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Multiquadric collocation methods in the numerical solution of Volterra integral and integro-differential equations

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ABSTRACT. Volterra integral equations and integro-differential equations with continuous kernel are considered and are solved by applying discrete collocation methods based on multiquadric approximations and quadrature rules. Applications to Hammerstein-type equations are also included with the methods applied in the light of the new collocation methods of Kumar and Sloan.

1. Introduction

This paper considers Volterra integral equations (VIEs) of the form

$$y(x) = f(x) + \int_0^x P(x,s)H(s,y(s))ds, \qquad 0 \le x \le X, \tag{1.1}$$

and Volterra integro-differential equations (VIDEs) of the form

$$y'(x) = G(x, y(x), z(x)), \qquad y(0) = y_0, \qquad 0 \le x \le X, \tag{1.2}$$

where

$$z(x) = \int_0^x K(x,\ s,\ y(s))ds. \tag{1.3}$$

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Detailed versions of parts of this paper will be submitted elsewhere.

We apply discrete collocation methods based on multiquadrics (MQ). For equations of the form (1.1) the methods will employ the approach used in the so-called new collocation methods (Kumar and Sloan [**9**], also Brunner [**3**], Makroglou [**12**]).

The plan of this paper is as follows. Section 2 gives a description of the application of the MQ approximations to equations (1.1), (1.2). In §3 are given numerical results, and in §4 conclusions and comments on remaining problems.

2. Description of the methods

First we consider the integral equation (1.1). Setting $z(x) = H(x, y(x))$, we obtain the following integral equation in z:

$$z(x) = H(x, f(x) + \int_0^x P(x,s)z(s)ds). \tag{2.1}$$

Then $y(x)$ can be obtained from (1.1) by replacing $H(s, y(s))$ by $z(s)$. In a piecewise collocation method we consider mesh points $0 = x_0 < x_1 < \cdots < x_N = X$ and seek approximate values of the solution of (2.1) for $0 \le x_{nj} \le X$, where $x_{nj} = x_n + c_j h_n$, $h_n = x_{n+1} - x_n$ $(n = 0, 1, \ldots, N-1)$, and the collocation parameters are such that $0 \le c_1 < \cdots < c_m \le 1$ (for more details, see for example Brunner and van der Houwen [**2**], Brunner [**4**], Baker [**1**], Makroglou [**11**]).

The form of a multiquadric augmented by a constant term which interpolates a set of points $(c_i, y_i), i = 1, 2, \ldots, m$, is

$$S(x) = \sum_{j=1}^{m} \alpha_j \tilde{u}_j(x), \tag{2.2}$$

where

$$\tilde{u}_j(x) = \begin{cases} 1 & \text{if } j = 1, \\ u_j(x) - u_1(x) & \text{if } j > 1, \end{cases} \tag{2.3}$$

$$u_j(x) = \sqrt{(x - c_j)^2 + R^2} \tag{2.4}$$

and the coefficients α_j are obtained from the solution of the linear system of equations

$$S(c_i) = y_i, \qquad i = 1, 2, \ldots, m. \tag{2.5}$$

For details on multiquadric interpolation, we refer for example to Hardy [**6, 7**], Powell [**13, 14**], Madych and Nelson [**10**], Buhmann and Dyn [**5**], Kansa and Carlson [**8**] and the references therein.

Quadrature rules based on multiquadric approximations can be derived for the approximation of the integrals . They are of the form

$$\int_0^1 \phi(s)ds = \sum_{\mu=1}^{m} w_\mu \phi(c_\mu), \tag{2.6}$$

where

$$w_\mu = \int_0^1 l_\mu(s)ds = \sum_{\lambda=1}^{m} Q_{\mu\lambda} \int_0^1 \tilde{u}_\lambda(s)ds, \tag{2.7}$$

the $Q_{\mu\lambda}$ being the coefficients of $\phi(c_\mu)$ in the expression of α_λ obtained from solving the linear system

$$\sum_{\lambda=1}^{m} \alpha_\lambda \tilde{u}_\lambda(c_j) = \phi(c_j), \qquad j = 1, 2, ..., m,$$

$$\int u_j(s)ds = \frac{R^2}{4}[0.5D + \ln|D| - \frac{1}{2D}], \tag{2.8}$$

$$D = \frac{2(s-c_j)^2}{R^2} + 1 + \frac{2(s-c_j)\sqrt{(s-c_j)^2 + R^2}}{R^2} \tag{2.9}$$

and

$$\int \tilde{u}_j(s)ds = \begin{cases} \int u_j(s)ds - \int u_1(s)ds & \text{if } j > 1, \\ s & \text{if } j = 1. \end{cases} \tag{2.10}$$

