201** (1999), 591–618.
- [48] ———, *Gelfand–Tsetlin bases for classical Lie algebras*. Univ. of Sydney, 2002, 120–181.
- [49] G. J. Murphy, *C*-algebras and operator theory*. Academic Press, Boston, MA, 1990.
- [50] M. A. Naimark, *Normed rings*. Wolters-Noordhoff, Groningen, 1970.
- [51] M. A. Naimark and A. I. Stern, *Theory of group representations*. Springer-Verlag, New York, 1982.
- [52] R. Narasimhan, *Analysis on real and complex manifolds*. North-Holland, Amsterdam, 1985.
- [53] Yu. A. Neretin, *Categories of symmetries and infinite-dimensional groups*. Oxford Univ. Press, New York, 1996.
- [54] S. P. Novikov and A. T. Fomenko, *Basic elements of differential geometry and topology*. Kluwer Academic Publishers, Dordrecht, 1990.
- [55] R. S. Pierce, *Associative algebras*. Springer-Verlag, New York, 1982.
- [56] L. S. Pontryagin, *Continuous groups*. Fourth ed., Nauka, Moscow, 1984; English transl. of the second ed. *Topological groups*. Gordon and Breach, London, 1966.
- [57] W. Rudin, *Functional analysis*. Second ed., McGraw-Hill, New York, 1991.
- [58] S. Sakai, *C*-algebras and W*-algebras*, Springer-Verlag, Berlin, 1998.
- [59] H. H. Schaefer and M. P. Wolff, *Topological vector spaces*. Second edition. Springer-Verlag, New York, 1999.
- [60] H. Seifert and W. Threlfall, *Seifert and Threlfall: a textbook of topology*. Academic Press, New York–London, 1980.
- [61] J.-P. Serre, *Lie algebras and Lie groups*. Second edition. Lecture Notes in Mathematics, vol. 1500. Springer-Verlag, Berlin, 1992.
- [62] ———, *Linear representations of finite groups*. Springer-Verlag, New York–Heidelberg, 1977.

- [63] N. N. Shapovalov, *On a bilinear form on the universal enveloping algebra of a complex semisimple Lie algebra*, Funktsional. Anal. i Prilozh. **6** (1972), no. 4, 65–70; English transl., Functional Anal. Appl. **6** (1972), 307–312.
- [64] I. R. Shafarevich, *Basic algebraic geometry*. 1, 2. Springer-Verlag, Berlin, 1994.
- [65] A. Shen and N. K. Vereshchagin, *Elements of set theory*. MCCME Publishers, Moscow, 1999; English transl., Amer. Math. Soc., Providence, RI, 2002.
- [66] G. E. Shilov, *Mathematical analysis. A special course*. Fizmatgiz, Moscow, 1960; English transl., Pergamon Press, Oxford–New York–Paris, 1965.
- [67] ———, *Mathematical analysis. A second special course*. Nauka, Moscow, 1965. (Russian)
- [68] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*. Princeton Univ. Press, Princeton, NJ, 1971.
- [69] S. Sternberg, *Lectures on differential geometry*. Prentice Hall, 1964.
- [70] A. Weil, *L'intégration dans les groupes topologiques et ses applications*. Actual. Sci. Ind., no. 869. Hermann et Cie., Paris, 1940.
- [71] H. Weyl, *Classical groups. Their invariants and representations*. Princeton University Press, Princeton, NJ, 1997.
- [72] E. B. Vinberg, *A course in algebra*. Faktorial, Moscow, 1999; English transl., Amer. Math. Soc., Providence, RI, 2003.
- [73] E. B. Vinberg and A. L. Onishchik, *Seminar on Lie groups and algebraic groups*, URSS, Moscow, 1995. (Russian)
- [74] F. Warner, *Foundations of differentiable manifolds and Lie groups*. Springer-Verlag, 1983.
- [75] D. P. Zhelobenko, *Compact Lie groups and their representations*. Nauka, 1970; English transl., Translation of Mathematical Monographs, vol. 40. Amer. Math. Soc., Providence, RI, 1973.
- [76] ———, *Harmonic analysis on complex semisimple Lie groups*. Nauka, Moscow, 1974. (Russian)
- [77] ———, *Representations of reductive Lie algebras*. Nauka, Moscow, 1992. (Russian)
- [78] ———, *Introduction to representation theory*. Faktorial, Moscow, 2001. (Russian)
- [79] ———, *Classical groups. Spectral analysis of finite-dimensional representations*. Russian Math. Surveys **17** (1962), 27–120.
- [80] ———, *Constructive modules and extremal projectors over Chevalley algebras*. Funktsional. Anal. i Prilozh. **27** (1993), no. 3, 5–14, 95; English transl., Funct. Anal. Appl. **27** (1993), no. 3, 158–165.
- [81] ———, *On Weyl algebras over quantum groups*. Teoret. Mat. Fiz. **118** (1999), no. 2, 190–204; English transl., Theoret. Math. Phys. **118** (1999), no. 2, 152–163.
- [82] O. Zariski and P. Samuel, *Commutative algebra*. I, II, Springer-Verlag, 1975.

