

Stochastic Processes in Mathematical Physics and Engineering



**PROCEEDINGS OF
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APPLIED MATHEMATICS**

VOLUME XVI

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EDITOR**

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CONTENTS

| | |
|--|-----|
| INTRODUCTION | vii |
| Stochastic Green's functions By G. ADOMIAN | 1 |
| On the theory of random equations By A. T. BHARUCHA-REID | 40 |
| Reynolds operators By G. C. ROTA | 70 |
| On propagation in random media of discrete scatterers By VICTOR TWERSKY | 84 |
| Wave propagation in a general random continuous medium By W. C. HOFFMAN | 117 |
| Stochastic equations and wave propagation in random media By J. B. KELLER | 145 |
| Stochastic transformations and functional equations By RICHARD BELLMAN | 171 |
| The application of stochastic approximation to the optimization of random circuits By K. B. GRAY | 178 |
| Random walks on lattices By E. W. MONTROLL | 193 |
| On statistical spectral analysis By EMANUEL PARZEN | 221 |
| Stability in single detection problems By W. L. ROOT | 247 |
| The construction of a class of stationary Markoff processes By EUGENE WONG | 264 |
| Probability bounds via dynamic programming By DAVID BLACKWELL | 277 |

| | |
|---|-----|
| Markovian sequential decision processes | 281 |
| By CYRUS DERMAN | |
| Application of truncated hierarchy techniques | 290 |
| By J. M. RICHARDSON | |
| AUTHOR INDEX | 303 |
| SUBJECT INDEX | 307 |

INTRODUCTION

In the study of the classification and recognition of mathematical processes, dichotomies can be introduced in several different fashions. Two of the most useful have been “deterministic vs. stochastic” and “descriptive vs. variational.” In recent years, we have seen greater and greater emphasis laid upon stochastic and decision processes, and even a blending of the two in the form of modern control theory.

The gradual shift of interest from the classical deterministic descriptive process has been largely a consequence of the continuing challenges to the mathematician to explain phenomena, produce numbers, and guide research in such fields as physics, engineering, economics, biology, medicine and operations research. Combinations of complexity and uncertainty have forced a more frequent use of probabilistic concepts.

In this volume, the emphasis is upon stochastic processes, divided a bit unevenly between descriptive and decision aspects. In dealing with random effects, we can use deterministic or stochastic equations, just as in dealing with deterministic effects we can employ stochastic or deterministic equations or in treating descriptive processes, we can use either descriptive or variational equations. The choice of a particular analytic formulation is a tribute to the existing analytic techniques and the computational resources available, as well as a reflection of the underlying physical process and the prevalent scientific philosophy.

The papers by Adomian, Bharucha-Reid, Hoffman, Keller, Richardson, and Twersky deal with various and overlapping aspects of the general theory of stochastic equations, principally ordinary and partial differential equations. Much of this work is motivated by particular equations arising in wave propagation and scattering theory, as the papers by Hoffman, Keller, and Twersky indicate. The objective of obtaining specific analytic results leads to the development of perturbation techniques and the consideration of questions of closure, as, for example, in Richardson’s paper.

The paper by Bellman deals with stochastic iteration, a topic which arises from the consideration of stochastic differential equations in very much the same way as classical iteration theory arises from deterministic differential equations. A particularly important type of stochastic iteration arises in the theory of stochastic approximation. This approach is used by Gray to study a design problem arising from circuits with random components. In their papers, Blackwell and Derman use dynamic programming, the first to obtain estimates in probability theory by identification with a stochastic decision process, and the second to treat an interesting class of sequential decision processes of Markovian type.

Classical estimation and identification problems arising from stochastic processes are treated by Parzen and Root using new and powerful approaches.

Finally, the papers by Montroll, Rota, and Wong consider deterministic equations associated respectively with random walk, turbulence, and diffusion theory.

Looking over the papers in this volume, we see many new, fascinating, and difficult classes of problems, ideal areas for the young analyst to test his talents, develop his techniques, and plant his flag.

RICHARD BELLMAN
Santa Monica, California
1964

AUTHOR INDEX

Italic numbers refer to pages on which a complete reference to a work by the author is given.

Roman numbers refer to pages on which a reference is made to a work of the author. For example, under Minkowski would be the page on which a statement like the following occurs: "This theorem was proved earlier by Minkowski [7, §2] in the following manner . . ."

Boldface numbers indicate the first page of articles in this volume.

- Achieser, N. I., 238, 245
Adomian, G., 1, 38, 56, 66
Akutowicz, E. J., 250, 263
Altman, M., 66
Ament, W. S., 126, 127, 140, 143, 146, 149, 170
Andersen, E. S., 72, 82
Antosiewicz, H. A., 66
Arbault, J., 81
Ashcroft, W. D., 192
Atkinson, F. V., 72, 83
- Babuška, I., 55, 58, 66
Bachelier, L., 193, 219
Bahadur, R. R., 81
Balakrishnan, A. V., 247, 263
Barrett, J. F., 273, 275
Batchelor, G. K., 144
Baxter, G., 72, 82
Bazer, J., 116
Beard, C. I., 116
Belayev, Yu. K., 249, 263
Bellman, R., 143, 144, 171, 172, 175, 177, 281, 288
Bergman, P. G., 39, 136, 143
Bernstein, S., 55, 66
Bertram, J. E., 66
Bharucha, B. H., 57, 65, 66
Bharucha-Reid, A. T., 40, 61, 66, 67, 144, 302
Billik, M., 81
Birkhoff, G. D., 5, 38, 70, 74, 81
Blackman, R. B., 239, 245
Blackwell, D., 277, 277, 280, 288
Blanc-Lapierre, A., 38, 121, 142
Blum, J. R., 182, 192
Bochner, S., 272, 275
Bogolyubov, N. N., 147, 170, 302
Booker, H. G., 127, 144
Bourret, R. C., 146, 149, 151, 154, 170, 302
Bowhill, S. A., 128, 144
Boyce, W. E., 61, 67
Brainerd, B., 81
- Bremmer, H., 127, 128, 140, 141, 143
Brooks, C. E. P., 223, 245
Buchan, J. F., 39
Bugnolo, D. S., 123, 127, 140, 143
Bush, R. R., 177
- Carruthers, N., 223, 245
Carstensten, E. I., 116
Chandrasekhar, S., 193, 219, 275
Chelpanov, B., 39
Chernov, L. A., 39, 136, 143, 153, 170
Chung, K. L., 288
Cole, G. H. A., 116
Courant, R., 67, 194, 220
Cramer, H., 27, 38, 275
Crawford, A. B., 128, 144
- Damask, A. C., 219
Dambis, K. Z., 62, 67
Dantzig, G., 281, 286, 288
Darling, D. A., 39, 273, 275
Davenport, W. B., 27, 39
Derman, C., 281, 288
Dienes, C. J., 219
Dieudonné, J., 81
Dobrushin, R. D., 249, 263
Dolph, C., 144
Domb, C., 193, 205, 220
Doob, J. L., 38, 67, 117, 142, 276, 278, 280
Driml, M., 67
Dubins, L. E., 277, 280
Dubreil-Jacotin, M. L., 74, 81
Duffin, R. J., 220
Dunford, N., 81
Dvoretzky, A., 177, 182, 183, 192, 194, 210, 220
Dynkin, E. B., 67
Dyson, F. J., 47, 67, 219, 302
- Eaton, J. H., 282, 289
Edwards, D. A., 58, 67
Engleman, R., 47, 67

