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Finite Frame Theory
A Complete Introduction to Overcompleteness

AMS Short Course
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January 8–9, 2015
San Antonio, Texas

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Editor

American Mathematical Society

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LECTURE NOTES PREPARED FOR THE AMERICAN MATHEMATICAL SOCIETY SHORT COURSE ON FINITE FRAME THEORY: A COMPLETE INTRODUCTION TO OVERCOMPLETENESS HELD IN SAN ANTONIO, TEXAS, JANUARY 8–9, 2015

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10 9 8 7 6 5 4 3 2 1 16 15 14 13 12 11

Contents

Preface

The formula used to define Fourier frames is written explicitly by Paley and Wiener, [**10**] page 115, in the context of one of their fundamental theorems dealing with non-harmonic Fourier series. Fledgling forms of Fourier frames go back to Dini (1880) and then G. D. Birkhoff (1917), and leap to the profound results of Beurling and H. J. Landau in the 1960s, [**2**], [**9**]. Fourier frames lead naturally to non-uniform sampling formulas, and this is far from a complete story, e.g., [**1**].

Notwithstanding the importance of Fourier frames, and research such as that by Paley, Wiener, Beurling, and Landau, frames are with us today because of the celebrated work of Duffin and Schaeffer [**5**] in 1952. They explicitly found and featured the mathematical power of Fourier frames, and did the *right thing* mathematically by formulating such frames for Hilbert spaces, extracting central features of frames such as the decomposition of functions in terms of frames and understanding the role of overcomplete systems such as frames as opposed to orthonormal bases. It should also be pointed out that parallel to this development, a major analysis of bases was under way by the likes of Bari and Köthe; and their results could be rewritten in terms of frames, see, e.g., [**11**].

And then wavelet theory came along! More precisely, with regard to frames, there was the important work of Daubechies, Grossmann, and Meyer (1986) [**4**]; and there was the subsequent wonderful mix of mathematics and engineering and physics providing new insights as regards the value of frames. We now understood a basic role for frames with regard to noise reduction, stable decompositions, and robust representations of signals. In retrospect, frame research in the 1990s, besides its emerging prominence in wavelet theory and time-frequency analysis and their applications, was an *analytic* incubator, with all the accompanying excitement, that led to *finite* frames!

By the late 1990s and continuing today as an expanding mysterious universe, finite frame theory has become a dominant, intricate, relevant, and vital field. There were specific topics such as frame potential energy theory, $\Sigma - \Delta$ quantization, quantum detection, and periodic approximants in ambiguity function behavior, all with important applications. This has brought to bear a whole new vista of advanced technologies to understand frames and to unify ideas. The power of harmonic analysis and engineering brilliance are still part and parcel of frames, whether finite or not, but now we also use geometry and algebraic geometry, combinatorics, number theory, representation theory, and advanced linear and abstract algebra. There are major influences from compressive sampling, graph theory, and finite uncertainty principle inequalities.

The time was right just a few years ago *to stop and smell the roses*, and the volume on finite frames, edited by Casazza and Kutyniok [**3**] appeared (2013).

Amazingly and not surprisingly, given the talent pool of researchers, the intrigue and intricacies of the problems, and the applicability of the subject, the time is *still* right. Kasso Okoudjou's 2015 AMS Short Course on Finite Frame Theory was perfectly conceived. He assembled the leading experts in the field, not least of whom in my opinion was Okoudjou himself, to explain the latest and deepest results. This book is the best step possible towards the future. Enjoy!

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Introduction

The present volume is the proceedings of the AMS Short Course on Finite Frame Theory: A Complete Introduction to Overcompleteness organized in San-Antonio, January 8 & 9, 2015, prior to the Joint Mathematical Meetings.

