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PREFACE

This book is the result of the 1986 American Mathematical Society Short Course entitled Approximation Theory given at the annual meeting at New Orleans, on January 5-6, 1986.

Approximation Theory is properly a subfield of Analysis, but derives much of its impetus from applications such as data fitting, the representation of curves and surfaces for design and display, the reconstruction of functions from partial information, the numerical solution of functional equations and the like. For this reason, Approximation Theory offers ready-made applications of the basic ideas of Analysis.

The first lecture describes and illustrates the basic concerns of Approximation Theory. The other lectures are intended to provide a quick introduction into some of the areas of current research interest. Topics highlighted are: Approximation in the complex domain, n-width, optimal recovery, interpolation, algorithms for approximation, and splines, with strong emphasis on a multivariate setting in the last three topics.

I thank the authors very much for the considerable and selfless effort they have put into the preparation of the lectures and these notes.

Carl de Boor
Madison, Wisconsin
March, 1986
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