

Proceedings of Symposia in
APPLIED MATHEMATICS

Volume 74

**Rigorous Numerics
in Dynamics**

AMS Short Course
Rigorous Numerics in Dynamics
January 4–5, 2016
Seattle, Washington

Jan Bouwe van den Berg
Jean-Philippe Lessard
Editors

Rigorous Numerics in Dynamics

Proceedings of Symposia in
APPLIED MATHEMATICS

Volume 74

**Rigorous Numerics
in Dynamics**

AMS Short Course
Rigorous Numerics in Dynamics
January 4–5, 2016
Seattle, Washington

Jan Bouwe van den Berg
Jean-Philippe Lessard
Editors



AMERICAN
MATHEMATICAL
SOCIETY

Providence, Rhode Island

Editorial Board

Suncica Canic Tasso J. Kaper (Chair) Daniel Rockmore

LECTURE NOTES PREPARED FOR THE AMERICAN MATHEMATICAL
SOCIETY SHORT COURSE ON RIGOROUS NUMERICS IN DYNAMICS
HELD IN
SEATTLE, WASHINGTON, JANUARY 4–5, 2016

The AMS Short Course Series is sponsored by the Society's Program Committee for National Meetings. The series is under the direction of the Short Course Subcommittee of the Program Committee for National Meetings.

2010 *Mathematics Subject Classification*. Primary 37M99, 65G99, 37B30, 65L99, 35B32, 65N35, 34K13.

Library of Congress Cataloging-in-Publication Data

Names: Berg, Jan Bouwe van den, 1973- editor. | Lessard, Jean-Philippe, 1978, editor.

Title: Rigorous numerics in dynamics : AMS Short Course, Rigorous Numerics in Dynamics, January 4-5, 2016, Seattle, Washington / Jan Bouwe van den Berg, Jean-Philippe Lessard, editors.

Description: Providence, Rhode Island : American Mathematical Society, [2018] | Series: Proceedings of symposia in applied mathematics ; volume 74 | Includes bibliographical references and index.

Identifiers: LCCN 2017043850 | ISBN 9781470428143 (alk. paper)

Subjects: LCSH: Nonlinear mechanics. | Topological dynamics. | AMS: Dynamical systems and ergodic theory – Approximation methods and numerical treatment of dynamical systems – None of the above, but in this section. msc | Numerical analysis – Error analysis and interval analysis – None of the above, but in this section. msc | Dynamical systems and ergodic theory – Topological dynamics – Index theory, Morse-Conley indices. msc | Numerical analysis – Ordinary differential equations – None of the above, but in this section. msc | Partial differential equations – Qualitative properties of solutions – Bifurcation. msc | Numerical analysis – Partial differential equations, boundary value problems – Spectral, collocation and related methods. msc | Ordinary differential equations – Functional-differential and differential-difference equations – Periodic solutions. msc

Classification: LCC QA321.5 .R54 2018 | DDC 515/.39–dc23

LC record available at <https://lcn.loc.gov/2017043850>

DOI: <http://dx.doi.org/10.1090/psapm/074/00645>

Copying and reprinting. Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy select pages for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication is permitted only under license from the American Mathematical Society. Requests for permission to reuse portions of AMS publication content are handled by the Copyright Clearance Center. For more information, please visit www.ams.org/publications/pubpermissions.

Send requests for translation rights and licensed reprints to reprint-permission@ams.org.

© 2018 by the American Mathematical Society. All rights reserved.

The American Mathematical Society retains all rights
except those granted to the United States Government.

Printed in the United States of America.

∞ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.

Visit the AMS home page at <http://www.ams.org/>

10 9 8 7 6 5 4 3 2 1 23 22 21 20 19 18

Contents

Preface	vii
Introduction to rigorous numerics in dynamics: General functional analytic setup and an example that forces chaos JAN BOUWE VAN DEN BERG	1
Validated numerics for equilibria of analytic vector fields: Invariant manifolds and connecting orbits J.D. MIRELES JAMES	27
Continuation of solutions and studying delay differential equations via rigorous numerics JEAN-PHILIPPE LESSARD	81
Computer-assisted bifurcation diagram validation and applications in materials science THOMAS WANNER	123
Dynamics and chaos for maps and the Conley index SARAH DAY	175
Rigorous computational dynamics in the context of unknown nonlinearities KONSTANTIN MISCHAIKOW	195

