AMS SHORT COURSE LECTURE NOTES
Introductory Survey Lectures
A Subseries of Proceedings of Symposia in Applied Mathematics

Volume 40  MATRIX THEORY AND APPLICATIONS
   Edited by Charles R. Johnson (Phoenix, Arizona, January 1989)

Volume 39  CHAOS AND FRACTALS: THE MATHEMATICS BEHIND THE
   COMPUTER GRAPHICS
   Edited by Robert L. Devaney and Linda Keen (Providence, Rhode Island, August 1988)

Volume 38  COMPUTATIONAL COMPLEXITY THEORY
   Edited by Juris Hartmanis (Atlanta, Georgia, January 1988)

Volume 37  MOMENTS IN MATHEMATICS
   Edited by Henry J. Landau (San Antonio, Texas, January 1987)

Volume 36  APPROXIMATION THEORY
   Edited by Carl de Boor (New Orleans, Louisiana, January 1986)

Volume 35  ACTUARIAL MATHEMATICS
   Edited by Harry H. Panjer (Laramie, Wyoming, August 1985)

Volume 34  MATHEMATICS OF INFORMATION PROCESSING
   Edited by Michael Anshel and William Gewirtz (Louisville, Kentucky, January 1984)

Volume 33  FAIR ALLOCATION
   Edited by H. Peyton Young (Anaheim, California, January 1985)

Volume 32  ENVIRONMENTAL AND NATURAL RESOURCE MATHEMATICS
   Edited by R. W. McKelvey (Eugene, Oregon, August 1984)

Volume 31  COMPUTER COMMUNICATIONS
   Edited by B. Gopinath (Denver, Colorado, January 1983)

Volume 30  POPULATION BIOLOGY
   Edited by Simon A. Levin (Albany, New York, August 1983)

Volume 29  APPLIED CRYPTOLOGY, CRYPTOGRAPHIC PROTOCOLS,
   AND COMPUTER SECURITY MODELS
   By R. A. DeMillo, G. I. Davida, D. P. Dobkin, M. A. Harrison, and R. J. Lipton
   (San Francisco, California, January 1981)

Volume 28  STATISTICAL DATA ANALYSIS
   Edited by R. Gnanadesikan (Toronto, Ontario, August 1982)

Volume 27  COMPUTED TOMOGRAPHY
   Edited by L. A. Shepp (Cincinnati, Ohio, January 1982)

Volume 26  THE MATHEMATICS OF NETWORKS
   Edited by S. A. Burr (Pittsburgh, Pennsylvania, August 1981)

Volume 25  OPERATIONS RESEARCH: MATHEMATICS AND MODELS
   Edited by S. I. Gass (Duluth, Minnesota, August 1979)

Volume 24  GAME THEORY AND ITS APPLICATIONS
   Edited by W. F. Lucas (Biloxi, Mississippi, January 1979)

Volume 23  MODERN STATISTICS: METHODS AND APPLICATIONS
   Edited by R. V. Hogg (San Antonio, Texas, January 1980)

Volume 22  NUMERICAL ANALYSIS
   Edited by G. H. Golub and J. Oliger (Atlanta, Georgia, January 1978)

Volume 21  MATHEMATICAL ASPECTS OF PRODUCTION AND DISTRIBUTION OF
   ENERGY
   Edited by P. D. Lax (San Antonio, Texas, January 1976)
PROCEEDINGS OF SYMPOSIA IN APPLIED MATHEMATICS

Volume 20  THE INFLUENCE OF COMPUTING ON MATHEMATICAL RESEARCH AND EDUCATION
Edited by J. P. LaSalle (University of Montana, August 1973)

Volume 19  MATHEMATICAL ASPECTS OF COMPUTER SCIENCE
Edited by J. T. Schwartz (New York City, April 1966)

Volume 18  MAGNETO-FLUID AND PLASMA DYNAMICS
Edited by H. Grad (New York City, April 1965)

Volume 17  APPLICATIONS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS IN MATHEMATICAL PHYSICS
Edited by R. Finn (New York City, April 1964)