This page intentionally left blank

Index

- A-module
 - Artinian, 93
 - diagonal, 57
 - free, 55
 - induced, 97
 - Noetherian, 94
 - primitive, 56
 - semisimple, 52
 - simple, 52
- action
 - adjoint, 352
 - transitive, 42
- admissible W -module, 88
- algebra, 5
 - $\mathcal{A}(G)$, 270, 283
 - $\mathcal{A}_q(\mathfrak{g})$, 370
 - $D_0(G)$, 169
 - $F[[x]]$, 80
 - $L_1(G)$, 165
 - $M_0(G)$, 168
 - $S_q(\mathfrak{g})$, 360
 - $U(\mathfrak{g})$, 115
 - $U_q(\mathfrak{g})$, 359
 - associative, 5
 - Banach, 301
 - Calkin, 332
 - Clifford, 79, 222
 - commutative, 5
 - contragredient, 378
 - Drinfeld–Jimbo, 360
 - exterior, 78
 - free, 103
 - Gaussian, 377, 379
 - graded, 72
 - Grassmann, 78
 - normed, 301
 - of bounded operators, 25, 302
 - of Cartan type, 381
 - of continuous functions, 301
 - of differential operators, 74, 224, 353
 - of polynomials, 181
 - of rational functions, 181
 - opposite, 51
 - quotient, 54
 - semisimple, 54, 59
 - simple, 54
 - structure, 182
 - symmetric, 77
 - tensor, 38, 75
 - unital, 5
 - universal enveloping, 115
 - Weyl, 86, 355
 - Wiener, 301
 - with involution, 304
 - with unit, 347
 - Witt, 132
- algebraic structure, 6
- annihilator, 54
- antipode, 350
- approximate identity, 320
 - strict, 320
- associativity axiom, 3
- atlas, 187
- average, 64
- basis
 - Cartan–Weyl, 231
 - Gelfand–Tsetlin, 297
 - orthonormal, 29
- basis of topology, 140
- bialgebra, 349
- bicommutant, 46
- bilinear form
 - canonical, 11
 - invariant, 109
 - nondegenerate, 11
- bimodule, 52
- Borel extension, 317
- Borel measure
 - invariant, 163
 - quasiinvariant, 163
- Borel subalgebra, 232
- Borel subgroup, 279
- boundary, 139
- bounded functional, 320
- branching rules, 297