Using the above type of approximations for $z(x)$ in $[x_n, x_{n+1}], n = 0, 1, \ldots, N-1$ and the quadrature rules (2.6)–(2.10), we obtain the following systems in the a_{nk}:

$$\sum_{k=1}^{m} a_{nk}\tilde{u}_k(c_j) = H(x_{nj}, f(x_{nj}) + \sum_{i=0}^{n-1} h_i \sum_{\lambda=1}^{m} w_\lambda P(x_{nj}, x_{i\lambda}) z_{i\lambda} \tag{2.11}$$

$$+ h_n c_j \sum_{k=1}^{m} a_{nk} \sum_{\lambda=1}^{m} w_\lambda P(x_{nj}, x_n + h_n c_j c_\lambda)\tilde{u}_k(c_j c_\lambda)), j$$

$$= 1, 2, .., m; n = 0, 1, ..., N-1.$$

Consider now the Volterra integro-differential equations (1.2)–(1.3). Proceeding similarly with approximations of the type (2.2)–(2.4) for both $y(x)$ and $z(x)$, and using the quadrature rules (2.6)–(2.10), we obtain the following systems of equations in the a_{nk}, b_{nk}:

$$\sum_{k=1}^{m} a_{nk}\tilde{u}_k(c_j) = \sum_{i=0}^{n-1} h_i \sum_{l=1}^{m} w_l G(x_{il}, y_{il}, z_{il}) + y_0 \tag{2.12}$$

$$+ h_n \sum_{l=1}^{m} w_l^j G(x_n + h_n c_l, \sum_{k=1}^{m} a_{nk}\tilde{u}_k(c_l), \sum_{k=1}^{m} b_{nk}\tilde{u}_k(c_l))$$

and

$$\sum_{k=1}^{m} b_{nk}\tilde{u}_k(c_j) = \sum_{i=0}^{n-1} h_i \sum_{l=1}^{m} w_l K(x_{nj}, x_{il}, y_{il}) \tag{2.13}$$

$$+ h_n c_j \sum_{l=1}^{m} w_l K(x_{nj}, x_n + h_n c_j c_l, \sum_{k=1}^{m} a_{nk}\tilde{u}_k(c_j c_l)),$$

$$j = 1, 2, ..., m; \quad n = 0, 1, ..., N-1,$$

where

$$w_l^j = \int_0^{c_j} l_l(s)ds.$$

3. Numerical Results

Numerical results are presented for the following examples:

1) $y(x) = 1+\sin^2(x) - \int_0^x 3\sin(x-s)y^2(s)ds$ with true solution $y(x) = \cos x$ (cf. Brunner and van der Houwen [**2,** p. 507]);
2) $y'(x) = -\sin x + \frac{x}{3}[\cos^3(x) - 1] + \int_0^x x\sin(s)y^2(s)ds, \quad y(0) = 1$ (constructed), with true solution $y(x) = \cos x$.

All computations were done in double precision.

The entries in Tables 1–3 represent maximum absolute errors for a number of stepsizes. The resulting nonlinear equations were solved by Newton's method. A tolerance equal to 1.D-10 was used. The results with heading "MQ" were obtained with the use of multiquadric quadrature rules (Eqs. 2.6–2.10). Example 1 was solved with both the Radau II points ($c_1 = 1/3, c_2 = 1$) and the Gauss points ($c_1 = \frac{3-\sqrt{3}}{6}, c_2 = \frac{3+\sqrt{3}}{6}$).

