Convexity

Volume VII

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SYMPOSIA IN
PURE MATHEMATICS

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PREFACE

Of the thirty-two papers in this volume, seventeen were presented at the Symposium on Convexity and the others were submitted later. (Symposium speakers were Besicovitch, Coxeter, Danzer, Davis, Day, Dvoretzky, Fan, Gale, Grünbaum, Hammer, Hoffman, Karlin, Klee, Motzkin, Phelps, Pták, Schaefer, and Valentine.) The thirty-third "paper" included here is a report on unsolved problems, based on the Symposium's session devoted to them, on informal discussions during the Symposium, and on later communications from the participants.

The papers are arranged alphabetically by author, since this seems most convenient for reference purposes. Interrelationships of the various papers, and their relation to the theory as a whole, are discussed in the Introduction. Since some of the individual bibliographies were so long and in such a state of flux, a common list of references did not seem feasible. However, the Author Index (in conjunction with the individual bibliographies) should be a fair substitute for such a list, and also makes it easy to learn which of the thirty-three papers cite the work of a given author. There are also a Subject Index and an Index of Unsolved Problems.

The editor is indebted to Professors Gale and Grünbaum for their assistance in planning the Symposium, to Dr. Pták and Professors Besicovitch, Coxeter, Day, Fan, and Motzkin for presiding at Symposium sessions, and to Dr. Danzer and Professors Besicovitch, Corson, Firey, Grünbaum, McMinn, and Motzkin for refereeing some of the papers. In particular, the advice and assistance of Branko Grünbaum have been invaluable.

The details of publication have been capably handled by Miss Ellen Swanson, Head of the American Mathematical Society's Editorial Department.

Victor Klee

INTRODUCTION

The systematic study of convex sets was initiated by H. Brunn and H. Minkowski. For most of the important notions in the field, at least a germ can be found in the latter's collected works (1911). Not only does the theory of convexity play a central role in Minkowski's geometry of numbers, but it also shares some of the nontechnical aspects of elementary number theory. Its basic notions are simple, natural, and of strong intuitive appeal. The subject is primarily one of ideas rather than machinery, and does not lend itself readily to unified treatment. It abounds in attractive special problems, and many mathematicians working mainly in other fields have published one or two papers on convexity. These aspects have accounted for the rapid but disorganized growth of the theory.

The 1934 survey by T. Bonnesen and W. Fenchel was an excellent summary of a large body of material, and is still a standard source of information in the field. Though selective in coverage, they cited more than 450 references; a current survey of the same degree of completeness would be a tremendous undertaking, probably not feasible. More than half of their book emphasized various quantitative notions such as diameter, area, volume and mixed volumes. Since 1934 these same notions have continued to play an important role. However, more striking (since less predictable) has been the intensive development of several qualitative aspects of the theory, including the combinatorial geometry associated with intersection and covering properties, the refinement and application (especially in functional analysis and game theory) of such notions as extremal structure and separation properties, the study of convexity in infinite-dimensional spaces, increasing use of convexity as a descriptive tool, and the evolution of various analogues and generalizations of convexity.

Though several quantitative investigations are included here, the Symposium was intended primarily to emphasize the more qualitative aspects of the theory. In particular, the five aspects listed above are all represented in the present volume. Among the unavoidable omissions, two are especially regretted by the editor. There is nothing here about the geometry of convex surfaces and the associated development of metric methods in differential geometry, carried out by A. D. Aleksandrov and his students in the Soviet Union and in this country by H. Busemann. Also omitted are the important results on infinite-dimensional simplexes, boundaries and extremal structure which have been developed in the past few years by G. Choquet and others.

In addition to the wide range of topics treated here, there is much variety of approach. Some of the shorter papers treat a single problem in full detail, while at the other extreme are several long papers which include very few proofs but survey broad areas in the field of convexity.