- d'Epenoux, F., 281, 289
 Erdélyi, A., 220, 276
 Erdős, P., 52, 67, 194, 210, 220
 Eringen, A. C., 39, 55, 69, 177
- Feller, W., 204, 206, 220, 276, 280, 283, 284, 289
 Feynman, R. P., 143
 Firstman, S. I., 192
 Fisher, R. A., 223, 245
 Foldy, L. L., 85, 86, 94, 98, 116, 146, 153, 160, 170
 Fortet, R., 9, 25, 38, 39, 121, 142
 Foster, F. G., 220
 Freidlin, M. I., 62, 67
 Freimer, M., 289
 Frenkiel, F. N., 122, 142
 Friedrichs, K., 220
 Friis, H. T., 128, 144
 Frisch, H. L., 60, 67
 Furstenberg, H., 175, 177
- Gaudin, M., 47, 68
 Gel'fand, I. M., 143
 de Ghellink, G., 289
 Girsanov, I. V., 62, 67
 Gnedenko, B. V., 276
 Good, I. G., 220
 Goodman, N. R., 245
 Gray, H. J., Jr., 192
 Gray, K. B., 178
 Grenander, U., 47, 55, 67, 177, 252, 263
- Hadley, G., 36, 39
 Hajek, J., 247, 263
 Halmos, P. R., 81
 Hammersley, J., 52, 67
 Hamming, R. W., 230, 241, 245
 Hannan, E. J., 241, 245
 Hanš, O., 38, 42, 44, 45, 50, 61, 64, 67
 Hardy, G. H., 220
 Harris, T. E., 172, 177
 Herman, R., 220
 Hille, E., 68, 83
 Hirschfeld, R., 81
 Hochwald, W., 192
 Hoffman, W. C., 2, 38, 117, 130, 143
 Hogg, D. C., 128, 144
 Hopf, E., 147, 170
 Howard, R. A., 281, 289
 Hufford, G. C., 81
- Ito, K., 61, 68
- Jacobs, K., 143
 Jansen, L., 92, 116
 Jenkins, G. M., 245
 Johns, M. V., 288
 Juréen, L., 69
- Kac, M., 2, 9, 38, 51, 68, 276
 Kalaba, R., 143, 144, 177
 Kalianpur, G., 257, 263
 Kalman, R. E., 68
 Kampé de Fériet, J., 58, 66, 68, 70, 74, 81, 82, 117, 118, 122, 124, 142
 Kantorovič, L. V., 68
 Karal, F. C., 151, 154, 170
 Karlin, S., 272, 276
 Kay, I., 134, 143
 Keller, J. B., 2, 38, 85, 86, 116, 124, 126, 135, 136, 143, 145, 147, 151, 154, 170
 Kelley, J. L., 82
 Kelly, E. J., 249, 259, 263
 Kesten, J., 175, 177
 Kharanen, V. Ya., 136, 143
 Khinchin, A. Ya., 38
 Kiefer, J., 180, 182, 192
 Klein, M., 289
 Koenigs, M. G., 172, 173, 177
 Kolmogorov, A. N., 38, 82, 120, 142, 267, 276
 Koopmans, T., 68
 Kraichnan, R. H., 58, 60, 68, 125, 143, 147, 170
 Krasnosel'skiĭ, M. A., 68
- Lampard, D. G., 273, 275
 Lanczos, C., 230, 241, 245
 Lax, M., 85, 98, 116, 146, 153, 159, 160, 161, 170
 Lawson, J. L., 2, 38
 van Lear, G. A., Jr., 69
 Lebedev, V. L., 55, 68
 Levy, P., 39
 Lewis, R. M., 147, 170
 Lewy, H., 220
 Lieberman, G. J., 288
 Lifshitz, I. M., 220
 Littlewood, J. E., 51, 68
 Lloyd, S. P., 60, 67
 Loève, M., 38, 142, 189, 192
 Lumley, J. J., 68
- Magnus, W., 220
 Mandel, M., 92, 116
 Manne, A. S., 281, 286, 289
 Maradudin, A., 220
 Maravell Casenoves, D., 58, 68

