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The Radon Transform, Inverse Problems, and Tomography

American Mathematical Society
Short Course
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Atlanta, Georgia

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Contents

Preface

This volume brings together six articles on the mathematical aspects of tomography and related inverse problems. They are based on the lectures in the Short Course, *The Radon Transform and Applications to Inverse Problems*, at the American Mathematical Society meeting in Atlanta, GA, January 3-4, 2005. They covered introductory material, theoretical problems, and practical issues in 3-D tomography, impedance imaging, local tomography, wavelet methods, regularization and approximate inverse, sampling, and emission tomography. All contributions are written for a general audience, and the authors have included references for further reading.

Tomography and inverse problems are active and important fields combining pure and applied mathematics with strong interplay between applications and the diverse mathematical problems that have emerged since the first article in the field appeared almost a century ago. The applied side is best known for medical and scientific applications, in particular, medical imaging, radiotherapy, and industrial non-destructive testing. Doctors use tomography to see the internal structure of the body or to find functional information, such as metabolic processes, noninvasively. Scientists discover defects in objects, the topography of the ocean floor, and geological information using X-rays, geophysical measurements, sonar, or other data. Thus, tomography consists of a broad range of inverse problems. These are called inverse problems because information about an object is obtained from indirect data.

X-ray tomography is the most basic modality, and it can be described in the following way: a beam of X-rays is emitted with a known intensity from a source outside the material to be scanned, usually some part of the human body. A detector on the other side of the body picks up the intensity after the ray has traveled along a straight line segment, L, through the body. Some X-rays are lost due to scattering and absorption because of the attenuation effects of the material. Let f be the linear attenuation coefficient of the body. If the X-rays are monochromatic, then the attenuation coefficient is proportional to the density of the object (the proportionality depends on the energy of the photons). Choosing units so the proportionality is 1 we can view f as the density function of the object. Then, a simple derivation (see e.g., [**8**, (2.1)]) shows that the logarithm of the intensity ratio is proportional to the line integral of the attenuation function, so in appropriate units,

$$\ln\left[\frac{I(\text{source})}{I(\text{detector})}\right] = \int_L f(x)\,dx =: Rf(L)\,. \tag{1}$$

The right-hand side of (1) is exactly the Radon line transform of f, $Rf(L)$. In short, the $Rf(L)$ averages the function f over the line L. The problems now becomes the recovery of f from Rf.

This leads us back to the beginning of the last century, more than forty years before the first CT-scanner emerged. The Radon line transform in $\mathbb{R}^2$, which is the case used in many X-ray tomography scanners, was treated by Johann Radon in 1917, and he also introduced this integral transform, that now bears his name, in arbitrary dimensions [**9**]. The Radon transform in $\mathbb{R}^n$ integrates over hyperplanes, and the case $n = 3$ was first considered by H.A. Lorentz before 1906, but it was never published (see [**4**, p. 51]).

In the early 1960s, Allan Cormack made the first CT (computerized tomography) scanner [**2**], and he developed mathematics to image objects from this X-ray data. His successful algorithm is based on a singular value decomposition, and it took a long time on the computers of that time. He received the Nobel Prize in 1979 for this seminal research. Subsequently, mathematicians, scientists, and engineers developed many fast algorithms for X-ray data, and they developed applications of the Radon transform to a broad range of tomography problems.

X-ray tomography is so useful, in general, because of an efficient, easy to implement, stable inversion method, filtered back projection, that gives excellent reconstructions, at least if complete data are given (the inversion formula and general remarks on implementation are given in [**8**, Section 2] and a detailed description of the algorithm is given in [**6**, Section 4]). The concept of complete data is understood using sampling theory, and this will be discussed in [**3**]. Loosely, complete data are tomographic data over a fairly evenly spaced set of lines through the object in a fairly equi-spaced set of directions. Even if filtered back projection is mathematically a simple and beautiful formula, the applied problems starts here! In particular, it is a simple fact, that no finite set of lines can determine the function f uniquely [**10**]. On the other hand, every real life CT apparatus can only use a finite number of lines, and the data are noisy and not exact. This brings up the connection to sampling theory and the importance of error estimates [**3**]. Also, physical limitations such as beam hardening (e.g., [**10**]) and other inaccuracies in the model require considerations that are not related to the Radon transform.

Many intriguing tomography problems involve limited tomographic data, when data over some lines are missing. One of the most important is the 3-D X-ray tomography problem discussed in [**6**]. Reconstruction from limited tomographic data is more difficult and less successful precisely because the missing data take away important information about the object, and in any case, the algorithms for complete data cannot be used. What is missing from the reconstruction can be analyzed in several ways, including microlocal analysis [**8**], singular value decompositions [**6**], and sampling theory [**3**], and this analysis is important since limited data problems occur in industry, science, and medicine.

This is not the whole story; many tomographic problems are modeled by Radon transforms that integrate in non-standard weights or over sets besides lines. Such transforms are called *generalized* Radon transforms. For example, the Radon transform of SPECT (single photon emission tomography) integrates over lines in a non-standard weight that depends on the object being scanned. One transform in sonar and geological testing integrates over circles or spheres, and several of the most important examples will be given in [**5**].