* * * * * * * * * * * *

Four of the papers are set in the Euclidean plane E^2. **BESICOVITCH**'s first paper gives a short proof of the known fact that a set of given constant width

has minimum area when it is a Reuleaux triangle. His second paper solves affiirmatively a special case of the following problem: Must a set of constant width w contain a semicircle of diameter w? **DANZER** gives a short proof of the known result that if C is a closed convex curve in E^2 which does not contain exactly three vertices of any rectangle, then C is a circle. In his first paper, **DAVIS** characterizes rectangles by means of an extremal area property involving inscribed crosses and also discusses a related conjecture of Ungar on extremal perimeters.

HAMMER's first paper is set in an arbitrary Minkowski plane where by the use of outwardly simple line families he is able to give an analytic representation for all convex curves of constant Minkowski width. He also summarizes his earlier work on diametral lines and associated convex bodies.

BESICOVITCH's third paper discusses Coxeter's problem of finding the smallest cage (edges of a convex polyhedron) which will hold a unit-sphere in E^3 without permitting it to escape. His other two papers give new proofs of known results concerning smoothness properties of a convex body K in E^3 and concerning directions of line segments in the boundary of K. In **GALE**'s first paper he uses the Borsuk-Ulam mapping theorem (involving antipodal points) to prove that if a convex body of width w' in E^n is obtained from one of width w by means of a homeomorphism which decreases distances, then $w' \leqq w$.

COXETER proposes an exact upper bound for the number of equal non-overlapping spheres in E^n that can touch another of the same size. The difficulty of this problem is indicated by the following quotation: "$\cdots$ Can a rigid material sphere be brought into contact with 13 other such spheres of the same size? Gregory said 'Yes' and Newton said 'No', but 180 years were to elapse before a conclusive answer was given." His historical survey of the problem in E^n extends from a paper by Kepler in 1611 to the latest published works. The problem is treated as the case $\phi = \pi/6$ of the problem of packing $(n-2)$-spheres of angular radius ϕ on an $(n-1)$-sphere, and the proposed upper bound is attained when the $(n-2)$-spheres are inscribed in the cells of a regular polytope $\{p, 3, \cdots, 3\}$. Though the bound is not fully established, much supporting evidence is given. Some related material is also discussed, such as the growth of the number of spheres as $n \to \infty$ and the known results for other values of ϕ.

PORITSKY treats a system of linear inequalities of the form $x_1 f_1(\theta) + \cdots + x_n f_n(\theta) \leqq g(\theta)$, where g and the f_i's are real analytic functions of the real variable θ ranging over a bounded or unbounded interval I. He studies the convex region consisting of all points $x = (x_1, \cdots, x_n) \in E^n$ which satisfy the given system of inequalities (for all $\theta \in I$), and is especially concerned with describing the region's boundary in terms of the envelope curve C and its tangent and osculating flats of various dimensions, where C is the set of all points x such that for some $\theta \in I$, $\sum_{i=1}^{n} x_i f_i^{(j)}(\theta) = g^{(j)}(\theta)$ for $0 \leqq j \leqq n-1$ ($^{(j)}$ indicating the jth derivative).

DVORETZKY reviews his earlier results on near-sphericity in E^n, one of which asserts that for each $\varepsilon \in]0, 1[$ and each positive integer k there exists $N(k, \varepsilon)$ such that every convex body of dimension $\geqq N(k, \varepsilon)$ admits a k-dimen-

sional section which is spherical to within ε. He derives new corollaries, including some on orthogonal projections, and discusses some open problems.

