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**Sum of Squares: Theory
and Applications**

AMS Short Course

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January 14–15, 2019

Baltimore, Maryland

Pablo A. Parrilo

Rekha R. Thomas

Editors

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Preface

This book is a compilation of the lecture notes from the 2019 AMS Short Course on the theory and applications of sum of squares polynomials. Over the past two decades, the theory of nonnegative and sums of squares polynomials has become relevant to several areas of mathematics and related fields. The aim of this book is to showcase some of the recent developments in the field. The introductory chapter gives a brief overview of sums of squares and their connections to real algebraic geometry and polynomial optimization, and also places all chapters in context. The remaining five chapters explore different aspects of sums of squares. The discussion begins with the geometry of spectrahedra, which are the feasible regions of semidefinite programs. This ties in with recent results on expressing convex sets as projections of spectrahedra, a topic further explored in the chapter on lifts of convex sets. The next chapter explains recent generalizations, using classical methods in algebraic geometry, of Hilbert's 1888 theorem cataloging the situations under which all nonnegative polynomials are sums of squares. This is followed by a chapter on recent advances in theoretical computer science that have been obtained by viewing sums of squares as a meta-algorithm for many standard computational tasks. Since sums of squares have found applications in a wide array of practical areas, the book ends with a survey of several examples from engineering, statistics, and operations research.

Pablo A. Parrilo

Rekha R. Thomas

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This volume is based on lectures delivered at the 2019 AMS Short Course “Sum of Squares: Theory and Applications”, held January 14–15, 2019, in Baltimore, Maryland.

This book provides a concise state-of-the-art overview of the theory and applications of polynomials that are sums of squares. This is an exciting and timely topic, with rich connections to many areas of mathematics, including polynomial and semidefinite optimization, real and convex algebraic geometry, and theoretical computer science.

The six chapters introduce and survey recent developments in this area; specific topics include the algebraic and geometric aspects of sums of squares and spectrahedra, lifted representations of convex sets, and the algorithmic and computational implications of viewing sums of squares as a meta algorithm. The book also showcases practical applications of the techniques across a variety of areas, including control theory, statistics, finance and machine learning.

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