Volume 16  STOCHASTIC PROCESSES IN MATHEMATICAL PHYSICS AND ENGINEERING
Edited by R. Bellman (New York City, April 1963)

Volume 15  EXPERIMENTAL ARITHMETIC, HIGH SPEED COMPUTING, AND MATHEMATICS
Edited by N. C. Metropolis, A. H. Taub, J. Todd, and C. B. Tompkins (Atlantic City and Chicago, April 1962)

Volume 14  MATHEMATICAL PROBLEMS IN THE BIOLOGICAL SCIENCES
Edited by R. Bellman (New York City, April 1961)

Volume 13  HYDRODYNAMIC INSTABILITY
Edited by R. Bellman, G. Birkhoff and C. C. Lin (New York City, April 1960)

Volume 12  STRUCTURE OF LANGUAGE AND ITS MATHEMATICAL ASPECTS
Edited by R. Jakobson (New York City, April 1960)

Volume 11  NUCLEAR REACTOR THEORY
Edited by G. Birkhoff and E. P. Wigner (New York City, April 1959)

Volume 10  COMBINATORIAL ANALYSIS
Edited by R. Bellman and M. Hall, Jr. (New York University, April 1957)

Volume 9  ORBIT THEORY
Edited by G. Birkhoff and R. E. Langer (Columbia University, April 1958)

Volume 8  CALCULUS OF VARIATIONS AND ITS APPLICATIONS
Edited by L. M. Graves (University of Chicago, April 1956)

Volume 7  APPLIED PROBABILITY
Edited by L. A. MacColl (Polytechnic Institute of Brooklyn, April 1955)

Volume 6  NUMERICAL ANALYSIS
Edited by J. H. Curtiss (Santa Monica City College, August 1953)

Volume 5  WAVE MOTION AND VIBRATION THEORY
Edited by A. E. Heins (Carnegie Institute of Technology, June 1952)

Volume 4  FLUID DYNAMICS
Edited by M. H. Martin (University of Maryland, June 1951)

Volume 3  ELASTICITY
Edited by R. V. Churchill (University of Michigan, June 1949)

Volume 2  ELECTROMAGNETIC THEORY
Edited by A. H. Taub (Massachusetts Institute of Technology, July 1948)

Volume 1  NON-LINEAR PROBLEMS IN MECHANICS OF CONTINUA
Edited by E. Reissner (Brown University, August 1947)
PROCEEDINGS OF SYMPOSIA
IN APPLIED MATHEMATICS
Volume 40

MATRIX THEORY
AND APPLICATIONS

Charles R. Johnson, Editor

American Mathematical Society
Providence, Rhode Island
# Table of Contents

Preface ................................................................................................................................. ix

The Many Facets of Combinatorial Matrix Theory
   Richard A. Brualdi ........................................................................................................ 1

Patterned Matrices
   Persi Diaconis ............................................................................................................ 37

Tangential Interpolation Problems for Rational Matrix Functions
   Joseph A. Ball, Israel Gohberg, and Leiba Rodman .............................................. 59

The Hadamard Product
   Roger A. Horn ............................................................................................................ 87

Matrix Completion Problems: A Survey
   Charles R. Johnson .................................................................................................... 171

The Role of Nonnegative Idempotent Matrices in Certain Problems in Probability
   Arunava Mukherjea .................................................................................................. 199

Interface Between Statistics and Linear Algebra
   Ingram Olkin ............................................................................................................ 233
Preface

As a subfield of mathematics, matrix theory continues to enjoy a renaissance that has accelerated during the past decade, though its roots may be traced much further back. This is due in part to stimulation from a variety of applications and to the considerable interplay with other parts of mathematics, but also to a great increase in the number and vitality of specialists in the field. As a result, the once popular misconception that the subject has been fully researched has been largely dispelled. The interest on the part of the American Mathematical Society and the approximately 140 participants in the Short Course (at the January 1989 Phoenix Meeting) on which this volume is based is a reflection of this change. The steady growth in quality and volume of the subject's three principal journals, Linear Algebra and its Applications, Linear and Multilinear Algebra, and the SIAM Journal on Matrix Analysis and Applications is another. Approximately 500 different authors have published in one of these three journals in the last two years. Geographically, strong research centers in matrix theory have developed recently in Portugal and Spain, Israel, the Netherlands, Belgium, and Hong Kong.