- Burnside identity
 - first, 67
 - second, 68
- canonical filtration, 116
- canonical projection, 13
- Cartan decomposition, 238
- Cartan lattice, 282
- Cartan matrix, 398
 - irreducible, 400
 - reducible, 400
 - symmetrizable, 399
- Cartan subalgebra, 111, 227
- Cartan subgroup, 267
- Casimir element, 120
- category, 7
 - monomial, 52
 - of groups, 7
 - of sets, 7
 - of vector spaces, 7
 - semisimple, 52
- category \mathcal{O} , 247
- Cayley–Hamilton equation, 16
- cell, 396
- center, 54
 - of a Lie algebra, 101
- central series, 102
- centralizer, 267
- character
 - central, 242
 - of a representation, 156
 - primitive, 68
- Chevalley basis, 232
- Chevalley generators, 232
- Clebsch–Gordan rule, 299
- closure, 139
- coaction, 349
- coalgebra, 348
 - cocommutative, 348
 - opposite, 348
- coassociativity, 198
- commutator, 52, 102, 175
- commutator subgroup, 175
- comodule, 349
- composition series, 95
- comultiplication, 198, 348
 - coassociative, 348
- connected component, 143
- constant term, 80
- convex hull, 146
- convolution, 43, 164
 - of measures, 168
- coordinates, 28
 - exponential, 207
- cotangent space, 189
- counit, 198, 348
- countability axiom
 - first, 141
 - second, 141
- Coxeter graph, 400
- decomposition reduced, 395
- derivation, 53, 83
 - inner, 100
- diffeomorphism, 188
- differential, 189
- dimension
 - Hilbert, 31
 - of a vector space, 9
- direct sum, 12
 - orthogonal, 31
 - topological, 174
- directed ordered set, 140
- distributivity, 4
- Dynkin diagram, 400
- eigenvalue, 14
- eigenvector, 14
- element
 - central, 101
 - even, 78
 - generalized nilpotent, 304
 - group-like, 127
 - homogeneous, 72
 - integral, 282
 - invertible, 3
 - normal, 305
 - odd, 78
 - positive, 317
 - regular, 276
- equivalent seminorms, 26
- extremal point, 328, 410
- field, 4
 - algebraically closed, 14
 - skew, 4
- field of fractions, 96
- flag of subspaces, 15
 - invariant, 15
- free structure, 70
- formal power series, 80
 - convergent, 85
- Fourier transform, 341
- function
 - Möbius, 126
 - modular, 163
 - positive definite, 155
- functional calculus, 21
- functor, 7
- fundamental group, 210
- fundamental sequence, 150
- fundamental weight, 246
- G -module
 - admissible, 169
 - differentiable, 218
 - holomorphic, 218

- topological, 152
- Gårding subspace, 218
- Gauss decomposition, 178, 265
 - binary, 291
- Gaussian binomial coefficients, 359
- Gelfand transform, 309
- Gelfand–Naimark–Segal construction (GNS), 324
- genetics, 51, 71
- group, 3
 - abelian, 3
 - algebraic, 182, 183
 - formal, 194
 - Gaussian, 268
 - general linear, 142
 - fundamental, 210
 - linearly reductive, 263
 - local, 193
 - nilpotent, 176
 - one-parameter, 154
 - orthogonal, 220
 - solvable, 176
 - spinor, 222
 - symplectic, 221
 - topological, 142
 - unimodular, 163
 - unitary, 142, 221
 - universal covering, 213
- group algebra, 64, 169
- Hermitian form, 26
- homeomorphism, 140
- homomorphism
 - diagonal, 119
 - Harish-Chandra, 252
- homotopy, 209
- homotopy group, 209
- Hopf algebra, 350
 - dual, 352
- ideal, 54
 - derived (in a Lie algebra), 102
 - left, 54
 - maximal, 96, 307
 - of a Lie algebra, 100
 - right, 54
 - two-sided, 54
- identity, 3
- inequality
 - Bessel, 28
 - Cauchy–Bunyakovsky, 27
 - Hölder, 164
 - Schwarz, 27
- interior, 139
- irreducible variety, 183
- Jacobi identity, 54, 99
- Jacobson radical, 96
- Jordan block, 18
- Jordan normal form, 19
- Jordan–Hölder series, 95
- Killing form, 110
- Kostant function, 242
- LCS, 146
- lemma
 - Dini, 158
 - Quillen, 46
 - Schur, 45
 - Urysohn, 144
- Lie algebra, 99
 - $\mathfrak{sl}(2)$, 103
 - associated with a formal group, 195
 - associated with a Lie group, 195
 - free, 104
 - Kac–Moody, 136
 - linear, 130
 - nilpotent, 105
 - orthogonal, 130
 - quotient, 102
 - reductive, 114
 - semisimple, 106
 - simple, 114
 - solvable, 105
 - symplectic, 131
 - tangent, 195
- Lie group, 192
 - local, 194
 - nilpotent, 263
 - reductive, 263
 - semisimple, 263
 - simple, 263
 - solvable, 263
- local coordinates, 187
- local isomorphism, 190
- manifold, 187
 - k -smooth, 188
 - analytic, 188
- map
 - l -smooth, 188
 - continuous, 140
 - covering, 211
 - exponential, 207
 - open, 140
 - regular, 191
- matrix elements, 63, 155
- measure
 - σ -finite, 158
 - Borel, 158
 - left-invariant, 159
 - right-invariant, 159
 - two-side invariant, 159
- finite, 158
- projection, 313
- regular, 158
- spectral, 313