Preface

Nonlinear dynamics shape the world around us. It shapes biology, from the electrophysiological properties of neurons, via the spiralling waves in contracting heart muscles, to gene regulatory networks. It shapes physics, from the swirling motions in fluid flows, via the creation of complex patterns in materials, to the harmonious motions of celestial bodies. It shapes chemistry, from the rich reaction kinetics phenomena, via the chemical basis of morphogenesis at the origin of patterns on animals, to the complicated biochemistry in the living cell. Mathematically, these beautiful and highly complicated phenomena can be described by nonlinear dynamical systems in the form of ordinary differential equations (ODEs), partial differential equations (PDEs), delay differential equations (DDEs), finite and infinite dimensional maps. Characterizing quantitatively (or even only qualitatively) the solutions of these models presents severe complications. While some difficulties are partially explained by the presence of nonlinearities in the models, the challenges are even greater for PDEs and DDEs, which are naturally defined on infinite dimensional function spaces. The recent availability of high-performance computers and sophisticated softwares made numerical simulations the primary tool used by scientists to study the complicated dynamics arising from the equations. However, the ubiquitous presence of computational methods in modern day science naturally leads to the fundamental question of validity of the outputs. Even for finite dimensional ODEs, this question inevitably arises if the system under study is chaotic, as small differences in initial conditions (such as those due to rounding errors in numerical computations) yield widely diverging outcomes. To address this fundamental issue in the context of nonlinear dynamics, the recent field of rigorous numerics emerged at the intersection of several fields of pure and applied mathematics. This field draws inspiration from the ideas in scientific computing, nonlinear analysis, numerical analysis, algebraic topology, functional analysis and approximation theory. In a nutshell, rigorous computations are mathematical theorems formulated in such a way that the assumptions can be rigorously verified on a computer. In this book, we introduce novel and exciting techniques within the field of rigorous numerics to study finite and infinite dimensional discrete and continuous dynamical systems.

The book begins with the first introductory chapter by Jan Bouwe van den Berg entitled *Introduction to rigorous numerics in dynamics: general functional analytic setup and an example that forces chaos*. In this chapter, we introduce some basic concepts of rigorous computing in a dynamical systems context, and as first application, we demonstrate how topological braids can be combined with rigorous numerics to prove the existence of chaos in ODEs. The second chapter *Validated*

numerics for equilibria of analytic vector fields: invariant manifolds and connecting orbits by J.D. Mireles James introduces a rigorous computational approach to compute rigorously stable and unstable manifolds of steady states of ODEs using the parameterization method and Taylor expansions. Once the local information from the invariants manifold is obtained, a projected boundary value problem is solved yielding computer-assisted proofs of existence of connecting orbits. The third chapter *Continuation of solutions and studying delay differential equations via rigorous numerics* by Jean-Philippe Lessard demonstrates how rigorous numerics helps studying the dynamics of delay equations. We present a rigorous continuation method for solutions of finite and infinite dimensional parameter dependent problems, which is then applied to compute branches of periodic solutions of two DDEs. In the fourth chapter *Computer-assisted bifurcation diagram validation and applications in materials science* by Thomas Wanner, it is demonstrated how rigorous computational techniques can be used to validate bifurcation diagrams, both in finite- and certain infinite-dimensional problems. We focus particularly on the verification of branches, as well as on saddle-node and symmetry-breaking pitchfork bifurcations. The method is applied to a finite dimensional discrete dynamical systems and to a PDE. In the fifth chapter *Dynamics and chaos for maps and the Conley index* by Sarah Day, we examine methods for constructing outer approximations, finite representations of discrete-time models that are amenable to computational studies and computer-assisted proofs. These methods rely heavily on Conley index theory, an algebraic topological generalization of Morse Theory. In the final and sixth chapter *Rigorous computational dynamics in the context of unknown nonlinearities* by Konstantin Mischaikow, we provide a brief introduction to a novel approach to dynamics whose aim is to provide, in a computationally tractable manner, an accurate description of the dynamics of models without precise nonlinearities. We characterize the dynamics using the language of combinatorics and algebraic topology.

