Translations of
MATHEMATICAL
MONOGRAPHS

Volume 247

Inverse Problems in the Theory of Small Oscillations

Vladimir Marchenko
Victor Slavin

AMS
American
Mathematical
Society

10.1090/mmono/247

Inverse Problems in the Theory of Small Oscillations

Translations of
MATHEMATICAL
MONOGRAPHS

Volume 247

Inverse Problems in the Theory of Small Oscillations

Vladimir Marchenko
Victor Slavin

American Mathematical Society
Providence, Rhode Island

This work was originally published in Russian by Наукова Думка, Киев, under the title "Обратные Задачи Теории Малых Колебаний" © 2015. The present translation was created under license for the American Mathematical Society and is published by permission.

2010 *Mathematics Subject Classification.* Primary 70F17, 65N21, 35P25, 35Q70, 65R32.

For additional information and updates on this book, visit
www.ams.org/bookpages/mmono-247

Library of Congress Cataloging-in-Publication Data

Names: Marchenko, V. A. (Vladimir Aleksandrovich), 1922– author. | Slavin, Victor, 1970– author.
Title: Inverse problems in the theory of small oscillations / Vladimir Marchenko, Victor Slavin.
Other titles: Obratnye zadachl teorii malykh kolebaniy. English
Description: Providence, Rhode Island : American Mathematical Society, [2018] | Series: Translations of mathematical monographs ; volume 247 | Originally published in Russian: Obratnye Zadachl Teorii Malykh Kolebaniy (Kiev : Naukova Dumka, 2015). | Includes bibliographical references.
Identifiers: LCCN 2018037179 | ISBN 9781470448905 (alk. paper)
Subjects: LCSH: Matrices. | Oscillations. | Frequencies of oscillating systems. | AMS: Mechanics of particles and systems – Dynamics of a system of particles, including celestial mechanics – Inverse problems. msc | Numerical analysis – Partial differential equations, boundary value problems – Inverse problems. msc | Partial differential equations – Spectral theory and eigenvalue problems – Scattering theory. msc | Partial differential equations – Equations of mathematical physics and other areas of application – PDEs in connection with mechanics of particles and systems. msc | Numerical analysis – Integral equations, integral transforms – Inverse problems. msc
Classification: LCC QA188 .M347513 2018 | DDC 512.9/434–dc23
LC record available at https://lccn.loc.gov/2018037179

10 9 8 7 6 5 4 3 2 1   23, 22, 21, 20, 19, 18

Contents

Preface

Inverse problems of spectral analysis deal with the reconstruction of operators in a specified form, given certain spectral characteristics of the operators. Interest in such problems was initially inspired by quantum mechanics. The main inverse spectral problems have already been solved for Schrödinger operators and their finite-difference analogues, the Jacobi matrices (see V. A. Ambartsumian [**11**], G. Borg [**12**], N. Levinson [**14**], V. A. Marchenko [**1**], M. G. Krein [**6**], I. M. Gelfand and B. M. Levitan [**3**], B. M. Levitan and M. G. Gasymov [**7**], L. D. Faddeev [**10**], R. Newton [**16**], N. I. Akhiezer [**2**], A. R. Its and V. B. Matveev [**5**], V. E. Zakharov and A. B. Shabat [**4**], P. Deift and X. Zhou [**13**], V. A. Yurko [**17**], Yu. I. Lyubarskii and V. A. Marchenko [**15**], etc.). On the other hand, little is known about inverse spectral problems for wider classes of operators, such as arbitrary Hermitian matrices.

The present monograph focuses on inverse problems in the theory of small oscillations of systems with finitely many degrees of freedom. Given data obtained from observations of these oscillations, to solve an inverse problem means to find the potential energy of the system in question. Since the oscillations are small, the potential energy is given by a positive definite quadratic form, whose matrix is called the matrix of potential energy. Hence, the problem is to find a matrix belonging to the quite wide set of all positive definite matrices. This is a principal difference between the inverse problems studied in this monograph and inverse problems for discrete analogues of the Schrödinger operators, where only tridiagonal Hermitian matrices are considered.