- Markoff, A., 38
 Martin, P. C., 290, 302
 Maruyama, G., 124, 143
 Mazur, P., 92, 116
 McCrea, W. H., 193, 205, 220
 McGregor, J., 272, 276
 Meecham, W. C., 151, 152, 154, 170
 Mehta, M. L., 47, 68
 Milnes, H., 220
 Mintzer, D., 39
 Mizel, V. J., 82
 Molinaro, I., 82
 Montroll, E. W., 193, 193, 204, 220
 Morrison, J. A., 60, 68
 Mosteller, F., 177
 Mourier, E., 68
 Moy, S.-T. C., 74, 82
 Moyal, J. E., 39, 58, 67, 68, 142
 Müller, C., 143

 Nagai, T., 62, 68
 Nagy, B., 83
 Nakamura, M., 82
 Nemytskii, V. V., 68

 Oberhettinger, F., 220
 Offord, A. C., 51, 52, 67, 68
 Opachowski, W., 219
 Oseen, C. W., 82

 Parzen, E., 221, 225, 227, 233, 238, 245, 247,
 249, 263
 Phillips, R. S., 68, 83
 Pitcher, T., 260, 263
 Pogorzelski, W., 68
 Pólya, G., 193, 204, 220
 Potts, R., 220
 Priestley, M. B., 241, 245, 246
 Primas, H., 61, 68
 Prins, J. A., 92, 95, 116
 Prokhorov, Yu. V., 257, 263
 Pugachev, V. S., 55, 69

 Rao, M. M., 82
 Raven, R. S., 39
 Rayleigh, L., 153, 170
 Reed, I. S., 249, 259, 263
 Reiche, F., 153, 170
 Reynolds, O., 70, 74, 82
 Rheinboldt, W. C., 66
 Richardson, J. M., 177, 290
 Riesz, F., 83
 Root, W. L., 39, 247, 249, 259, 260, 263

 Rosenblatt, J. R., 192
 Rosenblatt, M., 67, 246
 Rosenbloom, A., 39, 69
 Rota, G.-C., 69, 70, 81, 82
 Rutickii, Ya. B., 68
 Ryll-Nardzewski, C., 46, 69

 Sacks, J., 182, 192
 Samal, G., 52, 69
 Samuels, J. C., 9, 28, 29, 39, 55, 69, 177
 Sarachik, P. E., 66
 Savage, L. J., 277, 280
 Schauder, J., 69
 Schuster, A., 222, 223, 246
 Schwartz, J., 81
 Schwinger, J., 290, 302
 Šefl, O., 69
 Seidman, T. I., 143
 Sidak, Z., 82
 Siegert, A. J. F., 9, 38, 39, 273, 275
 Silver, S., 118, 142
 Silverman, R. A., 122, 134, 142, 143
 Singer, I. M., 74, 82
 Skinner, T. J., 124, 143
 Skorohod, A. V., 62, 69, 257, 263
 Sopka, J., 82
 Špaček, A., 38, 44, 45, 67, 69
 Šparo, D. I., 53, 69
 Spitzer, F., 72, 82
 Srinivasan, S. K., 57, 69
 Staras, H., 127, 143
 Sundstrom, M., 28, 39
 Šur, M. G., 53, 69
 Szegő, G., 276

 Tamm, I., 290, 302
 Tatarski, V. I., 144
 Thomas, J. B., 276
 Tikhonov, V. I., 39
 Toda, M., 220
 Tricomi, F., 220
 Tukey, J. W., 192, 239, 245, 246
 Twersky, V., 84, 85, 86, 95, 116, 142, 146, 160,
 161, 170

 Uhlenbeck, G. E., 2, 3, 38, 69, 276
 Uhlmann, W., 50, 69
 Ullrich, M., 61, 69
 Urbanik, K., 58, 69

 Villars, F., 39
 Vineyard, G. H., 194, 210, 220

- Wagner, H. M., 289
Walker, G., 223, 246
Wang, M. C., 3, 38, 276
Watson, G. N., 202, 205, 220
Weiss, G., 220
Weisskopf, V. F., 39
Wermer, J., 74, 82
Wheelon, A. D., 122, 127, 143
Whipple, F. J. W., 193, 205, 220
Whittle, P., 241, 246
Wiener, N., 223, 246
Wigner, E. P., 47, 69
Wilcox, C. H., 87, 116
Wing, G. M., 144
Wold, H., 69
Wolfe, P., 281, 286, 288
Wolfowitz, J., 180, 182, 192, 278, 280
Wong, E., 264, 276
Wright, F. B., 82
Yaglom, A. M., 143, 249, 263
Yeh, K. C., 128, 144
Zaanen, A. C., 69
Zadeh, L., 175, 177, 282, 289
Zernike, F., 92, 95, 116
Zumino, B., 147, 170

SUBJECT INDEX

- A_r , 127
- Abel–Schroder equation, 171
- abstract Cauchy problem, stochastic analogue of, 61
- acoustic waves through a turbulent medium, 117
- algebraic equation(s)
 - of degree n , 51
 - random, 51
 - random, 51
- almost
 - certain convergence, 119
 - certainly twice continuously differentiable, 120
- amplitude, 130
 - fluctuations, 133
 - function, 135
 - multiple scattered, 89
 - scattering, 87
 - window generators, 236
- analysis
 - of random functions, 118
 - regression, 241
 - spectral, 223
- analytic random function, 133
- angular
 - brackets, 292
 - emergent ——— power spectrum, 128
- anisotropic, 127, 128
 - randomly inhomogeneous medium, 128
- approximate solutions of random operator equations, 50
- approximation, 37
 - methods, 32
 - stochastic, 180
 - theory, 221
- associated
 - gain function $G_i(\theta, \varphi)$, 127
 - solution of ——— deterministic operator equation, 49
 - relation to expected solution, 49
- assumption of local independence, 149
- asymptotic
 - behavior, 173
 - expansions, 131
- attenuation coefficient
 - coherent, 105
 - attenuation coefficient (*continued*)
 - for the coherent wave, 152
- autocorrelation, 9, 22
 - function, 14
 - statistical, 12
- autoregressive process, 54
- average
 - scale of turbulence, 140
 - spectral, 227
- averaged hierarchy, 294
- averaging
 - identity, 70
 - operator, 70
- Banach space of generalized random variables, 50
- Banach–Cacciopoli, 44
 - fixed point theorem, 44, 50
- Bernoulli slabs, stack of, 134
- bias, 229
- Birkhoff–Khinchine theorem, 123
- Blackwell, 75
- Bloch’s relaxation theory, 61
- Borel space, 46
- Born approximation, 117, 124, 126
- boundary
 - condition(s)
 - expected value of random, 60
 - realizable, 58
 - stochastic ——— value problem, 58
- boundedness with probability, 1, 130
- brackets, angular, 292
- branching processes, 172
- Brownian
 - equation for ——— oscillator, 58
 - motion, 272
 - Langevin equation for simple, 58
- chain, Markov, 57, 284
- Chapman–Kolmogorov equation, 137, 141
- characteristic function, 7
 - joint, 9
 - multivariate, 8
 - n -dimensional, 10
- circuit design, statistical, 179
- classical integral equations, random solutions of, 62