- morphism, 7
- neighborhood, 139
- net, 140
- nilradical, 106
- normalizer, 111, 267
- objects, 7
- one-parameter group, 154
- operator
 - anti-Hermitian, 33
 - bounded, 25
 - compact, 33
 - finite-dimensional, 38
 - Fredholm, 34
 - Hermitian, 33
 - Hilbert–Schmidt, 414
 - linear, 5
 - multilinear, 5
 - of trace class, 416
 - one-dimensional, 38
 - positive, 33
 - projection, 19
 - unitary, 33
- Ore condition, 96
- parity function, 78
- Pfaffian, 294
- Poincaré series, 125
- Poisson bracket, 132
- Pontryagin duality theorem, 345
- prebasis of topology, 141
- precompact set, 145
- primitive exponents, 251
- principal affine variety, 285
- quantum deformation, 372
- quantum determinant, 373
- quantum factorial, 359
- quantum integer, 359
- quantum Serre algebra, 360
- quantum Serre conditions, 360
- quotient manifold, 192
- quotient module, 43
- quotient space, 13
- radical, 106
- rank of a root system, 391
- reflection, 397
- representation, 40
 - absolutely continuous, 153
 - adjoint, 100
 - coadjoint, 120
 - completely reducible, 44
 - Gelfand–Naimark, 324
 - irreducible, 44
 - of a C^* -algebra, 323
 - nondegenerate, 323
 - of a Lie algebra, 100
 - regular, 65
 - topologically irreducible, 157
 - unitary, 155
 - weakly differentiable, 169
- resolvent, 302
- resolvent set, 302
- ring, 4
 - Artinian, 93
 - commutative, 4
 - Jacobson-semisimple, 96
 - Noetherian, 94
 - of formal exponents, 241
 - of fractions, 96
 - semisimple, 93
 - simple, 93
 - with identity, 4
- root, 108, 391
 - negative, 393
 - positive, 393
 - simple, 393
- root subspace, 108
- root system, 391
 - reduced, 391
- root vector, 108
- semidirect product, 164, 217
- semigroup, 3
 - commutative, 3
 - one-parameter, 154
 - with identity, 3
- semigroup algebra, 43
- seminorm, 23
 - countably additive, 403
- series
 - Campbell–Hausdorff, 128
 - convergent, 85
- Serre relations, 232
- Shapovalov form, 253
- similar matrices, 12
- space
 - Banach, 24
 - complete, 150
 - double dual, 150
 - dual, 149
 - Euclidean, 26
 - Hilbert, 29
 - homogeneous, 42
 - locally Euclidean, 145
 - quasicomplete, 150
 - sequentially complete, 150
 - unitary, 26
- spectral radius, 304
- spectrum, 302
 - joint, 311
- spinor modules, 293
- state, 325
 - pure, 327
- structure constants, 102

