

Translations of  
**MATHEMATICAL  
MONOGRAPHS**

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Volume 231

**Essentials of  
Stochastic Processes**

Kiyosi Itô



**American Mathematical Society**

# Essentials of Stochastic Processes

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Translations of

# MATHEMATICAL MONOGRAPHS

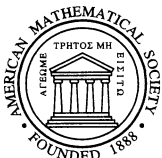
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Volume 231

## Essentials of Stochastic Processes

Kiyosi Itô

Translated by Yuji Ito



**American Mathematical Society**  
Providence, Rhode Island

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## Author's Preface

The present volume, *Essentials of Stochastic Processes*, is an English translation of my book written in Japanese and issued by Iwanami Shoten in 1957 in two parts: Stochastic Processes I (from Chapter 1 to 3) and II (from Chapter 4 to 5). In this work, I provide a unified and comprehensive account of additive processes (or Lévy processes), stationary processes, and Markov processes, which remain to this day the three most important classes of stochastic processes. I had sent the Japanese original at the time of its publication to Eugene B. Dynkin, and I was very pleased to see A. D. Wentzell's Russian translation published in 1960 (Part I) and 1963 (Part II). I am also grateful to Dynkin for editing the translation and adding some important clarification footnotes. In 1959 Shizuo Kakutani at Yale University, noting the significance of my description of the one-dimensional diffusions, advised Yuji Ito, then one of his graduate students, to produce a translation of Part II into English, which was distributed among a limited circle of mathematicians around Yale University as a typewritten mimeograph. On the occasion of my receiving the Kyoto Prize in 1998, Shinzo Watanabe and Masatoshi Fukushima encouraged me to have the entire 1957 book translated into English and published by the American Mathematical Society. Yuji Ito graciously agreed to take on this arduous task and revisited his earlier partial translation, not only adding Part I, but also fully revising his original translation of Part II.

Although almost half a century has passed since the initial publication in Japanese, I hope there is enough of value in this work to merit its publication in English at this time. It should be noted that some detailed introductions to additive processes and Markov processes are given in two of my lecture notes published later on:

- *Lectures on Stochastic Processes*, Tata Institute of Fundamental Research, Bombay, 1960.
- *Stochastic Processes*, edited by Ole E. Barndorff-Nielsen and Ken-iti Sato, Springer, 2004 (originally published as Lecture Notes from Aarhus University in 1969).

However, the present volume is the only one among my English textbooks that includes an introduction to stationary processes.

Chapter 5 is devoted to the one-dimensional diffusion theory which is important as a basic prototype of the study of Markov processes. This chapter starts with a presentation of the local generator of a one-dimensional diffusion process as a generalized second order differential operator discovered by William Feller several years before I wrote this book. It then proceeds to a detailed description of the



boundary behaviors of the solutions of the associated homogeneous and inhomogeneous equations in an analytical way, followed by their probabilistic implications on the path properties of the diffusion near the boundaries.

My lecture notes from the Tata Institute mentioned above contain another detailed explanation of the Feller local generator. Section 4.6 of my joint book with H. P. McKean (*Diffusion Processes and Their Sample Paths*, Springer, 1965; in *Classics in Mathematics*, Springer, 1996) also exhibits the boundary behaviors with some probabilistic proof, while sections 5.12, 5.13, and 5.14 of the present volume are readily understood even by readers unfamiliar with probability theory.

When I wrote the original Japanese version of this book, the real study of stochastic processes had just begun, and not much related literature was available as noted in the Postscript. In the five decades since then, there have been significant developments in the theory of stochastic processes with many important subsequent publications, some of which are listed in the Preface to the Original and the Foreword by the Editors in the above-mentioned *Stochastic Processes* published in 2004 based on my Aarhus Lecture Notes.

I am very much indebted to those who have helped me bring this translation project to a successful completion. My gratitude, first and foremost, goes to Yuji Ito for the precise yet elegant translation which far exceeded my expectations, and I sincerely wish to thank him once again for his time and efforts. My thanks are due to M. Fukushima, K. Ichihara, and S. Watanabe for the meticulous care they took in proof-reading and editing the translated manuscript. This English version is in many ways superior to the original in that it eliminates minor inconsistencies and updates some of the discussion. In particular, the original version in Japanese, written when I had just started my work on paths in Markov processes, contains discussions of the general theory in Chapters 4 and 5 that are in hindsight somewhat unclear and misleading. I am grateful to M. Fukushima and S. Watanabe for suggesting the appropriate amendments in these chapters.

In view of the fact that Professor Shizuo Kakutani had first suggested, shortly after its Japanese publication in 1957, that my book be translated into English, I had hoped to be able to finally present him with this English version published by the American Mathematical Society. It was with great sadness that I learned of his passing away in the summer of 2004 in New Haven. In order to express my deep respect and admiration for his teaching and his important contributions to mathematics, I wish to dedicate this book to the late Professor Shizuo Kakutani.