- coefficients, stochastic, 290
 coherent
 attenuation coefficient, 105
 field, 84, 93, 166
 exact series for, 92
 heuristic equation for, 93
 intensity, 84
 and flux, 166
 scattered wave, 153
 wave, 150
 attenuation coefficient for, 152
 compact
 representations for intensity, 96
 support of ν , 129
 compound Poisson process, 140
 condition(al)
 expectation, 72
 of a generalized random variable, 48
 of $x(\omega)$, 48
 for continuity, 257
 probability distribution, 8
 consecutive scattering at m equidistant planes, 141
 consistency relations, 120
 constant-parameter medium, 18
 continuous, 226
 eigenvalues, 266
 spectrum $C\sigma(T(\omega))$ of $T(\omega)$, 46
 continuity
 condition, 259
 for, 257
 ordinary, 119
 continuum, random, 117
 contraction
 Principle of ——— Mappings, 44
 random ——— operator, 44, 64
 convergence
 almost certain, 119
 weak ——— of measures, 257
 correlation, 37, 175, 295
 discard, 295
 function, 8, 12, 128, 150, 223, 293
 ellipsoidally symmetric, 128
 of Gaussian form, 128
 pair, 156
 length, 154
 covariance, 122
 approach, 223
 function(s), 9, 123, 127, 224
 for logarithmic amplitude, 128
 for phase fluctuations, 128
 of Θ , 131
 of the random solution of . . ., 60
 covariance
 function(s) (*continued*)
 $\rho(x, y)$, 60
 sample, 223
 stationary, 122
 table of ——— and associated spectra, 122
 stationarity, 225
 criteria for ergodic properties, 124
 cross-correlation, 9
 cross section scattering, 139
 per unit solid angle . . ., 140
 cumulant, 295
 discard, 295

 $\Delta\epsilon$, 126, 127
 density
 function
 multivariate, 9
 spectral, 132
 of the refractive index, 131
 sample, 224
 spectral, 11, 12
 determinants, random, 53
 deterministic
 equation, 130
 operator equation, 43
 single slab case, 141
 derivative of a random function, 118
 deviation(s), 74, 132
 time series, 227
 dielectric
 constant, 126, 127
 noise, 139
 difference equations, random, 53, 54
 differentiability, 119
 differential
 cross-section profile, 141
 equation(s)
 functional, 165
 Ito equation in the study of elliptic, 62
 partial ——— using semigroup theory, 61
 random, 55
 ordinary, 55
 their role in engineering, 55
 their role in physics, 55
 their role in technology, 55
 with piecewise constant coefficients, 56
 satisfied by the moments, 168
 stochastic, 27, 28
 Langevin type, 272
 linear, 290
 form of Poynting's theorem, 138
 stochastic ——— operator, 1, 13

- differentiation, functional, 164
- diffusion, 125, 127, 140
 - equation(s), 140
 - approach, 117
 - for multiple scattering, 138
 - in operator form, 141
 - theory, 264
- Dirichlet problem(s)
 - for Laplace's equation . . . , 58
 - generalized (Wiener) solution of ——— for Laplace's equation, 59
 - random, 58
 - random Wiener solution of, 59
- discrete, 226, 241
 - eigenvalues, 266
 - scatterers, 84
 - wave propagation in media of, 155
- distribution(s)
 - finite-dimensional, 120
 - function, 91
 - first probability, 3
 - n th joint, 22
 - spectral, 224
 - multivariate, 120
 - of functionals, 273
 - of the roots of the equation, 52
 - stable, 272
- dynamic programming, 171, 175, 277

- economic time series analysis, 222
- effective
 - field hypothesis, 161
 - propagation constant, 151
 - refractive index, 158
- eigenvalues
 - continuous, 266
 - discrete, 266
- eikonal equation, 135, 136
- electromagnetic
 - total ——— energy, 138
 - turbulent scattering of ——— waves, 123
 - waves through a turbulent medium, 117
- elliptic differential equation, Ito equation in the study of, 62
- emergent angular power spectrum, 128
- endomorphism
 - random ——— of \mathcal{X} , 42
 - Reynolds, 70
- energy
 - approximation, 97, 107
 - density, 84
 - total electromagnetic, 138
- equation(s)
 - differential ——— satisfied by the moments, 168
 - for generating functionals, 163
 - method of hierarchy, 163
 - of the Sturm–Liouville type, 265
 - random algebraic ——— of degree n , 51
 - stochastic, 145, 147
- equivalence, strong, 260
- equivalent
 - diffracting screens, 128
 - spatial Riccati equation, 124
- ergodic, 5, 8, 20
 - properties, 123
 - criteria for, 124
 - of wave propagation . . . , 124
 - theorems, 73
- ergodicity, 124
- error probabilities, 253
- estimation of time delay, 258
- exact series
 - for coherent field, 92
 - for intensity, 93
- expansion, Karhunen–Loève, 251
- expectation
 - conditional, 72
 - \bar{x} of a generalized random variable $x(\omega)$, 48
- expected
 - solution(s), 48
 - of random equations, 47
 - relation to solution of associated deterministic operator equation, 49
 - value, 147
 - of the random boundary condition, 60
- extrapolation of a positive definite function, 250