- subgroup, 4
 - Lie, 192
 - local, 203
 - one-parameter, 206
 - virtual, 208
- submanifold, 191
- submodule, 43
- subring, 4
- subset
 - bounded, 149
 - closed, 139
 - connected, 143
 - convex, 146
 - everywhere dense, 140
 - multiplicatively closed, 96
 - open, 139
- subspace
 - graded, 72
 - invariant, 13
- substitution rule, 82
- superalgebra, 78
 - commutative, 79
- symmetric subalgebra, 306
- system of generators, 51, 70

- tangent
 - map, 189
 - space, 188
 - vector, 188
- Taylor formula, 21
- tensor
 - antisymmetric, 75
 - symmetric, 75
- tensor product, 35, 36
- theorem
 - Artin, 57
 - Baire, 403
 - Banach
 - on closed graph, 404
 - on inverse operator, 405
 - Banach–Steinhaus, 405
 - Burnside, 62
 - Calkin, 332
 - Campbell–Hausdorff, 127
 - Cartan, 113, 205
 - Chevalley, 251
 - Engel, 107
 - Fitting, 108
 - Frobenius, 14
 - on reciprocity, 97
 - Gelfand–Mazur, 304
 - Gelfand–Naimark, 309, 325
 - Godement, 179
 - Haar, 159
 - Hahn–Banach, 148
 - Hamel, 9
 - Harish-Chandra, 252
 - Hilbert, 34, 95
 - Hilbert–Schmidt, 35
 - Jacobson, 62
 - Kadison, 334
 - Kaplansky, 333
 - Kolchin, 176
 - Krein–Milman, 410
 - Levi, 122
 - Levi–Maltsev, 123
 - Lie, 106, 177, 204
 - Maschke, 69
 - Peter–Weil, 170
 - Poincaré–Birkhoff–Witt (PBW), 117
 - Pontryagin, 345
 - Raikov, 341
 - Stone–Weierstrass, 406
 - Tikhonov, 144
 - von Neumann on bicommutant, 330
 - Wedderburn, 61
 - Wedderburn–Artin, 93
 - Weil, 162
 - Weyl, 274, 279
 - Weyl on semisimple representations, 121
- topological module, 152
- topological space, 139
 - compact, 144
 - connected, 143
 - Hausdorff, 139
 - linearly connected, 143
 - locally compact, 145
 - separable, 139
 - simply connected, 210
- topological vector space (TVS), 145
 - locally convex (LCS), 146
- topology
 - bounded convergence, 150
 - discrete, 140
 - quotient, 140
 - reflexive, 150
 - semireflexive, 150
 - simple convergence, 150
 - strong, 149, 329
 - Tikhonov, 141
 - trivial, 140
 - uniform, 329
 - weak, 149, 329
 - Zariski, 182
- torus, 272
- transition function, 187
- TVS, 145

- uniform norm, 24
- unital normed algebra, 301

- vector
 - k -smooth, 218
 - analytic, 218
 - cyclic, 18, 323
 - differentiable, 218
 - infinitely differentiable, 218

vector field, 132, 189
 left-invariant, 196
 smooth, 132
vector space, 4
 dual, 10
 graded, 72
Verma module, 366, 382
 universal, 239, 364
 with highest weight λ , 239
weight lattice, 244

Weyl
 chamber, 396
 character formula, 248
 group, 274
Yang–Baxter
 equation, 373
 matrix, 373
Young product, 286
zerodivisor, 96