Without loss of generality it can be assumed that the systems consist of finitely many material points (particles) $\alpha, \beta, \dots$ of masses $m_\alpha, m_\beta, \dots$, interacting with each other and with an external field. It is assumed that only a small portion of the particles is available for observation. The aim is to use observations obtained from this subset of particles to compute the reduced matrix of potential energy (**L**-matrix of the system), whose elements $\mathbf{L}(\alpha, \beta)$ are expressed in terms of elements $\mathbf{U}(\alpha, \beta)$ of the matrix of potential energy via the formula $\mathbf{L}(\alpha, \beta) = \mathbf{U}(\alpha, \beta)(m_\alpha m_\beta)^{-1/2}$, and to compute the masses of the particles, if possible.

The main results obtained in the monograph are the following:

- necessary and sufficient conditions are found for a portion of particles with observed oscillations to enable computation of the **L**-matrix of the entire system;
- conditions are found for extracting the required information on oscillations of an observable part of the system from its oscillations in a neighborhood of infinity;
- conditions are found for computing the **L**-matrix of the entire system by using the spectra of oscillations of the system and some of its perturbations.

For the reader's convenience, Chapters 1–7 of the monograph contain a detailed presentation of well-known results on inverse spectral problems for tridiagonal matrices, i.e., Jacobi matrices; see also [**18, 19**]. The subsequent Chapters 8–17 contain proofs of necessary and sufficient conditions for a subset of observable particles to determine uniquely the system **L**-matrix and give a method for its computation. Here a complete description is provided of the class of matrices that can be found from observable data on q particles, and some model examples are given. In particular, in the case $q = 1$ it is possible to find the tridiagonal matrix only. The problem of computation of the particle masses is also considered. Chapter 18 presents a solution of the inverse problem of reconstructing a Hermitian matrix given its spectrum and the spectra of some of its perturbations. Chapter 19 deals with the inverse problem of multichannel scattering. The final six chapters (Chapters 20–25) describe numerical methods for solving the inverse problem of the theory of small oscillations. To make the exposition clear to a wide range of readers, the material relating to numerical modeling of solutions is presented in detail. This content also includes topics which we believe may be of interest to experts in numerical analysis. Specific examples are given to illustrate the features of the methods described.

The authors are grateful to V. Kotlyarov, L. Pastur, S. Sinelshchikov, and V. Tkachenko for substantial assistance in the preparation of this monograph for publication.

Bibliography

[1] Vladimir A. Marchenko, *Sturm-Liouville operators and applications*, Operator Theory: Advances and Applications, vol. 22, Birkhäuser Verlag, Basel, 1986. Translated from the Russian by A. Iacob. MR897106

[2] N. I. Ahiezer, *Continuous analogues of orthogonal polynomials on a system of intervals* (Russian), Dokl. Akad. Nauk SSSR **141** (1961), 263–266. MR0140970

[3] I. M. Gel′fand and B. M. Levitan, *On the determination of a differential equation from its spectral function* (Russian), Izvestiya Akad. Nauk SSSR. Ser. Mat. **15** (1951), 309–360. MR0045281

[4] V. E. Zakharov and A. B. Shabat, *Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media* (Russian, with English summary), Ž. Èksper. Teoret. Fiz. **61** (1971), no. 1, 118–134; English transl., Soviet Physics JETP **34** (1972), no. 1, 62–69. MR0406174

[5] A. R. Its and V. B. Matveev, *Hill's operator with finitely many gaps.* Func. Anal. Appl. **9** (1975), 65–66. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, vol. 9, no. 1, pp. 69–70, January–March, 1975.