Many other tomographic problems are not directly modeled by Radon transforms, but they are tomographic because the goal is to find the internal structure of an object or the location of objects from indirect wave-based data such as in radar or seismic testing [**1**] or from electromagnetic data [**7**].

The first contribution, [**8**] by Todd Quinto, gives an introduction to the mathematics of X-ray tomography including a description of the range of the Radon transform and a basic inversion formula, filtered back projection. If $n = 2$, which is one important case in X-ray tomography, this inversion formula involves a non-local operator, which gives extra complications in the reconstruction of f. In many applications one does not need an exact reconstruction of f, instead one just needs to see the shape of structures in the object—the singularities of the object. Boundaries and imperfections in the object are singularities, so a reconstruction of singularities can show such structure. Furthermore, standard algorithms cannot be used in limited data problems. The author describes specific limited data problems in electron microscopy and industrial non-destructive evaluation, and he gives reconstructions. The theme of the article is that the microlocal analysis of the Radon transform can be used to understand what singularities of objects (wavefront set) are visible from limited tomographic data. The basic microlocal analysis is introduced, and then the author shows how these tomographic reconstructions reflect the microlocal understanding.

Alfred Louis discusses important algorithms in X-ray tomography, including limited angle tomography, a limited data problem that is important in industry, and 3-D cone-beam X-ray tomography, the tomography of many modern CT scanners [**6**]. He uses the approximate inverse to put these algorithms in the same general context. He develops a singular value decomposition for the limited angle transform, and he uses the approximate inverse to develop an inversion algorithm. Many modern CT scanners are so-called cone beam scanners; an X-ray source emits X-rays in a beam shaped like a three-dimensional cone. Typically, the source moves in a circle or a spiral around the object and the scanner generates a three-dimensional data set. The problem is more difficult geometrically and analytically. Louis describes the mathematics of 3-D cone beam tomography and he gives a mathematical description of his exact algorithm as well as reconstructions from real data.

In the third contribution to this volume [**3**], Adel Faridani applies sampling theory to X-ray tomography. He describes how many and which line integrals should be measured in order to achieve a desired resolution in the reconstructed image. This question for two-dimensional fan-beam tomography leads to a detailed discussion of problems in sampling theory on the torus, $\mathbb{T}^2$. The contribution provides an excellent example of the interplay between geometry, finite subgroups of the two-dimensional torus, the Shannon sampling theory and the Poisson summation formula. The focus is on the construction of efficient sampling schemes, the identification of algorithms for accurate reconstruction from efficiently sampled data, and the qualitative understanding of artifacts. The theory is based on the classification of all finite subgroups of $\mathbb{T}^2$ to give efficient sampling sets and secondly a non-convex set K such that the translates of K by a certain lattice are all disjoint. Numerical experiments with a simple mathematical phantom are used to show the efficiency of the construction as well as artifacts coming from undersampling. While direct reconstruction with the standard filtered backprojection algorithm is found

to be suboptimal, interpolating the data first to a denser lattice via the sampling theorem leads to good results. The last section contains several problems and ideas for those who are interested in research in this area.

Peter Kuchment's article [**5**] describes several of the most important examples of generalized Radon transforms in tomography. The Radon transform used in X-ray tomography is not weighted; the attenuation coefficient (or density) of the object is integrated over lines in the data set without any multiplicative weight factor in equation (1). However, some of the most interesting Radon transforms (so-called generalized Radon transforms) in tomography integrate over sets besides lines or in weights besides the canonical ones. For example, the Radon transform in single photon emission tomography (SPECT) integrates over lines but with a weight depending on the material on the line. The goal of SPECT is to detect the distribution of sources of radioactivity using this emission data. A broad range of beautiful pure and applied results about SPECT are described in this article. The Radon transform that comes up in thermoacoustic tomography, sonar, and geophysical testing involves integrals over circles and spheres, and much progress, including new inversion formulas and a better understanding of the underlying mathematics, has occurred recently. The author provides an overview of this progress as well as reconstructions for this problem. Finally, the author discusses a model of electrical impedance tomography that involves the geodesic Radon transform on the hyperbolic plane.

Peter Massopust's article [**7**] shows how tomographic methods are used in a specific important industrial problem, pipeline inspection. Detecting corrosion or fatigue in a pipe carrying natural gas or oil, particularly along distribution lines, can prevent natural disasters. Sensors are placed on a device that fits into the pipe and travels down the pipe. The device induces a magnetic field in the pipe wall that is measured. The author describes mathematics and physics of the problem including details of the pipeline inspection device, introduces a model for the measured data for oil pipelines, and describes the resulting inverse problem. He describes B-splines and wavelets and uses them to develop an inversion method and deals well with problems inherent in the data acquisition method, including detector sensitivity and noise. He describes limitations in the model and the data acquisition method, and he gives reconstructions.

The last article in this volume [**1**] is written by Liliana Borcea and describes applications of inverse problems in imaging in random media. The question is how to determine the location of sources for wave propagation or strong scatters buried in a cluttered random medium. These kinds of problems show up in reflection seismology, synthetic aperture radar imaging, interferometric radar imaging and inverse scattering of time harmonic acoustic or electromagnetic waves. The author describes the difference between homogeneous and random media and explains where classical methods break down in random media. The simplified model of a single point source buried in a finely layered medium is discussed in detail.

August 2005 Gestur Ólafsson and Eric Todd Quinto

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