- far-zone approximation, 127
- Feller, 80
- Fermat's principle, 135
- feuillelet, 127, 128
- field
 - coherent, 84, 93, 166
 - effective ——— hypothesis, 161
 - incoherent, 166
- filter, 221
 - stochastic, 17
- finite-dimensional distribution, 120
- first
 - passage times, 203
 - probability distribution function, 3
- fixed point
 - methods, 65
 - theorem(s)

- fixed point
 theorem(s) (*continued*)
 Banach–Cacciopoli, 44, 50
 random, 45
 Schauder, 44
- flux, 84
 coherent intensity and, 166
 incoherent, 166
 transport theory of, 165
- Fokker–Planck equation, 117, 138, 264, 267, 268, 269, 270, 271
 for the rotational motion of a Brownian particle, 138
- forward
 scatter, 127, 128, 140
 scattering amplitude per unit volume, 153
 -type scatterers, 102
- Fourier transform of $\Gamma(\xi_1, \xi_2)$, 122
- Fréchet differential of $T(\omega)$, 50
- free-space
 Green's function, 128
 wave number, 124
- Fredholm
 equations, 130
 integral, 129
 equation(s), 134
 mean square, 120
 random, 53, 64, 65
 operator, 64
 theory, 130
- functional(s), 164
 analysis, 118, 141, 142
 probabilistic, 41
 differential equation, 165
 differentiation, 164
 distribution of, 273
 equation(s), 171
 method involving ——— for a certain
 generating functional, 163
 equations for generating, 163
 integral, 140
 linear, 296
 of the random refractive index process, 118
 Taylor series, 164
- gain function of the receiving antenna, 127
- gambling, 277
- Gauss hypergeometric series, 270
- Gaussian
 cross-section profile, 141
 process, 4, 58
 random process, 130
 stationary ——— inhomogeneities, 128
- general
 question of random equations, 142
 statistical problem, 11, 18
- generalized
 gradient of ——— phase function Θ , 131
 random variable(s), 41
 Banach space of, 50
 conditional expectation of, 48
 expectation \bar{x} of, 48
 (Wiener) solution of the Dirichlet problem
 for Laplace's equation, 59
- generating
 equations for ——— functionals, 163
 stochastic Green's functions, 38
- generator
 lag window, 233
 window, 233
- geometric optics
 application of ——— to random continuous
 media, 136
 of a random medium, 135, 136
 of a randomly inhomogeneous medium, 137
- Gibbs phenomenon, 230
- globally stationary process, 122
- gradient of generalized phase function Θ , 131
- Green's
 function(s), 1, 56
 generating stochastic, 38
 stochastic, 1, 11, 12, 14, 15, 17, 21, 28, 38, 56
 theorem, 126
- Hamiltonian operator, stochastic, 1
- Hamiltonians, random, 61
- Hammerstein equation, random, 65
- Hamming, 239
- hanning, 239
- harmonic time dependence, 118
- heat equation for an infinite rod, random
 solutions of, 58
- Hermite polynomials, 268
- heuristic equation for coherent field, 93
- hidden periodicities, 221, 223
- hierarchy, 290
 averaged, 294
 equations, 160, 163, 164
 method of, 163
 truncated, 290
 unaveraged, 293
- Hilbert–Schmidt theory, 130
- homomorphisms, 74
- Huyghen's principle, 137
- hypergeometric series, Gauss, 270

- identity, averaging, 70
- incoherent
 - field, 166
 - flux, 166
 - intensity, 84, 92, 166
- indefinite integral, mean square, 120
- independence, assumption of local, 149
- index, refractive, 117, 125, 130, 131
 - effective, 158
- inference problems, nonsingular statistical, 249
- initial value problem, 135
- integral
 - equation(s), 128
 - for intensity, 100
 - formulation, 117, 124
 - Fredholm, 134
 - nonlinear, 65
 - nonsingular, 129
 - random, 61, 128
 - associated with Markov process, 61
 - solutions of classical, 62
 - systems of, 64
 - 2-body, 96
 - of a random function, 118
 - stochastic, 122
- integration in functional spaces, 125, 142
- integro-differential equation, 139
- intensity, 84
 - coherent, 84
 - and flux, 166
 - compact representations for, 96
 - exact series for, 93
 - incoherent, 84, 92, 166
 - transport theory of, 165
 - integral equations for, 100
- invariant
 - function, 123
 - imbedding, 125, 141
 - measure, 73
 - set, 123
- inverse of the random operator, 42
- irreducible, 284
- iterated kernel, 129
- iteration(s)
 - method of, 148
 - procedures for random operator equations, 50
 - second approximation to ———-perturbation solution, 127
 - stochastic, 172
- iterative solution, 133
- Ito
 - equation, 62
 - equation (*continued*)
 - in the study of elliptic differential equation, 62
 - random integral equation, 61
- Jackson-de la Vallée Poussin kernel, 238
- Jacobi polynomials, 269
- joint
 - characteristic function, 9
 - moment, 8, 9
 - n th ——— distribution function, 22
 - probability
 - density $h_{\zeta}(x, y; \xi, \eta)$, 140
 - distributions, 3
- jump function, spectral, 226
- Karhunen-Loève expansion, 38, 251
- Keller formulation of the radiation problem, 124
- kernel
 - iterated, 129
 - Jackson-de la Vallée Poussin, 238
 - not symmetric, 130
 - Parzen, 239
 - random, 64
 - stochastic, 35
 - Tukey-Hamming, 239
- Kiefer-Wolfowitz (KW) procedure, 181
- Kolmogoroff's fundamental theorem, 120
- Kolmogorov, 70-74
- Kronecker product, 174, 175
- lag window generator, 233
- Laguerre polynomials, 269
- Lanczos' σ factors, 241
- Langevin
 - equation, 57, 58
 - for simple Brownian motion, 58
 - type stochastic differential equation, 272
- lattices, random walks on, 1
- least mean square error, 296
- Lebesgue
 - random ——— measure, 63
 - stochastic ——— integrals, 120
- Lévy distance, 257
- likelihood ratio, 260
- linear
 - functional, 296
 - perturbation theory of ——— stochastic equations, 147
 - properties, 18
 - stability of ——— systems, 56