Table 1. Example 1, collocation methods, Radau II points
max. abs. error in $0 \le x \le 2$

h	R	Polynom. Spline	MQ
0.1	6	0.94 D − 4	0.32 D − 4
0.05	12	0.12 D − 4	0.39 D − 5
0.025	24	0.14 D − 5	0.48 D − 6
0.0125	48	0.18 D − 6	0.60 D − 7
0.00625	96	0.22 D − 7	0.75 D − 8
3.125 D − 3	192	0.28 D − 8	0.93 D − 9

Table 2. Example 1, collocation methods, Gauss points
max. abs. error in $0 \le x \le 5$

h	R	Iter.Polyn. Spline	MQ
0.1	6	0.63 D − 4	0.16 D − 4
0.05	12	0.40 D − 5	0.22 D − 6
0.025	24	0.25 D − 6	0.11 D − 7
0.0125	48	0.16 D − 7	0.13 D − 8
6.25 D − 3	96	0.99 D − 9	0.55 D − 9

Table 3. Example 2, collocation methods, Radau II points
max. abs. error in $0 \le x \le 5$

h	R	Polynom. Spline	MQ
0.1	19	0.37 D − 3	0.47 D − 5
0.05	38	0.47 D − 4	0.61 D − 6
0.025	76	0.60 D − 5	0.79 D − 7
0.0125	152	0.75 D − 6	0.99 D − 8
6.25 D − 3	304	0.95 D − 7	0.12 D − 8

4. Concluding Remarks

In this paper the application of collocation methods based on approximations with multiquadrics to Volterra integral and integro-differential equations was considered. The results obtained show that the new methods can produce very accurate results when used in combination with multiquadric quadrature rules. Orders of convergence $O(h^3)$ and $O(h^4)$ were observed with $m = 2$ for the Radau II and Gauss points, respectively. The accuracy of the results depends on the successful choice of the multiquadric parameter R. We are continuing research on the derivation of a method for choosing R, on an error analysis for obtaining the order of convergence, and on applications of the methods to equations with weakly-singular kernels.

References

1. Baker, C.T.H., *The Numerical Solution of Integral Equations*, Clarendon Press, Oxford, 1977.
2. Brunner, H. and Van der Houwen, P.J., *The numerical solution of Volterra equations*, CWI Monographs **3** (1986), North Holland, Amsterdam.
3. Brunner, H., *Implicitly linear collocation methods for nonlinear Volterra equations*, Appl. Numer. Math. **9** (1992), 235–247.
4. Brunner, H., *Iterated collocation methods and their discretizations for Volterra integral equations*, SIAM J. Numer. Anal. **20** (1984), 1132–1145.
5. Buhmann, M.D. and Dyn, N., *Error estimates for multiquadric interpolation*, pp. 51–58 in *Curves and Surfaces* , Laurent, P.J., LeMéhauté, A., Schumaker, L.L. (eds.), Academic Press, 1991.
6. Hardy, R.L., *Multiquadric equations of topography and other irregular surfaces*, J.Geophys. Res. **76** (1971), 1905–1915.
7. Hardy, R.L., *Theory and applications of the multiquadric-biharmonic method*, Comput. Math. Appl. **19** (1990), 163–208.
8. Kansa, E.J. , Carlson, R.E., *Improved accuracy of multiquadric interpolation using variable shape parameters*, Comput. Math. Appl. **24** (1992), 99–120.
9. Kumar, S. and Sloan, I.H., *A new collocation-type method for Hammerstein integral equations*, Math. Comp. **48** (1987), 585–593.
10. Madych, W.R. and Nelson, S.A., *Multivariate interpolation and conditionally positive definite functions. II*, Math. Comp. **54** (1990), 211–230.
11. Makroglou, A., *Convergence of a block-by-block method for nonlinear Volterra integro-differential equations*, Math. Comp. **35** (1980), 783–796.
12. Makroglou, A., *Radial basis functions in the numerical solution of Fredholm integral and integro-differential equations*, pp. 478–484 in *Advances in Computer Methods for Partial Differential Equations - VII* , Vichnevetsky, R., Knight, D., Richter, G. (eds.), Proceedings of 7th IMACS Conf. on Computer Methods for PDEs, IMACS, N.Brunswick, NJ, 1992.
13. Powell, M.J.D., *Univariate multiquadric interpolation : some recent results*, DAMTP 1990/NA9, Dept. of Applied Mathematics and Theoretical Physics, Univ. of Cambridge, England, 1990.
14. Powell, M.J.D., *Radial basis function approximations to polynomials*, pp. 223–241 in *Numerical Analysis* , Griffiths, D.F. and Watson, G.A. (eds.), Longman Scientific and Technical, Harlow, 1988.

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