Kyoto, December 2005

K. Itô

## Translator's Foreword

It is my great pleasure to present an English translation of *Essentials of Stochastic Processes* written by Professor Kiyosi Itô. It was almost half a century ago when the original Japanese version of this book was published by Iwanami Shoten. As it is mentioned by Professor Itô in the Author's Preface, I took up the translation of Part II (Chapters 4 and 5 of the book) into English only a couple of years after the publication of the original with the urging of the late Professor Shizuo Kakutani of Yale University. I was a graduate student in mathematics at Yale at the time, trying to write a Ph.D. thesis under Professor Kakutani's supervision, and he probably thought that I should look into the possibility of working in the field of continuous parameter Markov processes, which was undergoing a rapid development at the time. No doubt, he felt that the best place to follow this development was to read the account by Professor Itô, who was one of the central figures spearheading this development. Professor Kakutani himself was very much interested in the materials contained in this book, and he thought there may be people around Yale and elsewhere in the United States who would benefit a great deal from learning the contents of this book, especially the part on diffusion processes. This was why he urged me to translate (rather than just read through) Part II of the book into English, and when I finished the translation, he decided to have it typed and copies mimeographed by a secretary of the Mathematics Department of Yale and put out as a part of the lecture note series circulated by the Department. I do not know how many copies of the translation were circulated in this manner, but I learned much later that there were a number of people, some of whom eventually became prominent probabilists, who have read the translation and benefitted from it. Although I ended up choosing a thesis topic in Ergodic Theory, a field related but not directly connected with the contents of this book, I certainly learned a great deal about Stochastic Processes in going through the book carefully in the process of translation.

A couple of years ago, Masatoshi Fukushima approached me and asked whether I would be interested in having my old translation (possibly adding a new translation of Part I) published in a more formal manner, as there are materials in it which had never been published in English elsewhere and continue to draw the interest of the specialists in the field. I was delighted to hear this proposal with the additional information that it was the wish of Professor Itô also to have a formal publication of an English translation of this book, and he would like me to take up the task of the actual translation of the entire book. As I was not sure whether Professor Kakutani had asked for permission from Professor Itô to translate the portion of the book before he told me to take up the task and decided to circulate copies of the product through the Mathematics Department of Yale, I was very pleased

and honored to learn Professor Itô's wishes, and decided to embark on the new translation project with his blessings.

I had thought that I would be able to finish the project within a year or so, but it took much longer than I had expected, partly because I decided, in addition to translating Part I, to retranslate Part II to make the entire manuscript consistent and easier to read. My lack of any previous experience in writing articles in AMS- $\LaTeX$  format also forced me to spend a considerable amount of extra time. I am truly grateful to Fukushima, Shinzo Watanabe, and Kanji Ichihara for proof-reading my manuscript very carefully. Although I tried, while I was translating, to correct minor errors in the original as much as I could, I still missed a few, and furthermore, I introduced new errors, typographical and otherwise, of my own (many of which were caused by my inexperience in AMS- $\LaTeX$  typesetting). All of these were found and corrected by Fukushima, Watanabe, and Ichihara. Furthermore, as it was explained by Professor Itô in the Author's Preface, Fukushima and Watanabe suggested a few amendments for arguments used in the original Japanese version, in order to eliminate minor inconsistencies and to update some of the discussion, which would have been impossible for me to do as a non-specialist in the field. I am very happy that with their great help, I was finally able to complete this translation project. I am grateful also to the American Mathematical Society for agreeing to publish this translation of Professor Itô's excellent account of the properties of stochastic processes.

Tokyo, January 2006

Yuji Ito

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## Postscript

Let us first list books which are good references for the topics covered in this book as a whole.

- A. Kolmogoroff: Grundbegriffe der Wahrscheinlichkeitsrechnung, Erg. der Math. (Berlin, 1933).
- W. Feller: An introduction to probability theory and its applications (1950).
- J. L. Doob: Stochastic processes (1952).
- P. Lévy [1]: Théorie de l'addition des variables aléatoires (Paris, 1937). [2]: Processus stochastiques et mouvement brownien (Paris, 1948).
- Y. Kawada: Probability Theory (in Japanese) (Kyouritsu Shuppan, 1948).
- K. Kunisawa: Modern Probability Theory (in Japanese) (Iwanami Zensho, 1951).
- K. Itô: Probability Theory (in Japanese) (Gendai Sugaku, Iwanami, 1952).
- G. Maruyama: Probability Theory (in Japanese) (Kyoritsu Gendai Sugaku Koza, 1957).

Let me explain briefly the contents of each chapter.

Chapter 1: Fundamental concepts of the probability theory are introduced in this chapter. The most convenient way to construct a mathematically rigorous theory of probability is to adopt the measure theoretic method of A. Kolmogorov. We followed this method in this book, but it is essential, in order to get familiar with applications to practical problems, to understand a more intuitive background for probability theory. For acquiring such an intuitive background, we recommend the book by W. Feller quoted above.