Titles in This Series

- 228 **D. Zhelobenko**, Principal structures and methods of representation theory, 2006
- 227 **Takahiro Kawai and Yoshitsugu Takei**, Algebraic analysis of singular perturbation theory, 2005
- 226 **V. M. Manuilov and E. V. Troitsky**, Hilbert C^* -modules, 2005
- 225 **S. M. Natanzon**, Moduli of Riemann surfaces, real algebraic curves, and their superanalogues, 2004
- 224 **Ichiro Shigekawa**, Stochastic analysis, 2004
- 223 **Masatoshi Noumi**, Painlevé equations through symmetry, 2004
- 222 **G. G. Magaril-Il'yaev and V. M. Tikhomirov**, Convex analysis: Theory and applications, 2003
- 221 **Katsuei Kenmotsu**, Surfaces with constant mean curvature, 2003
- 220 **I. M. Gelfand, S. G. Gindikin, and M. I. Graev**, Selected topics in integral geometry, 2003
- 219 **S. V. Kerov**, Asymptotic representation theory of the symmetric group and its applications to analysis, 2003
- 218 **Kenji Ueno**, Algebraic geometry 3: Further study of schemes, 2003
- 217 **Masaki Kashiwara**, D -modules and microlocal calculus, 2003
- 216 **G. V. Badalyan**, Quasipower series and quasianalytic classes of functions, 2002
- 215 **Tatsuo Kimura**, Introduction to prehomogeneous vector spaces, 2003
- 214 **L. Š. Grinblat**, Algebras of sets and combinatorics, 2002
- 213 **V. N. Sachkov and V. E. Tarakanov**, Combinatorics of nonnegative matrices, 2002
- 212 **A. V. Mel'nikov, S. N. Volkov, and M. L. Nechaev**, Mathematics of financial obligations, 2002
- 211 **Takeo Ohsawa**, Analysis of several complex variables, 2002
- 210 **Toshitake Kohno**, Conformal field theory and topology, 2002
- 209 **Yasumasa Nishiura**, Far-from-equilibrium dynamics, 2002
- 208 **Yukio Matsumoto**, An introduction to Morse theory, 2002
- 207 **Ken'ichi Ohshika**, Discrete groups, 2002
- 206 **Yuji Shimizu and Kenji Ueno**, Advances in moduli theory, 2002
- 205 **Seiki Nishikawa**, Variational problems in geometry, 2001
- 204 **A. M. Vinogradov**, Cohomological analysis of partial differential equations and Secondary Calculus, 2001
- 203 **Te Sun Han and Kingo Kobayashi**, Mathematics of information and coding, 2002
- 202 **V. P. Maslov and G. A. Omel'yanov**, Geometric asymptotics for nonlinear PDE. I, 2001
- 201 **Shigeyuki Morita**, Geometry of differential forms, 2001
- 200 **V. V. Prasolov and V. M. Tikhomirov**, Geometry, 2001
- 199 **Shigeyuki Morita**, Geometry of characteristic classes, 2001
- 198 **V. A. Smirnov**, Simplicial and operad methods in algebraic topology, 2001
- 197 **Kenji Ueno**, Algebraic geometry 2: Sheaves and cohomology, 2001
- 196 **Yu. N. Lin'kov**, Asymptotic statistical methods for stochastic processes, 2001
- 195 **Minoru Wakimoto**, Infinite-dimensional Lie algebras, 2001
- 194 **Valery B. Nevzorov**, Records: Mathematical theory, 2001
- 193 **Toshio Nishino**, Function theory in several complex variables, 2001
- 192 **Yu. P. Solov'yov and E. V. Troitsky**, C^* -algebras and elliptic operators in differential topology, 2001
- 191 **Shun-ichi Amari and Hiroshi Nagaoka**, Methods of information geometry, 2000