The purpose of the Short Course was to present a sample of the ways in which modern matrix theory is stimulated by its interplay with other subjects. Though the course was limited to seven speakers, the "other subjects" represented included combinatorics, probability theory, statistics, operator theory and control theory, algebraic coding theory, partial differential equations, and analytic function theory. Among other important examples, numerical analysis, optimization, physics and economics are, unfortunately, at most lightly touched. There is no limit to the specific examples that might be cited.

One of the ingredients in the recent vitality of matrix theory is the variety of points of view and tools brought to the subject by researchers in different areas. This is responsible for a number of important trends in current research. For example, the notion of majorization (mentioned in the talk by Olkin) has become pervasive in a historically brief period of time. The trend away from the "basis-free" point of view is illustrated by work in combinatorial matrix theory (Brualdi, Johnson), the Hadamard product
(Horn) and nonnegative matrices (Mukherjea). There are many quite worthy issues that are at least excruciatingly difficult to view in a basis-free way, and freedom from the basis-free view has opened many exciting avenues of research. On the other hand, recognition of the “right” problem dependant symmetries can provide vital insight (Diaconis). The synergy between matrix theory and systems theory has had a tremendous impact on both, and on the now highly mathematically driven field of electrical engineering (Gohberg). The immense variety of tools and problems illustrates a reason for use of the term “matrix theory” or “matrix analysis” in place of “linear algebra”. A large portion of current work is neither primarily linear nor primarily algebraic in nature. No point of view on what the subject is or where it is going could, or should, be without substantial disagreement. This only reflects the remarkable breadth of interest enjoyed by the subject. For an historical perspective on the nature and role of the subject the reader might enjoy the prefaces to each of the following: Recent Advances in Matrix Theory (Schneider 1964); A Survey of Matrix Theory and Matrix Inequalities (Marcus and Minc, 1964); Linear Algebra and its Applications, volume 1 (Alan Hoffman, 1968); and Matrix Analysis (Horn and Johnson, 1985). A glimpse of the contagious appeal of the subject is communicated by Olga Taussky in her November 1988 Monthly article “How I Became a Torchbearer for Matrix Theory”.

As organizer, I would like to again thank each of the speakers for a contribution that will advance both the subject and the general understanding of it. The significant time necessary to prepare both a talk and then subsequent paper is much appreciated. The community would also like to thank the American Mathematical Society for recognizing, and providing a forum for, the subject.

Charles R. Johnson
College of William and Mary
Subject Index

Abelian group designs, 42
Adjacent vertices, 179
Algebra
  adjacency, 143
  coherent, 143
Amitzur-Levitzki Theorem, 19, 21
Association scheme, 142
Atom in $T_\infty$, 221

Banded square partial matrix, 174, 177, 182
Basis, 200
Bergman kernel, 146
Bergstrom's inequalities, 157
Beta distribution,
  multivariate, 243
Bipartite graph, 10, 192, 195
Blaschke product, 65
Block
  banded patterns, 180
  design, 7
  matrix, 111
  triangular, 195
Bose-Messner algebra, 142

Cara\text{theodory-Fejer theorem}, 173–174, 184
Cayley-Hamilton theorem, 22
Characteristic polynomial, 23
Characteristic roots
  distribution of, 245
  estimation of, 243
Characterizations, 247
Chebychev inequalities, 248
Cholesky decomposition, 243
Chordal, 183–187, 195. See also Graph ordering, 181
Chromatic number of a graph, 26
Circuit, 179
Class circulants, 43
Class functions, 43
Clique, 180, 187–189
  number of a graph, 26
Cocktail party graphs, 24
Coherent algebra, 143
Combinatorial matrix
  algebra, 19
  analysis, 12
  problem, 172
  theory, 1, 27, 172
Completely monotone function, 145
Condition number, 137
Conference matrix, 30
Confoundable words, 25
Congruence, 112, 177
Connor's inequalities, 9
Contraction, 111, 182, 191, 192, 193
  completions, 190, 191, 192
  criterion, 112
Controllable pair, 84
Convolution, 39, 43, 209
Cyclic
  codes, 41
  designs, 42

