

# Elasticity



**PROCEEDINGS OF  
SYMPOSIA IN  
APPLIED MATHEMATICS**

**VOLUME III**

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SYMPOSIA IN APPLIED MATHEMATICS  
VOLUME III

# ELASTICITY

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EDITORIAL COMMITTEE  
R. V. Churchill   Eric Reissner   A. H. Taub

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## EDITOR'S PREFACE

The papers that were presented during the Third Symposium in Applied Mathematics of the American Mathematical Society are published in this volume. The Symposium was held at the University of Michigan from June 14 to 16, 1949. The subject of the Symposium was *Elasticity*; it included plasticity. The Symposium was cosponsored by the Applied Mechanics Division of the American Society of Mechanical Engineers, and joint sessions of the Symposium and that Division's Fifteenth Applied Mechanics Conference were held on June 15.

The papers in this volume are grouped roughly according to subjects. Owing to previous arrangements for publication, two of the papers appear here as abstracts containing references to the complete papers. Four papers from the program of the Applied Mechanics Conference were presented at the joint sessions with the Symposium. The place of publication of those four papers is announced at the end of this volume.

On behalf of the American Mathematical Society, the Editorial Committee, consisting of Eric Reissner, A. H. Taub, and the undersigned, wishes to express its gratitude to the McGraw-Hill Book Company, Inc., for undertaking the publication of this volume. The Committee extends its thanks to R. C. Gibbs of the National Research Council for his assistance in initiating these arrangements for publication and to the officers of McGraw-Hill and of the Society for carrying out this arrangement. In the editorial processing of this volume, the undersigned wishes to acknowledge the work of the New York Office of the Society as well as the assistance of the other two members of the Editorial Committee and the help given by G. E. Hay and P. G. Hodge, Jr.

R. V. CHURCHILL  
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REFERENCES TO PAPERS FROM THE PROGRAM OF THE APPLIED  
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JOINT SESSION WITH THE SYMPOSIUM

The following papers from the program of the Fifteenth Applied Mechanics Conference of the American Society of Mechanical Engineers were presented at the joint session of that Conference and the Third Symposium in Applied Mathematics of the American Mathematical Society on June 15, 1949. The last paper was presented by title. References to these papers are listed with their titles.

L. H. Donnell and C. C. Wan, *Effect of imperfections of thin cylinders and columns under axial compression*, Journal of Applied Mechanics vol. 17 (1950) pp. 73-83.

T. J. Jaramillo, *Deflections and moments due to a concentrated load on a cantilever plate of infinite length*, Journal of Applied Mechanics vol. 17 (1950) pp. 67-72.

C. C. Miesse, *Determination of the buckling load for columns of variable stiffness*, Journal of Applied Mechanics vol. 16 (1949) pp. 406-410.

S. Dokos, *A force applied to the median plane at the center of a circular insert in a plate*, Journal of Applied Mechanics vol. 16 (1949) pp. 411-413.

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