Hilbert space frames have traditionally been used to decompose, process and reconstruct signals/images. In this volume we shall focus on frames for finite dimensional Euclidean spaces. In this setting, a set of vectors $\{f_k\}_{k=1}^M \subset \mathbb{K}^N$ is a frame for $\mathbb{K}^N$ if and only if there exist $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_{k=1}^{M} |\langle x, f_k\rangle|^2 \leq B\|x\|^2$$

for each $x \in \mathbb{K}^N$, where $\mathbb{K} = \mathbb{R}$, or $\mathbb{K} = \mathbb{C}$. We refer to Section 4 of Chapter 1 for more details on frames. Today, frame theory is an exciting, dynamic subject with applications to pure mathematics, applied mathematics, engineering, medicine, computer science, and quantum computing. From a mathematical perspective, frame theory is at the intersection of many fields such as functional and harmonic analysis, numerical analysis, matrix theory, numerical linear algebra, algebraic and differential geometry, probability, statistics, and convex geometry. Problems in frame design arising in applications often present fundamental, completely new challenges never before encountered in mathematics. We refer to [**3**, **6**–**8**] for more on frame theory and its applications.

Finite unit norm tight frames (FUNTFs) are one of the most fundamental objects in frame theory with several applications. For example, in communication theory, FUNTFs are the optimal frames amongst unit norm frames to defeat additive white Gaussian noise, and amongst tight frames to defeat one erasure. Additionally, FUNTFs with minimal worst-case coherence are often optimal codes for synchronous multiple access communication channels, and equiangular FUNTFs can be used to embed fingerprints into media content to defeat piracy and common forgery attacks. Finally, compressed sensing matrices tend to (nearly) have FUNTF structure, and conversely, FUNTFs with small worst-case coherence typically perform well as compressed sensing matrices.

Perhaps the most beautiful theory behind FUNTFs concerns the *frame potential*. This quantifies the potential energy in a collection of particles on the sphere corresponding to a force between the particles which encourages pairwise orthogonality; while this *frame force* doesn't naturally arise from physics, the formulation still leads to a satisfying physical intuition. It's easy to show that FUNTFs are precisely the configurations which minimize the frame potential, and it turns out that all local minimizers are also global, meaning FUNTFs can be visualized as the steady state of a dynamical system governed by the frame force. In three dimensions, minimizers of the frame potential form the vertices of recognizable figures such as Platonic solids and the soccer ball.

To illustrate the many faces of frame theory, we observe that from a geometric point of view, the FUNTFs lie in the intersection of a product of spheres and a scaled Stiefel manifold. These restrictions can be viewed as a system of multivariate quadratic equations, making this space a quadratic algebraic variety. Though several basic and long standing problems have been tackled using algebro-geometric techniques, our understanding of these spaces remains incomplete. For example, the classification of the singular points of the FUNTF varieties remains one of the big open questions in this area. By contrast, the nonsingular points can be characterized as FUNTFs which admit a nontrivial decomposition into two mutually orthogonal collections. These "orthodecomposable" frames frequently present obstructions to nice theoretical arguments in finite frame theory. For example, the characterization of the local geometry of the FUNTF space near these points remains also an open and challenging question in the area. Nonetheless, algebro-geometric methods are increasingly being used to better understand frame theoretical questions.

Despite the promises and the power of the recently developed algebro-geometric methods for constructing FUNTFs, many questions in the area remain unsolved. For example, no effective method is known to construct FUNTFs when additional constraints on the frame vectors are imposed, e.g., the construction of equiangular FUNTFs (though some convex optimization techniques have been proposed). Furthermore, it seems desirable to have generic methods that would allow one to ideally transform a frame into a tight (or "nearly tight") one. These methods will be analogs of preconditioning methods prevalent in numerical linear algebra. Recently, techniques from convex geometry have been used to describe a class of frames called *scalable frames* which have the property that their frame vectors can be rescaled to result in tight frames.

Frames are intrinsically defined through their spanning properties. However, in real euclidean spaces, they can also be viewed as distributions of point masses. This point of view is also partially justified by the frame potential described above. In this context, the notion of *probabilistic frames* was introduced as a class of probability measures with finite second moment and whose support spans the entire space. This notion is a special case of continuous frames for Hilbert spaces that has applications in quantum computing. In this framework, probabilistic tools related to the Wasserstein metric can be appealed to in order to investigate questions in frame theory.