- linear (*continued*)
 stochastic ——— differential equation, 290
 stochastic ——— operators, 290
- localization principle, 141
- locally stationary processes, 122
- M*-sequence, 277
- majorants, stochastic, for the Neumann series . . ., 134
- many scatterers, 89
- mapping(s), 22, 38
 stochastic, 13
- Markoff process, 264
 stationary, 273
- Markov
 chain, 57, 284
 process, 137
 stationary ——— process, 140
 wide-sense ——— process, 137
- Markovian
 hypothesis, 138
 ray tracing, 137
- martingales, 73
- mathematical
 analysis for random functions, 118
 properties, 24
- matrices, random, 47, 171, 174
- matrix, stochastic, 14
- maximum-likelihood
 estimator, 256, 259
 test, 251
- mean
 free path for scattering, 140
 square, 296
 convergence, 118
 and the covariance function, 119
 of random functions, 118
 of a sequence, 118
 convergent Neumann series, 130
 derivative, 120
 n.a.s.c. for the existence of ——— of order p , 119
 deviation of optical path length . . ., 136
 Fredholm integral equations, 120
 indefinite integral, 120
 least ——— error, 296
 n.a.s.c. for ——— existence of the partial derivative of order p , 119
 power, 127
 properties, 119
 of a random function, 119
 rules for ——— differentiation, 120
 Volterra integral equation, 120
- mean (*continued*)
 value theorem, 151
 wave, 150
- measure(s)
 invariant, 73
 -preserving transformations, 73
 statistical, 1, 14, 17, 20, 21, 23, 24, 27, 28
- medium, random, 145
- method
 involving a functional equation . . ., 163
 of hierarchy equations, 163
 of iterations or successive substitutions, 148
- metric transitivity, 123
- metrically transitive, 123
- Mikusinski operator, random, 61
- mixed spectrum, 226
- mixing, 73
- moments, 7
 of order k of $x(\omega)$, 48
 of the solution of a given random equation, 47
- Monte Carlo, 180
- motion, Brownian, 272
- multiple
 scattered
 amplitude, 89
 rays, 140
 phase coherence of, 140
 wave, 89
 scattering, 125, 127
 diffusion
 equations for, 138
 function, 140
 of a plane wave . . ., 134
 transport equations for, 138
- multivariate
 characteristic functions, 8
 density function, 9
 distribution, 120
- n.a.s.c., 119
- n -dimensional characteristic function, 10
- n th
 joint distribution function, 22
 order correlation R_n , 10
- Navier–Stokes equations, 49, 72
- Neumann series
 mean square convergent, 130
 solution, 130, 134
- Newton–Cotes scheme, stochastic approach to, 50, 51
- Newton–Kantorovitch iteration procedure, stochastic analogue of, 51

- noise
 -in-noise, 248
 theory, 221
 white, 222
 process, 54
- nonlinear integral equations, 65
- non-self-adjoint
 problem, 142
 spectral representations for ——— equations,
 142
- nonstationary, 123
 process, 4, 12, 22
- nonstochastic operator, sure, 147
- Norton profile, 141
- Ω , space of all real functions, 120
- one-dimensional
 form of the spatial Riccati equation, 133
 problem, 134
- operator(s)
 averaging, 70
 equation(s)
 deterministic, 43
 random, 43
 approximate solutions of, 50
 Fredholm, 64
 random, 42
 inverse of, 42
 Reynolds, 70
 stochastic, 12, 13, 14, 27, 147
 sure
 nonstochastic, 147
 stochastic perturbations of, 147
 transition, 155
- optical path length, 135
- optimization, statistical, 27
- order(s)
 n th ——— correlation R_n , 10
 of scattering, 90
- ordinary
 continuity, 119
 of interest in engineering, 57
 of interest in mathematical physics, 57
 random ——— differential equations, 55
- Orlicz space, 62, 142
- Ornstein-Uhlenbeck process, 272, 273
- orthogonal polynomials, 272
- $\langle \Psi_c \rangle$ -consistent
 form, 99
 integral equation, 109
- pair correlation function, 156
- parameter of smallness, 291
- parametrix methods, 142
- partial differential equations
 random, 58
 integrals of, 118
 with constant coefficients, 61
 using semigroup theory, 61
- Parzen kernel, 239
- path length, optical, 135
- Pearson equation, 264, 265, 267, 268, 269, 271
- penetration distance, variance proportional to,
 140
- periodicities, hidden, 221, 223
- periodogram, 222
- permittivity, 126, 127
- perturbation
 expansion, 125
 theory, 117, 124, 125, 128, 291
 of linear stochastic equations, 147
- phase, 130
 coherence of multiply scattered rays, 140
 fluctuations, 133
- photon flux, 138
- point spectrum $P_\sigma(T(\omega))$ of $T(\omega)$, 46
- Poisson
 compound ——— process, 140
 problems for Laplace's equation . . . , 58
- polarization factors, 127
- polynomials
 Hermite, 268
 Jacobi, 269
 Laguerre, 269
 orthogonal, 272
- positive definite function, extrapolation of, 250
- potential, random, 60
- power spectrum, 14
- Poynting's theorem, differential form of, 138
- primitive of $X(\omega, \xi)$ in mean square, 120
- Principle of Contraction Mappings, 44
- probabilistic functional analysis, 41
- probabilities, error, 253
- probability
 distribution, 1, 3
 conditional, 8
 second ——— function, 3
 joint, 3
 measure space, 41
 that a ray is scattered n times in a distance
 R , 140
 transitional, 264
- process(es)
 autoregressive, 54
 branching, 172
 Gaussian, 58