Two papers treat the facial structure of convex polyhedra. **GALE**'s second paper is concerned with cyclic polytopes in R^{2m}, these being convex polyhedra which are combinatorially equivalent to the convex hull of an n-pointed subset of the moment curve $\{(t, t^2, \cdots, t^{2m}): t \in R\}$. They have the remarkable property of being m-neighborly in the sense that each m vertices determine a face. He computes the number of $(2m-1)$-dimensional faces of such a polytope and this is conjectured to be the maximum attainable for convex polyhedra in R^{2m} which have n vertices. Certain neighborly polytopes are proved to be cyclic, and regular cyclic polytopes are constructed in E^{2m}. **GRÜNBAUM AND MOTZKIN** call an abstract graph k-polyhedral provided it is isomorphic with the graph formed by the edges and vertices of a k-dimensional convex polyhedron. They prove that each k-polyhedral graph contains as subgraph a refinement of C_{k+1}, the complete graph with $k+1$ nodes. As Gale's result shows, the graph C_{k+1} is j-polyhedral whenever $4 \leqq j \leqq k$; however, this and other sorts of ambiguity are excluded for graphs which are 2-polyhedral or 3-polyhedral.

VALENTINE deals mainly with known results on the intersection properties of convex sets. He obtains refinements and new proofs for many of these, his aim being to show what can be accomplished by systematic exploitation of dual cones. His viewpoint is well expressed by the following quotation: "$\cdots$ since it is a rare coincidence for the proofs of a theorem and its dual to be of equal difficulty, there is a double reason to investigate the dual. One may gain either a simpler proof or a less obvious theorem."

Five of the papers are expository surveys of a sort which should be valuable in any field, and especially in the field of convexity where so many results have been rediscovered so many times and where there are so many elementary unsolved problems. Though including few proofs or none at all, they give rather complete descriptions of known results and existing literature in their respective areas. Some of them include new results as well, and most of them discuss many unsolved problems. Since the papers are themselves summaries, it is hardly feasible to summarize them here, but it may be helpful to list their section headings.

GRÜNBAUM, *Borsuk's problem and related questions*— reductions of the problem; partial solutions; universal covers; other results on partitions; coverings by translates; finite sets; related problems.

GRÜNBAUM, *Measures of symmetry for convex sets*— distance-functions for spaces of convex sets; invariant points and sets; a property of some measures of symmetry; general methods for geometric definitions of measures of symmetry; known results on special measures of symmetry; some extremal problems which possibly lead to measures of symmetry; an interesting functional; some generalizations.

DANZER, GRÜNBAUM AND KLEE, *Helly's theorem and its relatives*— proofs of Helly's theorem; applications of Helly's theorem; the theorems of Carathéodory and Radon; generalizations of Helly's theorems; common transversals; some covering problems; intersection theorems for special families;

other intersection theorems; generalized convexity. (The last section makes little contact with the others. It contains a rather complete survey of existing generalizations of the notion of convex set.)

KLEE, *Infinite-dimensional intersection theorems* — intersection theorems for infinite families (also in R^n); intersection theorems involving the weak topology; intersection properties of metric cells.

ÇUDIA, *Rotundity* — rotundity and smoothness properties; comparison of properties; product spaces, quotient spaces, and subspaces; duality; geometry and reflexivity.

Like those of Cudia and Klee, the papers by **BISHOP AND PHELPS** and by **PHELPS** are concerned with the geometry of infinite-dimensional convex sets. The principal result of Bishop and Phelps is that if C is a closed convex subset of a Banach space, then the support points of C are dense in the boundary of C. They show also that for each bounded closed convex subset C of a Banach space E, the members of the conjugate space E^* which attain their maximum on C are dense in E^* (norm topology). Several other interesting results are obtained by the same methods. The paper by Phelps treats some of the more technical points which arise when the space is not normable. In particular, he uses supporting cones to give a new proof of the existence of relative extreme points, where a convex cone K with vertex x is said to support the convex set C provided $C \cap K = \{x\}$.

CORSON AND KLEE show that the topological classification problem for closed convex bodies in a normed linear space E can be reduced to that for E's unit cell and its closed linear subspaces of finite deficiency. For all $\aleph_0$-dimensional spaces as well as for a wide variety of infinite-dimensional Banach spaces, the problem is solved by proving that all closed convex bodies in E are homeomorphic with E itself. The main tool is the fact that certain spaces are homeomorphic with their positive cones. Also obtained are some results on uniformly continuous transformations of convex sets.

