Quantum Field Theory, Supersymmetry, and Enumerative Geometry

Daniel S. Freed
David R. Morrison
Isadore Singer
Editors
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IAS/Park City Mathematics Institute runs mathematics education programs that bring together high school mathematics teachers, researchers in mathematics and mathematics education, undergraduate mathematics faculty, graduate students, and undergraduates to participate in distinct but overlapping programs of research and education. This volume contains the lecture notes from the Graduate Summer School program.

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Preface

The IAS/Park City Mathematics Institute (PCMI) was founded in 1991 as part of the “Regional Geometry Institute” initiative of the National Science Foundation. In mid 1993 the program found an institutional home at the Institute for Advanced Study (IAS) in Princeton, New Jersey.

The IAS/Park City Mathematics Institute encourages both research and education in mathematics and fosters interaction between the two. The three-week summer institute offers programs for researchers and postdoctoral scholars, graduate students, undergraduate students, high school teachers, undergraduate faculty, and researchers in mathematics education. One of PCMI’s main goals is to make all of the participants aware of the total spectrum of activities that occur in mathematics education and research: we wish to involve professional mathematicians in education and to bring modern concepts in mathematics to the attention of educators. To that end the summer institute features general sessions designed to encourage interaction among the various groups. In-year activities at the sites around the country form an integral part of the High School Teachers Program.

Each summer a different topic is chosen as the focus of the Research Program and Graduate Summer School. Activities in the Undergraduate Summer School deal with this topic as well. Lecture notes from the Graduate Summer School are being published each year in this series. The first eleven volumes are:

- Volume 1: *Geometry and Quantum Field Theory* (1991)
- Volume 3: *Complex Algebraic Geometry* (1993)
- Volume 11: *Quantum Field Theory, Supersymmetry, and Enumerative Geometry* (2001)

Volumes are in preparation for the subsequent years.

Some material from the Undergraduate Summer School is published as part of the Student Mathematical Library series of the American Mathematical Society.
We hope to publish material from other parts of the IAS/PCMI in the future. This will include material from the High School Teachers Program and publications documenting the interactive activities which are a primary focus of the PCMI. At the summer institute late afternoons are devoted to seminars of common interest to all participants. Many deal with current issues in education. Others treat mathematical topics at a level that encourages broad participation. The PCMI has also spawned interactions between universities and high schools at a local level. We hope to share these activities with a wider audience in future volumes.
Titles in This Series

11 Daniel S. Freed, David R. Morrison, and Isadore Singer, Editors, Quantum Field Theory, Supersymmetry, and Enumerative Geometry, 2006
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 3 János Kollár, Editor, Complex Algebraic Geometry, 1997
 1 Daniel S. Freed and Karen K. Uhlenbeck, Editors, Geometry and Quantum Field Theory, 1995
Each summer the IAS/Park City Mathematics Institute Graduate Summer School gathers some of the best researchers and educators in a particular field to present diverse sets of lectures. This volume presents three weeks of lectures given at the Summer School on Quantum Field Theory, Supersymmetry, and Enumerative Geometry, three very active research areas in mathematics and theoretical physics.

With this volume, the Park City Mathematics Institute returns to the general topic of the first institute: the interplay between quantum field theory and mathematics. Two major themes at this institute were supersymmetry and algebraic geometry, particularly enumerative geometry. The volume contains two lecture series on methods of enumerative geometry that have their roots in QFT. The first series covers the Schubert calculus and quantum cohomology. The second discusses methods from algebraic geometry for computing Gromov-Witten invariants. There are also three sets of lectures of a more introductory nature: an overview of classical field theory and supersymmetry, an introduction to supermanifolds, and an introduction to general relativity.

This volume is recommended for independent study and is suitable for graduate students and researchers interested in geometry and physics.