One of the fundamental application areas of frame theory remains modern signal processing. In this context, frame expansions and dual frames allow one to reconstruct a signal from its frame coefficients — the use of redundant sets of vectors ensures that this process is robust against noise and other forms of data loss. Although frame expansions provide discrete signal decompositions, the frame coefficients generally take on a continuous range of values and must also undergo a lossy step to discretize their amplitudes so that they become amenable to digital processing and storage. This analog-to-digital conversion step is known as quantization. A very well-developed theory of quantization based on finite frame theory is now part of the applied mathematics infrastructure.

An emerging topic in applied harmonic analysis is the question of nonlinear (signal) reconstruction. For example, frame design for phaseless reconstruction belongs to this class of problems. The problem of phaseless reconstruction can be simply stated as follows. Given the magnitudes of the coefficients of an output of a linear redundant system (frame), we want to reconstruct the unknown input. This problem has first occurred in X-ray crystallography starting from the early $20th$ century. In 1985 the Nobel prize in chemistry was awarded to Herbert Hauptman (a mathematician) for his contributions to the development of X-ray crystallography. The same nonlinear reconstruction problem shows up in speech processing, particularly in speech recognition. An age-old problem is about the importance of phase in signal processing and whether the magnitude of short-time Fourier transform encodes enough information to allow essentially unique reconstruction of the input signal. Generically, frame theory provides a unifying language to state and solve the problem.

The broader question of nonlinear signal analysis has also been investigated in the context of the new field of compressed sensing that arose as a response to inefficient traditional signal acquisition schemes. For example, assuming that the signal of interest is sparse (with respect to some fixed orthonormal basis), a typical problem is to algorithmically reconstruct this signal from a small number of linear measurements. During the last decade some deep results have been obtained in compressed sensing. However, in a number of applications, the signal of interest is sparse with respect to a (redundant) tight frame and many of the traditional compressed sensing methods are not applicable. A general theory of compressed sensing for signals that are sparse in tight frames has now emerged. Another problem related to compress sensing is the dictionary learning problem which consists of finding sparse and interpretable signal representations which are then used in applications such as compression, denoising, and super-resolution. In this context a trade-off is often made between analytic dictionaries which are backed by a very rich theoretical foundation, and the data-dependent dictionaries which are more flexible and are application based.

The seven chapters in this volume will cover most of the topics discussed above. In particular, in Chapter 1 Casazza and Lynch give an introduction to (finite dimensional) Hilbert space frame theory and its applications, and list a number of open problems. In chapter 2 Mixon gives a motivated account of the study of FUNTFs in finite dimensional spaces focusing on topics such as the frame potential, eigensteps, and equiangular tight frames. In Chapter 3, Strawn starts from a series of examples and explores algebraic varieties of FUNTFs leading to an exposition of the algebraic and differential geometry roots of certain of the known methods for constructing FUNTFs. In Chapter 4, Okoudjou through a series of motivating examples surveys certain preconditioning techniques recently introduced in frame theory and gives an account of a probabilistic interpretation of finite frame theory. In Chapter 5, Dunkel, Powell, Spaeth, and Yilmaz provide an expository introduction to the theory and practice of quantization for finite frame coefficients. In particular, the chapter focuses on memoryless scalar quantization (MSQ), the uniform noise model, first order Sigma-Delta ($\Sigma\Delta$) quantization, and the role of error diffusion in quantization. In Chapter 6, Balan reviews existing analytical results as well as algorithms for signal recovery for the phaseless reconstruction problem. In Chapter 7, Chen and Needell give an overview of recent results extending the theory of compressed sensing to the case of sparsity in tight frames, and describe another application of frame theory in the context of dictionary learning.

We would like to thank Sivaram Narayan for soliciting a proposal to organize the AMS 2015 Short Course. We also thank all the lecturers not only for the outstanding talks they gave, but also for their contributions to the present volume. Our gratitudes also go to the AMS staff for their help with the organizational aspects of the course. Finally, we would like to thank the reviewers for their useful comments which helped improve this volume.

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Index

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73

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