As basic material related to the measure theoretic method of Kolmogorov, we recommend, in addition to the book by Kolmogorov, the book by Y. Kawada mentioned above. However, neither book treats the detailed description of sample processes of stochastic processes. In order to investigate such topics, the notion of separability introduced by J. L. Doob is indispensable. The original paper of Doob, which took up this concept for the first time, however, was not written with sufficient rigor and was hard to go through. But the book by Doob mentioned above treats this topic rigorously and efficiently, and is quite readable.

Chapter 2: Discussions in this chapter were centered on the topics of continuous time parameter stochastic processes with independent increments (additive processes). Discussion of the case of discrete parameter (additive sequences) is limited to the situations, which have some bearing on the investigation of additive processes. For instance, detailed discussions of laws of large numbers, central limit theorems, laws of iterated logarithms, and so on were omitted. For these topics, we

refer the readers to the book by K. Kunisawa mentioned above. Also the English translation by K. L. Chung of the following book originally written in Russian gives a well-organized treatment of these topics:

- Gnedenko-Kolmogoroff: Limit distributions for sums of independent random variables (Moscow-Leningrad, 1949).

Concerning additive processes, the aforementioned books by P. Lévy have a wealth of material, but are difficult to read through. Basic parts of the theory covered in his books are explained from the view point of Kolmogorov's approach in the books by Doob and K. Itô mentioned above, and are more accessible.

Chapter 3: Basic facts concerning stationary processes have been treated in this chapter, but results on interpolation and extrapolation by N. Wiener and Kolmogorov, and on parameter estimation by U. Grenander, were omitted. We hope these topics would be taken up in the book by T. Kawata entitled "The Applications of Stochastic Processes" in this series. Let us list a few references on these topics.

- Chapter 12 of the book by Doob mentioned above.
- N. Wiener: Extrapolation, interpolation and smoothing of stationary time series (1949).
- A. Kolmogoroff: Interpolation und Extrapolation in stationären Zufälligen Folgen, Bull. Acad. Sci. U.R.S.S. Ser. Math. 5 (1941).
- U. Grenander: Stochastic processes and statistical inference, Arkiv. för Mat. 1 (1950).

As for the generalized harmonic analysis that we touched on without proofs in this book, we refer the readers to:

- N. Wiener: Generalized harmonic analysis, Acta. Math. 55 (1930).

Chapters 4 and 5: In these two chapters, we restricted ourselves to the discussion of temporally homogeneous Markov processes, and called them simply Markov processes. The reason for this restriction is that the temporally homogeneous case is where a more or less complete theory has been established. But even here, not everything worth mentioning has been covered. Chapter 4 dealt only with basic material on Markov processes, and in Chapter 5, Feller's recent treatment of diffusion processes was discussed by infusing with the methods developed by E. B. Dynkin. There are many results from olden days concerning ergodic properties of Markov processes, but we omitted discussions of this topic, because of the limitation of the pages available.

Classical results on this topic are summarized in

- M. Fréchet: Recherches théorétiques modernes sur le calcul des probabilités, second livre, methode des fonctions arbitraires, théorie des événement en chain dans le cas d'un nombre fini d'étas possibles (Paris, 1938).

This treats, as the title of the book indicates, the case of finite state Markov processes. A pioneering work on Markov processes with countable infinity of states is the following article by Lévy:

- P. Lévy: Systèmes markoviens et stationaires. Cas dénombrable. Ann. Sci. École Norm. Sup. 68 (1951).

Chung has published many results obtained by detailed studies trying to make the results in this article by Lévy more rigorous. The aforementioned book by G. Maruyama provides a good account in Japanese in this area.

There are many investigations for temporally non-homogeneous Markov processes as well. We refer the readers to the aforementioned books by Feller, Itô, and Doob on this topic. We recommend the readers to go through the following work of Kolmogorov, which started the studies in this direction:

- A. Kolmogoroff: Analytische Methoden in der Wahrscheinlichkeitsrechnung, Math. Ann. 104.

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This book is an English translation of Kiyosi Itô's monograph published in Japanese in 1957. It gives a unified and comprehensive account of additive processes (or Lévy processes), stationary processes, and Markov processes, which constitute the three most important classes of stochastic processes. Written by one of the leading experts in the field, this volume presents to the reader lucid explanations of the fundamental concepts and basic results in each of these three major areas of the theory of stochastic processes.

With the requirements limited to an introductory graduate course on analysis (especially measure theory) and basic probability theory, this book is an excellent text for any graduate course on stochastic processes.

Kiyosi Itô is famous throughout the world for his work on stochastic integrals (including the Itô formula), but he has made substantial contributions to other areas of probability theory as well, such as additive processes, stationary processes, and Markov processes (especially diffusion processes), which are topics covered in this book. For his contributions and achievements, he has received, among others, the Wolf Prize, the Japan Academy Prize, and the Kyoto Prize.



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