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Mathematical Foundations of Information Flow

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Information Flow in Physics,
Geometry, Logic and Computation
March 12–15, 2008
Tulane University
New Orleans, Louisiana

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Preface

This volume contains papers from the 2008 Clifford Lectures. The Clifford Lectures is an annual series sponsored by the Tulane University Mathematics Department in honor of A. H. Clifford, the father of algebraic semigroup theory and a longtime member of the Tulane mathematics department. The 2008 Clifford Lectures were delivered by Samson Abramsky, with the theme of *Information Flow in Physics, Geometry, Logic and Computation* [**1**]. The Lectures included five talks by Professor Abramsky, as well as invited talks by twelve colleagues on topics ranging from mathematics, and in particular topology, to computer science, physics, classical and quantum information, systems biology and finite model theory. This broad range of topics was deliberate in the design of the lectures, the aim of which was to encourage collaboration among a group of researchers, all of whom were working on some aspect of information flow. Rather than comprising a proceedings of that meeting alone, this volume represents the culmination of a series of meetings on the same theme. Indeed, the 2008 Clifford Lectures provided the impetus for a series of meetings focused on information flow, and the authors of the papers in this volume have been participants in most of these meetings. The meetings included two Workshops on Informatic Phenomena held at Tulane in the fall of 2008 and 2009 [**5, 6**], a Seminar on the *Semantics of Information* held at Schloß Dagstuhl, the German International Meeting Center for Computer Science, in June 2010 [**3**], the 2011 Clifford Lectures [**2**] in March 2011 which featured talks by Dr. Christopher Fuchs (Perimeter Institute) on quantum information, and finally, the forthcoming *Seminar on Information Flow* [**4**], to be held again at Schloß Dagstuhl in August 2012. These meetings maintain the broad representation of topics of the initial 2008 Clifford Lectures, and several fruitful collaborations have sprung up among the researchers who participated in the meetings, some of which are represented in this volume. This volume represents a significant component of the research presented at the series of meetings just described.

While the theme of the 2008 Clifford Lectures was deliberately broad, the focus of the research presented in this volume is narrower. The principal theme represented in this volume is information flow in classical and quantum physics and its mathematical underpinnings. This is quite appropriate, since Professor Abramsky's research in this area was the impetus for the 2008 Clifford Lectures and the basis for casting the wide net of research interests featured at that and the subsequent meetings. More precisely, the focus of much of Professor Abramsky's recent research has been the application of ideas from theoretical computer science to develop a novel categorical formulation of quantum mechanics, as the basis of a new approach to quantum physics and quantum information. In keeping with this theme, all the papers in this volume focus on information flow in quantum and classical physics and its mathematical underpinnings.

The focus of the AMS *Proceedings* of Symposia in Applied Mathematics series is on the application of mathematics to other disciplines. In that spirit, the papers in this volume comprise a broad representation of applications of mathematics to quantum physics and to classical and quantum information: The research reported here utilizes category theory, domain theory, harmonic analysis, probability theory, Shannon information theory, as well as topology, as tools for modeling quantum physics and classical and quantum information.

The papers in this volume

We shall give a brief indication of the contents of the papers which appear in this volume.

(1) The paper by Abramsky and Heunen addresses the issue of extending the categorical quantum mechanics paradigm to the infinite-dimensional case. At the same time, it makes connections with some classic topics in operator algebras, notably the work by Ambrose on H^*-algebras and an infinite-dimensional extension of the Wedderburn structure theorem. It relates these to Frobenius algebras, which have been studied in categorical quantum mechanics as an algebraic way of capturing orthonormal bases and measurements. It also characterizes Frobenius algebras in various categories of relations.

(2) The paper by Barnum, Barrett, Leifer and Wilce considers which probability theories support teleportation. Previous work of the authors showed that phenomena associated to quantum mechanics such as no-cloning and no-broadcasting are generic in all non-classical probabilistic theories. On the other hand, teleportation is not supported in most such theories, leading the authors in the present paper to explore which probabilistic theories support this protocol. They isolate a natural class of composite systems which they term *regular* and establish necessary and sufficient conditions for a regular tripartite system to support a conclusive teleportation protocol. They also give sufficient conditions for deterministic teleportation, yielding a large supply of composite state spaces that are neither classical nor quantum, but that do support such a protocol

(3) The paper by Brandenburger, Friedenberg and Keisler looks at another fruitful source of ideas about modelling information flow, coming from game theory, and the interaction of rational agents. In particular, it looks at epistemic game theory, where there is an explicit formal representation of the belief states of the agents, in terms of type spaces. In this context, order-theoretic fixpoints play a prominent rôle, in contrast to the topological theorems of Brouwer and Kakutani, which are widely used in the study of Nash equilibria. Order-theoretic fixpoints are also widely used in theoretical computer science. An interesting point of difference is that the fixpoints used in epistemic game theory may come from non-monotonic functions; this leads to a number of interesting mathematical questions and results.