- process(es) (*continued*)
 Markoff, 264
 nonstationary, 4, 12, 22
 Ornstein-Uhlenbeck, 272, 273
 stationary Markoff, 273
 white noise, 54
 Wiener, 61
 with orthogonal increments, 121
- programming, dynamic, 171, 175, 277
- propagation, 84
 effective ——— constant, 151
 vector, 127
 applications to ——— wave in continuous media, 149
- pursuit problems, 282
- quasilinearization, 171
- radiation
 ascending hierarchy of ——— problems, 125
 condition, 129
 problem, 128
 for an unbounded random medium, 124
 formulation by Keller, 124
 transport theory of, 169
- random
 adjoint, 42
 algebraic equations, 51
 of degree n , 51
 application of geometric optics to ———
 continuous media, 136
 continuum, 117
 contraction operator, 44, 64
 determinants, 53
 difference equation, 53, 54
 differential equations, 55
 their role in physics, 55
 their role in engineering, 55
 their role in technology, 55
 with piecewise constant coefficients, 56
 Dirichlet problem, 58
 endomorphism of \mathcal{X} , 42
 equation(s), 1
 associated with a stochastic pendulum, 58
 expected solutions of, 47
 general question of, 142
 moments of solution of given, 47
 expected value of ——— boundary condition, 60
 field, 10
 field point theorems, 45
 Fredholm integral equations 53, 64, 65
- random (*continued*)
 function(s), 2, 5
 analysis of, 118
 analytic, 133
 derivative of, 118
 integral of, 118
 mathematical analysis for, 118
 mean square convergence of, 118
 mean square properties of, 119
 regular, 118, 135
 second order, 130
- functional of ——— refractive index process, 118
- Hamiltonians, 61
- Hammerstein equation, 65
- integral equation(s), 61, 128
 associated with Markov process, 61
 Ito, 61
 systems of, 64
- integrals of partial differential equations, 118
- kernel, 64
- Lebesgue measure, 63
- matrices, 47, 171, 174
 sequence of ——— Φ_k , 57
- media, 1, 84
- medium, 145
 geometric optics of, 135
 geometrical optics of, 136
 ray propagation in, 136
 radiation problem for unbounded, 124
- Mikusinski operator, 61
- operator, 42
 inverse of, 42
 equations, 43
 approximate solutions of, 50
 iteration procedures for, 50
- operators
 semigroups of, 61
 spectral properties of concrete, 47
- ordinary differential equations, 55
 of interest in mathematical physics, 57
 of interest in engineering, 57
- partial differential equations, 58
 with constant coefficients, 61
- potential, 60
- process, 6, 12
 Gaussian, 130
 stationary, 291
- residual
 for permittivity, 126, 127
 for the dielectric constant, 126, 127
- resolvent operator, 47
- sampling, 27

- random
 sampling (*continued*)
 operator, 14
 series, 6
 set, 63
 solution(s), 43
 of some classical integral equations, 62
 of the heat equation for an infinite rod, 58
 stability of ——— linear systems, 56
 Sturm–Liouville problem, 61
 transformation, 41
 $T(\omega)$
 linear, 41
 bounded, 42
 variable(s), 1, 6, 145
 generalized, 41
 Banach space of, 50
 conditional expectation of, 48
 vibration problem, 61
 Volterra integral equations, 64
 walks on lattices, 1
 Wiener solution of the Dirichlet problem, 59
 randomly inhomogeneous
 anisotropic ——— medium, 128
 geometric optics of, 137
 Volume, V , 126
 rare gas, 102
 ratio, likelihood, 260
 ray
 curvature, 135
 direction, 135
 paths, 125, 135
 propagation in a random medium, 136
 theory, 117
 ray-tracing, 125, 127
 realizable boundary conditions, 58
 realization, or sample function, 120
 recurrence relation, 134
 reciprocity relation, 87
 reduced wave equation, 118, 120, 135
 one-dimensional, 124
 reflection coefficients, 134
 refractive index, 117, 125, 130, 131
 effective, 158
 Markov process, 136
 spectral density functions of, 131
 regression analysis, 241.
 regular
 functions, 121
 random functions, 118, 135
 regularity, 130
 relaxation theory, F. Bloch's, 61
 representation, spectral, 118, 119, 121, 131
 representation, spectral (*continued*)
 for non-self-adjoint equations, 142
 residual spectrum $R\sigma(T(\omega))$ of $T(\omega)$, 46
 resolvent
 random ——— operator, 47
 set $\rho(T(\omega))$ of a random endomorphism
 $T(\omega)$, 47
 Reynolds
 endomorphism, 70
 identity, 70
 operator(s), 49, 50, 70
 Riccati equation(s), 133, 141
 spatial, 130
 analogue of, 117
 one-dimensional form, 133
 three-dimensional analogue of, 131
 Riemann integrals, stochastic, 120
 roots of the equation, distribution of, 52
 Rytov
 method of, 117, 128
 transformation, 124, 130
 sample
 covariance function, 223
 function, 120
 space S , 6
 spectral density function, 224
 sampling, random, 27
 operator, 14
 scattered wave, coherent, 153
 scatterers
 discrete, 84
 forward type, 102
 scattering
 amplitude, 87
 forward ——— per unit volume, 153
 cross section, 139
 per unit solid angle . . . , 140
 total, 88
 isotropic, 127
 orders of, 90
 Schauder fixed point theorem, 44
 Schrödinger equation with random potential
 function, 60
 second
 approximation to the iteration-perturbation
 solution, 127
 distribution, 8
 order
 functions, 121
 random functions, 130
 stationary ——— processes, the spectral
 representation, 121