The remaining papers are not so directly concerned with convex sets as such, though in each case some sort of convexity is essential either in the paper itself or for its motivation. Both Karlin and Davis deal with convex functions. For real intervals X and Y, **KARLIN** considers the functional transformation T carrying a real function f on Y into the function $g = Tf$ on X given by the formula $gx = \int_Y K(x, y)f(y)dy$, the kernel K being a bounded measurable function on the rectangle $X \times Y$. He is especcially interested in conditions on K which insure that g is convex whenever f is bounded and convex; a similar problem for monotone functions is also considered. The conditions obtained involve the total positivity or sign-regularity of K, where K is said to be sign-regular of order r provided there exists a sequence of numbers (ε_m), each either $+1$ or -1, such that whenever $x_1 < x_2 < \cdots < x_m$, $y_1 < y_2 < \cdots < y_m$, $x_i \in X$, $y_j \in Y$, and $1 \leqq m \leqq r$, then $\varepsilon_m K(x_1, \cdots, x_m; y_1, \cdots, y_m) \geqq 0$, where the expression $K(\cdots;\cdots)$ is the determinant of the matrix which has $K(x_i, y_j)$ in the ith row and the jth column; K is totally positive of order r provided this condition holds with all the ε_m's equal to $+1$. Inter-relation-

ships among various classes of kernels are studied, and many examples are given.

In his second paper, **DAVIS** studies various classes of real-valued convex functions (of one or several real variables) where for each class the defining condition involves the class $\boldsymbol{H}_n$ of $n \times n$ (real) symmetric matrices. For example, if f is a function of one real variable and the matrix $A \in \boldsymbol{H}_n$ has spectral representation $A = \sum_1^n \lambda_i P_i$, it is customary to write $f(A) = \sum_1^n f(\lambda_i) P_i$. In this way f can be regarded as a function on $\boldsymbol{H}_n$ to $\boldsymbol{H}_n$. The function f is called matrix-convex provided $f((1-\lambda)A + \lambda B) \leqq (1-\lambda)f(A) + \lambda f(B)$ for all $\lambda \in [0, 1]$ and $A, B \in \boldsymbol{H}_n$, where the ordering is that induced in $\boldsymbol{H}_n$ by agreeing that a member of $\boldsymbol{H}_n$ is non-negative if and only if it is positive semidefinite. The matrix-convex functions form a proper subclass of the ordinary convex functions and are closely related to the matrix-monotone functions of Loewner. The paper is devoted to an exposition of Loewner's theory along with related ideas for several variables due to Korányi, Sherman, and Davis himself.

In addition to the paper of Poritsky mentioned earlier, two other papers are included here because of the close connections between convex sets and linear inequalities. **BELLMAN AND FAN** study systems of linear inequalities in which the variables are Hermitian matrices and the ordering is defined as in the paper of Davis just mentioned. They find consistency conditions for various systems of inequalities, the conditions being quite analogous to those in the classical situation except that in each case the consistency of an auxiliary system must be assumed. Also included are several interesting examples, as well as results on the minimum and maximum of the traces of certain matrices related to the systems in question.

HOFFMAN supplies a unified approach to some linear programming problems which are amenable to "obvious" solutions. His guide is the observation by Monge that if unit quantities are to be transported from points X and Y to points Z and W (not necessarily respectively) so as to minimize the total distance traveled, then the two routes cannot intersect. He defines a Monge sequence to be an ordering of the set $\{(i, j): 1 \leqq i \leqq m, 1 \leqq j \leqq n\}$ and introduces the notion of such a sequence being consonant with a given $m \times n$ matrix. An algorithm is given whereby a solution for the transportation problem associated with a given matrix can be derived from a Monge sequence consonant with the matrix. The warehouse problem of Cahn is transformed into one to which this algorithm is applied and many other problems are mentioned to which the same idea is applicable.