This volume is the proceedings of the AMS Short Course on Rigorous Numerics in Dynamics, organized for the Joint Mathematics Meetings in Seattle on January 4–5, 2016.

SELECTED PUBLISHED TITLES IN THIS SERIES

- 74 **Jan Bouwe van den Berg and Jean-Philippe Lessard, Editors**, Rigorous Numerics in Dynamics, 2018
- 73 **Kasso A. Okoudjou, Editor**, Finite Frame Theory, 2016
- 72 **Van H. Vu, Editor**, Modern Aspects of Random Matrix Theory, 2014
- 71 **Samson Abramsky and Michael Mislove, Editors**, Mathematical Foundations of Information Flow, 2012
- 70 **Afra Zomorodian, Editor**, Advances in Applied and Computational Topology, 2012
- 69 **Karl Sigmund, Editor**, Evolutionary Game Dynamics, 2011
- 68 **Samuel J. Lomonaco, Jr., Editor**, Quantum Information Science and Its Contributions to Mathematics, 2010
- 67 **Eitan Tadmor, Jian-Guo Liu, and Athanasios E. Tzavaras, Editors**, Hyperbolic Problems: Theory, Numerics and Applications, 2009
- 66 **Dorothy Buck and Erica Flapan, Editors**, Applications of Knot Theory, 2009
- 65 **L. L. Bonilla, A. Carpio, J. M. Vega, and S. Venakides, Editors**, Recent Advances in Nonlinear Partial Differential Equations and Applications, 2007
- 64 **Reinhard C. Laubenbacher, Editor**, Modeling and Simulation of Biological Networks, 2007
- 63 **Gestur Ólafsson and Eric Todd Quinto, Editors**, The Radon Transform, Inverse Problems, and Tomography, 2006
- 62 **Paul Garrett and Daniel Lieman, Editors**, Public-Key Cryptography, 2005
- 61 **Serkan Hoşten, Jon Lee, and Rekha R. Thomas, Editors**, Trends in Optimization, 2004
- 60 **Susan G. Williams, Editor**, Symbolic Dynamics and its Applications, 2004
- 59 **James Sneyd, Editor**, An Introduction to Mathematical Modeling in Physiology, Cell Biology, and Immunology, 2002
- 58 **Samuel J. Lomonaco, Jr., Editor**, Quantum Computation, 2002
- 57 **David C. Heath and Glen Swindle, Editors**, Introduction to Mathematical Finance, 1999
- 56 **Jane Cronin and Robert E. O'Malley, Jr., Editors**, Analyzing Multiscale Phenomena Using Singular Perturbation Methods, 1999
- 55 **Frederick Hoffman, Editor**, Mathematical Aspects of Artificial Intelligence, 1998
- 54 **Renato Spigler and Stephanos Venakides, Editors**, Recent Advances in Partial Differential Equations, Venice 1996, 1998
- 53 **David A. Cox and Bernd Sturmfels, Editors**, Applications of Computational Algebraic Geometry, 1998
- 52 **V. Mandrekar and P. R. Masani, Editors**, Proceedings of the Norbert Wiener Centenary Congress, 1994, 1997
- 51 **Louis H. Kauffman, Editor**, The Interface of Knots and Physics, 1996
- 50 **Robert Calderbank, Editor**, Different Aspects of Coding Theory, 1995
- 49 **Robert L. Devaney, Editor**, Complex Dynamical Systems: The Mathematics Behind the Mandelbrot and Julia Sets, 1994
- 48 **Walter Gautschi, Editor**, Mathematics of Computation 1943–1993: A Half-Century of Computational Mathematics, 1995
- 47 **Ingrid Daubechies, Editor**, Different Perspectives on Wavelets, 1993
- 46 **Stefan A. Burr, Editor**, The Unreasonable Effectiveness of Number Theory, 1992
- 45 **De Witt L. Sumners, Editor**, New Scientific Applications of Geometry and Topology, 1992

For a complete list of titles in this series, visit the
AMS Bookstore at www.ams.org/bookstore/psapmseries/.

ISBN 978-1-4704-2814-3



9 781470 428143

PSAPM/74