- second (*continued*)
 probability distribution function, 3
 self-consistency, 162
 semi-group property, 172
 semigroups of random operators, 61
 sequence of random matrices Φ_k , 57
 series, functional Taylor, 164
 set, random, 63
 signal
 detection, 222, 247
 measurement, 247
 simple Brownian motion, Langevin equation
 for, 58
 single-scattering, 127
 solution, 126
 random, 43
 of classical integral equations, 62
 Sommerfeld radiation condition, 124
 space(s)
 Borel, 46
 of all real functions, Ω , 120
 Orlicz, 62, 142
 spatial
 analogue of the Riccati equation, 117
 Riccati equation, 130
 equivalent, 124
 one-dimensional form of, 133
 spectra, 128
 spectral
 analysis, 223
 average, 227
 density, 11, 12, 125
 function, 132, 225
 sample, 224
 functions of the refractive index, 131
 Gaussian, 140
 distribution function, 224
 general form of ——— theorem for regular
 processes, 121
 jump function, 226
 properties of some concrete random opera-
 tors, 47
 representation(s), 118, 119, 121, 131
 for non-self-adjoint equations, 142
 window, 227
 spectrum, 226
 mixed, 226
 of a time series, 224
 stable, 254, 256
 distributions, 272
 stability, 173, 247, 256
 condition, 254
 criteria, 56
 stability (*continued*)
 of a random linear system . . . , 62
 of linear systems, 56
 of random linear systems, 56
 stack of Bernoulli slabs, 134
 stationarity, 5
 covariance, 225
 stationary, 9
 covariance function, 122
 Gaussian inhomogeneities, 128
 Markoff process, 273
 Markov process, 140
 process(es)
 locally, 122
 not necessarily ergodic, 123
 random process, 291
 second order processes . . . , 121
 statistical
 autocorrelation, 12
 circuit design, 179
 measures, 1, 14, 17, 20, 21, 23, 24, 27, 28
 nonsingular ——— inference problems, 249
 optimization, 27
 transforms, 20
 Stieltjes integrals, stochastic, 120
 stochastic
 analogue
 of the abstract Cauchy problem, 61
 of the Newton–Kantorovitch iteration
 procedure, 50
 approach to the Newton–Cotes scheme, 50,
 51
 approximation, 180
 boundary value problem, 58
 formulation of, 59
 case . . . reflection and transmission co-
 efficients, 141
 coefficients, 290
 differential equation(s), 27, 28
 Langevin type, 272
 differential operator, 1, 13
 equation, 145, 147
 filter, 17
 Green's function, 1, 11, 12, 14, 15, 17, 21,
 28, 38, 56
 Hamiltonian operator, 1
 integral, 122
 iteration, 172
 kernel, 35
 Lebesgue integrals, 120
 linear
 differential equation, 290
 operators, 290

- stochastic (*continued*)
 majorants for the Neumann series . . . ,
 134
 mappings, 13
 matrix, 14
 operator(s) 12, 13, 14, 27, 147
 perturbations of the sure operator, 147
 process associated with the random difference
 equation, 65
 processes, 2, 3, 5
 associated with random equations, 41, 65
 Riemann integrals, 120
 Stieltjes integrals, 120
 transformation, 171
 strong equivalence, 260
 Sturm–Liouville
 equation, 270, 271
 of the ——— type, 265
 random ——— problem, 61
 substitutions, method of successive, 148
 sure
 (nonstochastic) operator, 147
 stochastic perturbations of ——— operator,
 147
 -signal-in-noise, 248
 systems of random integral equations, 64
- 2-body integral equations, 96
 table of covariance functions and associated
 spectra, 122.
 Taylor series, functional, 164
 test functional, 252, 258
 three-dimensional analogue of the Riccati
 equation, 131
 time
 autocorrelation, 12
 estimation of ——— delay, 258
 series
 analysis, 221
 economic, 222
 deviation, 227
 spectrum of, 224
 spectrum, 12
 total
 electromagnetic energy, 138
 scattering cross-section, 88
 per unit volume at wave number k , 140
 transformation(s)
 measure-preserving, 73
 random, 41
 Rytov, 124, 130
 stochastic, 171
 theory, 24
- transforms, statistical, 20
 transition(al)
 operator, 155
 probabilities, 125
 probability, 140, 264
 transmission
 coefficient(s), 106, 112, 134
 line, 106, 112
 equation, 125, 134
 transmitted power, 127
 transport
 equation(s), 117, 127
 approach, 125
 for multiple scattering, 138
 phenomena, 141
 process, 135
 theory
 of incoherent intensity and flux, 165
 of radiation, 169
 truncated hierarchy, 290
 truncation
 approximations, 294
 point, 233, 237, 241
 Tukey–Hamming kernel, 239
 turbulence, 70, 128
 turbulent
 atmosphere, 127
 scatter, 117
 scattering, 139
 of electromagnetic waves, 123
- unaveraged hierarchy, 293
 unbounded, 130
 radiation problem for ——— random
 medium, 124
 uniformly bounded with probability 1, 129
 uniqueness theorem, 122
- V , 126, 127
 variable(s), random, 1, 6, 145
 variance, 7, 229
 of signal level fluctuations, 136
 of the wave, 165, 166
 of $x(\omega)$, 48
 proportional to penetration distance, 140
 vibration problem, random, 61
 Volterra integral equations
 mean square, 120
 random, 64
- wave
 coherent, 150
 attenuation coefficient for, 152

- wave (*continued*)
 - equation, 118
 - reduced, 118, 120, 135
 - to be one-dimensional, 124
 - multiple scattered, 89
 - propagation
 - applications to ——— in continuous media, 149
 - in media of discrete scatterers, 155
 - scattered, coherent, 153
 - variance of, 165, 166
- weak convergence of measures, 257
- weakly inhomogeneous medium, 132
- weight, 233
- white noise, 222
 - process, 54
- wide-sense
 - Markov process, 137
 - Markovian, 137
 - solution, 43
- Wiener
 - integrals, 125, 140, 142
 - process, 61
- Wiener-Hopf, 72
- window, 230, 232
 - generator, 233
 - amplitude, 236
 - lag, 233
- worst-case design, 179
- X**, 126
- $X(\omega, \xi)$, 119