TITLES IN THIS SERIES

- 190 **Alexander N. Starkov**, Dynamical systems on homogeneous spaces, 2000
- 189 **Mitsuru Ikawa**, Hyperbolic partial differential equations and wave phenomena, 2000
- 188 **V. V. Buldygin and Yu. V. Kozachenko**, Metric characterization of random variables and random processes, 2000
- 187 **A. V. Fursikov**, Optimal control of distributed systems. Theory and applications, 2000
- 186 **Kazuya Kato, Nobushige Kurokawa, and Takeshi Saito**, Number theory 1: Fermat's dream, 2000
- 185 **Kenji Ueno**, Algebraic Geometry 1: From algebraic varieties to schemes, 1999
- 184 **A. V. Mel'nikov**, Financial markets, 1999
- 183 **Hajime Sato**, Algebraic topology: an intuitive approach, 1999
- 182 **I. S. Krasil'shchik and A. M. Vinogradov, Editors**, Symmetries and conservation laws for differential equations of mathematical physics, 1999
- 181 **Ya. G. Berkovich and E. M. Zhmud'**, Characters of finite groups. Part 2, 1999
- 180 **A. A. Milyutin and N. P. Osmolovskii**, Calculus of variations and optimal control, 1998
- 179 **V. E. Voskresenskii**, Algebraic groups and their birational invariants, 1998
- 178 **Mitsuo Morimoto**, Analytic functionals on the sphere, 1998
- 177 **Satoru Igari**, Real analysis—with an introduction to wavelet theory, 1998
- 176 **L. M. Lerman and Ya. L. Umanskiy**, Four-dimensional integrable Hamiltonian systems with simple singular points (topological aspects), 1998
- 175 **S. K. Godunov**, Modern aspects of linear algebra, 1998
- 174 **Ya-Zhe Chen and Lan-Cheng Wu**, Second order elliptic equations and elliptic systems, 1998
- 173 **Yu. A. Davydov, M. A. Lifshits, and N. V. Smorodina**, Local properties of distributions of stochastic functionals, 1998
- 172 **Ya. G. Berkovich and E. M. Zhmud'**, Characters of finite groups. Part 1, 1998
- 171 **E. M. Landis**, Second order equations of elliptic and parabolic type, 1998
- 170 **Viktor Prasolov and Yuri Solov'yev**, Elliptic functions and elliptic integrals, 1997
- 169 **S. K. Godunov**, Ordinary differential equations with constant coefficient, 1997
- 168 **Junjiro Noguchi**, Introduction to complex analysis, 1998
- 167 **Masaya Yamaguti, Masayoshi Hata, and Jun Kigami**, Mathematics of fractals, 1997
- 166 **Kenji Ueno**, An introduction to algebraic geometry, 1997
- 165 **V. V. Ishkhanov, B. B. Lur'e, and D. K. Faddeev**, The embedding problem in Galois theory, 1997
- 164 **E. I. Gordon**, Nonstandard methods in commutative harmonic analysis, 1997
- 163 **A. Ya. Dorogovtsev, D. S. Silvestrov, A. V. Skorokhod, and M. I. Yadrenko**, Probability theory: Collection of problems, 1997
- 162 **M. V. Boldin, G. I. Simonova, and Yu. N. Tyurin**, Sign-based methods in linear statistical models, 1997
- 161 **Michael Blank**, Discreteness and continuity in problems of chaotic dynamics, 1997
- 160 **V. G. Osmolovskii**, Linear and nonlinear perturbations of the operator div, 1997
- 159 **S. Ya. Khavinson**, Best approximation by linear superpositions (approximate nomography), 1997
- 158 **Hideki Omori**, Infinite-dimensional Lie groups, 1997
- 157 **V. B. Kolmanovskii and L. E. Shaikhet**, Control of systems with aftereffect, 1996

For a complete list of titles in this series, visit the
AMS Bookstore at www.ams.org/bookstore/.

The main topic of this book can be described as the theory of algebraic and topological structures admitting natural representations by operators in vector spaces. These structures include topological algebras, Lie algebras, topological groups, and Lie groups.

The book is divided into three parts. Part 1 surveys general facts for beginners, including linear algebra and functional analysis. Part 2 considers associative algebras, Lie algebras, topological groups, and Lie groups, along with some aspects of ring theory and the theory of algebraic groups. The author provides a detailed account of classical results in related branches of mathematics, such as invariant integration and Lie's theory of connections between Lie groups and Lie algebras. Part 3 discusses semisimple Lie algebras and Lie groups, Banach algebras, and quantum groups.



For additional information
and updates on this book, visit

www.ams.org/bookpages/mmono-228

ISBN 0-8218-3731-1



9 780821 837313

MMONO/228

AMS *on the Web*
www.ams.org