Daleckii-Krein formula, 119
Design, 6, 10
Determinant, 1, 3, 177, 185
  inequalities, 172, 185, 187, 190
  maximizer, 174
  of a matrix, 29
Digraph, 1, 12, 17. See also Permutation
  cut set of, 12
  cycle of, 1
  of a matrix, 19
  strongly connected, 12, 18
Directed multigraph, 20-22

Edge, 179, 192
Eigenspace, 88
Eigenvalue, 88, 96
  Hermitian completion problem, 184
  inclusion region, 15
  of a matrix, 14
  of an irreducible matrix, 13
Electrical engineering/systems theory, 172

257
Elliptic operator, 94
Entropy, 172
Euclidean norm, 88, 92
Eulerian path, 20
Extremal problems, 233

Fisher’s inequality, 8, 9
Fourier
   analysis, 37
   inversion theorem, 44
   transform, 43
Frobenius, 195
   inner product, 101
   König theorem, 173
   norm, 92, 100
Full range pair, 84
Functional equations, 236

Gain matrix, 140
Gaussian elimination, 179
General inertia theorem, 136
Gersgorin’s theorem, 14–15
Graded poset, 3
Graph, 10, 17, 178, 187, 190. See also Intersection
   bipartite, 10, 192, 195
   decomposition, 11
   chordal, 27, 180–195
   chromatic number, 26
   complete, 10, 180
   distance transitive, 50
   line, 24–25
   stability number of, 25
   stacked, 108
   strong product, 25
   undirected, 179, 192, 196
Grothendieck’s theorem, 152
Group
   of nonnegative matrices, 202
   representations, 43
Grunsky inequalities, 145

Hadamard, 92, 93, 94, 154, 163, 185
   determinant theorem, 29
   factorization, 107
   Fischer inequalities, 187
   function of a matrix, 144
   inequality, 186
   matrix, 29
   multipliers, 93
   product, 88
   reciprocal, 145
   Schur product, 93
Hecke algebras, 49
Hermitian, 89, 171, 178
Hessian, 94
Hölder p-norms, 150

Homogeneous spaces, 49
Hopf Lemma, 94
Householder transformation, 253

Idempotent
   kernel, 201
   probability measure, 201
Idempotent matrices, 246
Ill-conditioned, 137
Image enhancement, 185
Incidence matrix, 6, 8
Inclusion-exclusion theorems, 250
Indecomposable pattern, 192
Index of primitivity, 142
Inertia, 149, 181
Inheritance property, 172, 191, 195
Inner product, 101
Interlacing eigenvalues theorem, 90, 97
Interpolation problem
   tangential LaGrange, 79
   tangential Nevanlinna-Pick, 60
Intersection graph, 187

König, 195
   theorem, 27
Krein, 120, 133
   conditions, 143
   parameters, 143
Kronecker product, 96, 151

Lagrange, 59
Laplace’s equation, 94
Laurent expansion, 66
Linear partial differential operator, 93
Linearly constrained positive definite completion problem, 184
Lipschitz norm-continuous, 127
Loewner, 93, 133, 144, 165, 166
   matrix, 119, 121
   ordering, 251
   partial order, 120
   theorem, 121
Log
   concave, 177
   concavity, 238
Lyapunov, 134
   equation, 74, 134

