Introduction to Mathematical Finance

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Preface

Nearly 100 years ago Bachelier, in his fundamental work "Théorie de la spéculation," laid the foundation for the subject now known as Mathematical Finance. In the same work, he provided the first treatment of Brownian motion. The pace of work in this area has grown rapidly. About 50 years ago, Markowitz developed his mean-variance based model for portfolio selection. A little over 25 years ago, the works of Black, Merton, Scholes and Samuelson identified and illuminated the important (and shocking) consequences of assuming that markets present no opportunities for arbitrage. A few years later, Harrison and Kreps demonstrated the fundamental role of martingales and stochastic calculus in constructing and understanding models for financial markets. This connection opened the door for a virtual flood of mathematicians to contribute to developments over the past 20 years.

Concurrently with these mathematical developments, markets have developed and grown. For example, the Chicago Board Options Exchange (CBOE), founded in 1973, revolutionized options trading by creating standardized, listed stock options. Financial institutions now write custom (derivative) contracts for other firms allowing these firms to reduce their interest rate, foreign currency, and credit risks. The level of activity has grown rapidly. The total notional of derivatives written by U.S. commercial banks was $20 trillion in 1996, an order of magnitude greater than the federal budget.

Research activity in this area, both in academia and in industry, has continued to grow. There are now several journals devoted to this subject, and many universities have developed special programs to educate students in this area. One manifestation of this activity was the Short Course on Mathematical Finance, given in San Diego, CA in January of 1997; papers delivered at this course constitute the contents of this volume.

We would like to thank the AMS for their enthusiastic support and their encouragement of the publication of this collection.
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