(4) The paper by Coecke and Edwards looks at the well-known ‘toy model’ of quantum mechanics developed by Rob Spekkens from the perspective

of categorical quantum mechanics. This model shows that many features held to be characteristic of quantum mechanics can be realized in an intriguingly simple model based on finite sets and relations. Previous work had shown that this model could be captured in an elegant fashion in the setting of categorical quantum mechanics. However, the constraints placed on the model by the 'knowledge balance principle' means that it is surprisingly difficult to give an explicit description of the full model, as opposed to an inductive construction. The present paper gives such a description, which can serve as a basis for further investigations.

(5) The paper by Hines and Scott uses the classical Sagnac interferometer as a thought experiment in single-photon linear optics, which leads to a general construction on Hilbert spaces. This construct has a close connection to constructions from algebraic and categorical program semantics, the so-called trace. The authors analyze their general construction in terms of a categorical trace which generalizes a 'particle-style' trace on Hilbert space they studied in an earlier paper. They show this general construction has a physical realization based on the thought experiment that motivated the work.

(6) The paper by Hofmann and Mislove has two aims. The first is to provide a self-contained, accessible account of some basic results in the theory of compact monoids and harmonic analysis, and to demonstrate how these results, when applied to the compact affine monoid of probability measures on a compact group, lead to Wendel's proof that such a group has a unique Haar measure. The second goal is to apply some of the same theory to analyze Shannon's classical information of discrete lossless noisy channels with finite inputs and outputs. Using domain theory as an additional tool, the main result generalizes work of Martin, Allwein and Moskowitz about the nature of Shannon capacity as a function on the family of binary channels.

(7) The paper by Martin introduces the notion of the *scope* of a unital quantum channel. Such a channel has a range of possible Shannon capacities for sending classical information, depending on the basis used to encode the information. The author calls this range the *scope* of the channel. He shows that, in the case of qubit channels, the scope is a compact interval, and he uses the algebraic structure of the family of channels as a monoid to characterize the scope of the channel, and how to calculate it. The author also presents an adaptive scheme for communication in which the participants can maximize the information transmitted after they first determine the state of the environment, for which a method is also presented. The author shows how this work can be applied in quantum cryptography to minimize the error rate over any time interval during which the environment remains stable.

(8) The paper by Martin and Panangaden is the second in which the authors have explored a partial order on the events in spacetime that is defined by the causal structure. In an earlier paper the authors used techniques from domain theory to show that the topology of globally hyperbolic spacetimes could be reconstructed from the causal structure. But the causal

structure determines the metric only up to a local rescaling (a conformal transformation); in a four-dimensional spacetime, the metric tensor has ten components, and thus effectively only nine are determined by the causal structure. To remedy this deficiency, the authors again apply domain theory. They first establish the relationship between measurement in domain theory, the concept of a global time function and the Lorentz distance. Then they are able to domain-theoretically recover the final tenth component of the metric tensor, thereby obtaining causal reconstruction of not only the topology of spacetime, but also its geometry.

(9) Finally, the paper by Pavlovic continues the theme of categorical models of quantum computation by considering how to identify classical data in a quantum computing setting. It is shown that polynomial extensions of (dagger-)monoidal closed categories capture exactly the classical data and admissible operations thereon, namely, copying, deleting and abstraction. A running example of Simon's algorithm is used to illustrate the results.

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Finally, thanks are due for the financial support that agencies provided in support of the meetings mentioned above. First and foremost, we thank the US Office of Naval Research, and especially Dr. Ralph Wachter, whose research program provided generous support to help underwrite the 2008 Clifford Lectures, as well as the other meetings listed above. Dr. Keye Martin (NRL) also is owed a debt of thanks for providing the funds to support the Workshops on Informatics Phenomena held in 2008 and 2009, and for providing support for the participants to attend the Seminar on the Semantics of Information at Schloß Dagstuhl in June 2010. The U.K. Engineering and Physical Sciences Research Council have supported Professor Abramsky's research through a Senior Fellowship.

Samson Abramsky
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December 2011

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