Majorization, 140, 148, 251
Matrix completion problem, 172
Matrix. See also Banded; Combinatorial;
   Convolution; Determinant; Digraph;
   Group; Hadamard; Hermitian; Loewner;
   Maximum; Partial
   adjacency, 1, 12, 24
   adjoint, 88
   Cauchy, 134
combinatorially symmetric, 179
completely positive definite, 144
completion problem, 172
convexity, 251
correlation, 89, 141, 148
covariance, 41, 140
diagonally
dominant, 14
signed, 149
similar, 17
difference quotient, 119
doubly stochastic, 28, 140, 251–252
embeddable, 223
exponent of, 13
factorization of the determinant, 5
formal incidence, 143
function, 82, 118, 120, 129
functional equations, 236
gain, 140
Gram, 89
idempotent, 246
identically distributed, 209
identities, 19
incidence, 4–8
independent, 199, 209
ininitely divisible, 144, 226–228
irreducible, 12, 14, 18, 44
partially defined, 143
completion of, 171, 194
primitive, 13
random, 199, 209, 253
reducible, 12
sparse, 179
square root of, 89
term rank of, 27
total support of, 28
transpose, 88
tridiagonal, 186
Mixed product rule, 96
Model reduction, 185
Monotone operator functions, 132
Monotonicity theorem, 90
Multivariate
beta distribution, 243
integrals, 233
Liouville-Dirichlet integrals, 244
normal distribution, 234
Norm, 91
inequalities, 92
spectral, 91, 190
Nullity, 149
Null-Kernel pair, 84
Numerical
analysis, 191
linear algebra, 179
radius, 99
range, 98, 184
Observable pair, 84
Operator theory, 191
Oppenheim, 154, 158, 167
inequality, 103
Optimization, 176, 190
Orderings, 250
Orthogonal
projection, 8
random matrix, 253
Orthostochastic, 140
Pairwise balanced design, 9
Partial
contraction, 191, 193
fraction expansion, 65
Hermitian matrix, 171, 178, 181
matrix, 171, 191–195
positive definite matrix, 173, 182, 185, 188
Pattern, 178
Perfect
codes, 47
graphs, 27
Permanent, 6
Permutation
digraph, 2–3
equivalence, 193–195
Pointwise multiplication, 93
Positive definite, 89, 174–177
completion problem, 176
Product rule, 96
Quadratic
forms, 245
inequalities, 182
Quasi-linearization, 90, 157
Radon transforms, 47
Random matrices, 199, 209
Random orthogonal matrix, 253
Rank, 96, 194–195
one factor, 104
one matrices, 109
Rational field, 190
Rayleigh-Ritz theorem, 89
Realization, 82, 84
Recurrent, 219
Relative gain array, 140
Representation, 43
Ring, 182
Schur, 91, 92, 93, 94, 95, 109, 154, 167, 168
complement, 155
Hadamard product, 93
multipliers, 93
product, 92
product theorem, 95, 183
unitary triangularization, 91
Seismic reconstruction, 172, 185
Selfadjoint norms, 92
Separator, 180
Shannon capacity, 26
problem of, 25
Shifted circulants, 55
Similar realizations, 84
Simple current, 179
Simulations, 233, 252
Singular
value, 91, 96, 190
vectors, 91
Smoothing data, 41
Spanning tree, 187, 188
Spectral
decomposition, 24
graph theory, 24
norm, 91, 190
radius, 88, 96
Spherical function, 50
State space isomorphism theorem, 84
Steiner system, 7
Stochastic. See also Matrix
forward chain, 217
Strongly ergodic, 219
Strong monotonicity theorem, 131
Subgraph, 180, 192
Submatrix, 186, 191, 194, 195
Subpattern, 194
Supergraphs, 182
Sylvester
determinantal identity, 174
equation of, 85
inertia theorem of, 89
Symmetric group, 44
Symmetric 2-design, 9
Systems theory, 82
Tail sigma-field, 221
Tensor product, 96
Term rank, 28
Toeplitz, 174, 184, 193
contraction, 193
Trace, 88
identity, 21
Transfer function, 82
Tree, 187, 189
Triangular minimum rank, 195
Triangular subpatterns, 195
Uniqueness theorem, 94
Unitarily invariant, 92
Unitary, 89
Univalent function, 144
Vertex separator, 180
Vertices, 179, 187
Weight, 1–5, 23
Wishart distribution, 235
Zero inertia, 149
Zero-pole interpolation, 65