[6] M. G. Kreĭn, *Solution of the inverse Sturm-Liouville problem* (Russian), Doklady Akad. Nauk SSSR (N.S.) **76** (1951), 21–24. MR0039895

[7] B. M. Levitan and M. G. Gasymov, *Determination of a differential equation by two spectra* (Russian), Uspehi Mat. Nauk **19** (1964), no. 2 (116), 3–63. MR0162996

[8] S. P. Novikov, *A periodic problem for the Korteweg-de Vries equation. I* (Russian), Funkcional. Anal. i Priložen. **8** (1974), no. 3, 54–66. MR0382878

[9] B. A. Dubrovin, *A periodic problem for the Korteweg-de Vries equation in a class of short-range potentials* (Russian), Funkcional. Anal. i Priložen. **9** (1975), no. 3, 41–51. MR0486780

[10] L. D. Faddeev, *The inverse problem in the quantum theory of scattering* (Russian), Uspehi Mat. Nauk **14** (1959), no. 4 (88), 57–119. MR0110466

[11] W. A. Ambarzumjan, *Über eine Frage der Eigenwerttheorie.* Zeitschrift für Physik **53** (1929), 690–695.

[12] Göran Borg, *Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe. Bestimmung der Differentialgleichung durch die Eigenwerte* (German), Acta Math. **78** (1946), 1–96, DOI 10.1007/BF02421600. MR0015185

[13] Percy Deift and Xin Zhou, *Direct and inverse scattering on the line with arbitrary singularities*, Comm. Pure Appl. Math. **44** (1991), no. 5, 485–533, DOI 10.1002/cpa.3160440502. MR1105873

[14] N. Levinson, *On the uniqueness of the potential in a Schrödinger equation for a given asymptotic phase*, Danske Vid. Selsk. Mat.-Fys. Medd. **25** (1949), no. 9, 29 pp. MR0030080

[15] Yu. I. Lyubarskii and V. A. Marchenko, *Inverse problem for small oscillations*, Spectral analysis, differential equations and mathematical physics: a festschrift in honor of Fritz Gesztesy's 60th birthday, Proc. Sympos. Pure Math., vol. 87, Amer. Math. Soc., Providence, RI, 2013, pp. 263–290, DOI 10.1090/pspum/087/01441. MR3087910

[16] Roger G. Newton, *Construction of potentials from the phase shifts at fixed energy*, J. Math. Phys. **3** (1962), 75–82, DOI 10.1063/1.1703790. MR0138364

[17] V. A. Yurko, *Inverse spectral problems for differential operators and their applications*, Analytical Methods and Special Functions, vol. 2, Gordon and Breach Science Publishers, Amsterdam, 2000. MR1769454

[18] Yu. M. Berezansky, *Expansions in eigenfunctions of selfadjoint operators*, Translations of Mathematical Monographs, vol. 17, American Mathematical Society, Providence, RI, 1969, 809 pp.

[19] Gerald Teschl, *Jacobi operators and completely integrable nonlinear lattices*, Mathematical Surveys and Monographs, vol. 72, American Mathematical Society, Providence, RI, 2000. MR1711536

