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San Antonio, Texas

Ingrid Daubechies
Editor

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Preface

With hindsight the wavelet transform can be viewed as a synthesis of ideas that have emerged since the 60-s (and for some aspects even earlier) in fields as diverse as mathematics (pure as well as applied), physics and electrical engineering. The basic idea is always to use a family of building blocks to represent the object at hand (a function, an operator, a signal or image,...) in an efficient and/or insightful way; the building blocks themselves come in different sizes, and are suitable for describing features with a resolution commensurate with their size. This no doubt sounds rather vague, and the different contributions in this book flesh it out in different ways.

The first chapter in this book is the most introductory. It gives some motivation for the wavelet transform (although the proof of the pudding is in the eating – the true motivation is not given by abstract considerations, but by the usefulness of wavelets in proving real-life theorems or analyzing real-life data; some such applications are the main message of later chapters) and it describes various different types of wavelet transform, with special emphasis on orthonormal wavelet bases, because that is what most of the other chapters need.

There are two important aspects to wavelets, which I shall call "mathematical" and "algorithmical". In their mathematical aspect, wavelets are rooted in the use of dilations and convolutions in Calderón-Zygmund theory in harmonic analysis. The techniques used and refined there in the past 25 years had led to powerful tools suited to proving hard theorems. But that was the full extent of their range; they had not led to any applications in numerical analysis or signal analysis before the advent of wavelets. Algorithmically, wavelets are related to subband filtering in electrical engineering. Subband filtering was developed from the 70-s on; exact reconstruction procedures were discovered in the early 80-s. These were obviously fast algorithms, meant as a front-end processing step before encoding or compressing information in various types of signals. A lot of effort went into optimizing the filters for various applications, and this subfield of electrical engineering is now quite mature. But the results were purely algorithmic; they were never intended or viewed as a powerful mathematical tool that could interest people other than signal processors. Another algorithmic ancestor of wavelets are the multiple algorithms in numerical analysis, closer to mathematics, but still ad hoc.

Wavelets then incorporate both aspects: the fast algorithms of subband coding and the powerful mathematical potential of the Calderón-Zygmund theory tools. Chapters 2 to 8 in this book all illustrate different aspects of (bases of) wavelets, some more mathematical, others more algorithmical. Chapters 2 to 4 are mostly from the mathematical point of view (although the ideas in chapters 3 and 4 are being implemented in numerical work, which would be impossible without the fast algorithms); chapters 5 to 7 place more emphasis on the algorithmic aspect, without losing sight of the mathematical properties which are essential for any of these applications to even make sense. Chapter 8 combines some of the deep mathematical properties of wavelets (their "adaptability" to many functional spaces) with their algorithmic ease for still different applications. All the chapters in this volume have been written for these short course notes and are in final form as such; it should be noted, however, that each constitutes the summary of several mathematical research articles, some of which still have to appear.

Even though the first chapter explains many of the basic properties of wavelets in detail, most chapters contain their own short review of wavelet bases. I have not tried to edit away this slight overlap: the small differences in emphasis on which properties are deemed most important by the different authors is an illustration of the versatility of wavelets. All the authors chose also the notation that suited their purpose best; the notation is therefore not uniform throughout the book, but every chapter defines its own notation carefully.

I am very glad that some of the chapters (especially chapter 6) contain constructions "beyond wavelets" (wavelet packets, local trigonometric bases). Wavelets are an exciting development, and they let us put a foot in many doors, but they are not the answer to everything. And even for problems where they seem perfect, we need to develop refined and efficient tools to make use of their properties, such as is done in chapters 4 and 5. This book is therefore a good illustration of the powerful potential of wavelets in different directions, a potential that we hope some of our readers will help develop!

I should add that there are many very interesting developments using wavelets that were not represented at the short course and are not included in these lecture notes – including some of my favorites. But there was only so much we could do in one short course, or in one volume of course notes.

I would like to use this opportunity to thank all the speakers for their participation. It was not easy to make a short course where everybody would find something to like, from wavelet novices to experts working with them for the past four years, and I think we succeeded. (Although the result was also a course where everybody found something to dislike!) I would like to thank especially Yves Meyer for still giving us his contribution for this volume, even though he had to cancel his lecture, and Pierre-Gilles Lemarié-Rieusset for substituting for Yves Meyer at very short notice.

Finally, I would like to thank Tina Sharp for the seemingly impossible task of massaging all the faxes, typescripts, diskettes, e-mail files and scribbled notes into one coherent whole.

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