Selected Published Titles in This Series

247 **Vladimir Marchenko and Victor Slavin,** Inverse Problems in the Theory of Small Oscillations, 2018
246 **Sergey Gorchinskiy and Constantin Shramov,** Unramified Brauer Group and Its Applications, 2018
245 **Takeshi Saito,** Fermat's Last Theorem, 2014
244 **Atsushi Moriwaki,** Arakelov Geometry, 2014
243 **Takeshi Saito,** Fermat's Last Theorem, 2013
242 **Nobushige Kurokawa, Masato Kurihara, and Takeshi Saito,** Number Theory 3, 2012
241 **O. A. Logachev, A. A. Salnikov, and V. V. Yashchenko,** Boolean Functions in Coding Theory and Cryptography, 2012
240 **Kazuya Kato, Nobushige Kurokawa, and Takeshi Saito,** Number Theory 2, 2011
239 **I. Ya. Novikov, V. Yu. Protasov, and M. A. Skopina,** Wavelet Theory, 2011
238 **Leonid L. Vaksman,** Quantum Bounded Symmetric Domains, 2010
237 **Hitoshi Moriyoshi and Toshikazu Natsume,** Operator Algebras and Geometry, 2008
236 **Anatoly A. Goldberg and Iossif V. Ostrovskii,** Value Distribution of Meromorphic Functions, 2008
235 **Mikio Furuta,** Index Theorem. 1, 2007
234 **G. A. Chechkin, A. L. Piatnitski, and A. S. Shamaev,** Homogenization, 2007
233 **A. Ya. Helemskii,** Lectures and Exercises on Functional Analysis, 2006
232 **O. N. Vasilenko,** Number-Theoretic Algorithms in Cryptography, 2007
231 **Kiyosi Itô,** Essentials of Stochastic Processes, 2006
230 **Akira Kono and Dai Tamaki,** Generalized Cohomology, 2006
229 **Yu. N. Lin′kov,** Lectures in Mathematical Statistics, 2005
228 **D. Zhelobenko,** Principal Structures and Methods of Representation Theory, 2006
227 **Takahiro Kawai and Yoshitsugu Takei,** Algebraic Analysis of Singular Perturbation Theory, 2005
226 **V. M. Manuilov and E. V. Troitsky,** Hilbert C^*-Modules, 2005
225 **S. M. Natanzon,** Moduli of Riemann Surfaces, Real Algebraic Curves, and Their Superanalogs, 2004
224 **Ichiro Shigekawa,** Stochastic Analysis, 2004
223 **Masatoshi Noumi,** Painlevé Equations through Symmetry, 2004
222 **G. G. Magaril-Il'yaev and V. M. Tikhomirov,** Convex Analysis: Theory and Applications, 2003
221 **Katsuei Kenmotsu,** Surfaces with Constant Mean Curvature, 2003
220 **I. M. Gelfand, S. G. Gindikin, and M. I. Graev,** Selected Topics in Integral Geometry, 2003
219 **S. V. Kerov,** Asymptotic Representation Theory of the Symmetric Group and its Applications in Analysis, 2003
218 **Kenji Ueno,** Algebraic Geometry 3, 2003
217 **Masaki Kashiwara,** D-modules and Microlocal Calculus, 2003
216 **G. V. Badalyan,** Quasipower Series and Quasianalytic Classes of Functions, 2002
215 **Tatsuo Kimura,** Introduction to Prehomogeneous Vector Spaces, 2002
214 **L. Š. Grinblat,** Algebras of Sets and Combinatorics, 2002
213 **V. N. Sachkov and V. E. Tarakanov,** Combinatorics of Nonnegative Matrices, 2002

For a complete list of titles in this series, visit the
AMS Bookstore at **www.ams.org/bookstore/mmonoseries/**.

Inverse problems of spectral analysis deal with the reconstruction of operators of the specified form in Hilbert or Banach spaces from certain of their spectral characteristics. An interest in spectral problems was initially inspired by quantum mechanics. The main inverse spectral problems have been solved already for Schrödinger operators and for their finite-difference analogues, Jacobi matrices.

This book treats inverse problems in the theory of small oscillations of systems with finitely many degrees of freedom, which requires finding the potential energy of a system from the observations of its oscillations. Since oscillations are small, the potential energy is given by a positive definite quadratic form whose matrix is called the matrix of potential energy. Hence, the problem is to find a matrix belonging to the class of all positive definite matrices. This is the main difference between inverse problems studied in this book and the inverse problems for discrete analogues of the Schrödinger operators, where only the class of tridiagonal Hermitian matrices are considered.

For additional information
and updates on this book, visit

www.ams.org/bookpages/mmono-247

ISBN 978-1-4704-4890-5

9 781470 448905

MMONO/247

AMS
www.ams.org