The many new notions in **MOTZKIN**'s paper are treated in 79 theorems distributed among 50 sections. The paper is concisely written and can hardly be summarized here, but we shall describe its basic idea. Let R be a (not necessarily commutative) ring with unit 1 and let V be a left module over R. Let $\tilde{R}$ be the set of all finite sequences $\lambda = (\lambda_1, \cdots, \lambda_k)$ of members of R. When $S \subset V$, the vector λ is said to be an endovector of S, or S is said to be endo-λ, provided S includes the point $\sum_{i=1}^k \lambda_i s_i$ for every choice of $s_1, \cdots, s_k \in S$. For $\Lambda \subset R$, S is said to be endo-Λ provided S is endo-λ for each $\lambda \in \Lambda$. Since the family of all endo-Λ sets in V is intersectional, the Λ-hull of S is defined

as the smallest endo-Λ set which contains S. The set Λ is said to be complete provided for some S, Λ is the set of all endovectors of S. These and related notions are studied in some detail, where of course the most important cases are those in which R is the real field and the condition $(\lambda_1, \cdots, \lambda_k) \in \Lambda$ is equivalent to one of the following: (i) $\lambda \in \tilde{R}$; (ii) $\sum_1^k \lambda_i = 1$; (iii) $\lambda_i \geqq 0$; (iv) $\sum_1^k \lambda_i = 1$ and $\lambda_i \geqq 0$. The corresponding endo-Λ sets are the linear subspaces (Θ-flats), the affine subspaces (flats), the positive cones (convex cones with vertex O), and the convex sets.

MOTZKIN AND STRAUS are concerned with representing the points of a set as linear combinations of boundary points. Their principal result asserts that if $\alpha_1 + \cdots + \alpha_n = 1$ and $\sum_{i \neq j} |\alpha_i| \geqq |\alpha_j|$ for $1 \leqq j \leqq n$, then for very general sets S it is true that each point of S can be represented in the form $p = \sum_1^k \alpha_i x_i$ for points x_i of the outer boundary of S.

PTÁK presents a unified treatment of several important results on weak compactness, all of which are shown to follow from a combinatorial lemma which gives conditions for the existence of certain convex means. For an infinite set S, let $M(S)$ denote the set of all functions λ on S to $[0, \infty[$ for which the set $N(\lambda)$ is finite and $\sum_{s \in S} \lambda(s) = 1$, where $N(\lambda) = \{s \in S: \lambda(s) > 0\}$. Let $\mathscr{W}$ be a family of subsets of S, and for $\varepsilon > 0$ and $H \subset S$ let $M(H, \mathscr{W}, \varepsilon)$ denote the set of all $\lambda \in M(S)$ such that $N(\lambda) \subset H$ and $\sum_{w \in W} \lambda(w) < \varepsilon$ for all $W \in \mathscr{W}$. The lemma asserts the equivalence of the following two conditions: (1) $M(H, \mathscr{W}, \varepsilon) = \emptyset$ for some infinite $H \subset S$ and some $\varepsilon > 0$; (2) there exists a sequence (s_n) of distinct points of S and a sequence (W_n) of members of $\mathscr{W}$ such that $\{s_1, \cdots, s_n\} \subset W_n$ for all n. With the aid of this lemma he proves that if A is a subset of a complete convex space E and A satisfies a certain double limit condition, then the closed convex hull of A is weakly compact. This includes the well-known theorems of Krein and Eberlein on weak compactness. The same lemma is employed to yield an extensive series of results on weak convergence and weak compactness in locally convex spaces and especially in spaces of continuous functions.

SCHAEFER is concerned with spectral properties in an ordered locally convex algebra A, where this is a locally convex algebra (usually over the complex field) with unit e and with an associated positive cone $K \ni e$ such that K is closed, proper, includes the product of any commuting pair of its elements, and is normal in the sense that there is a family of pseudonorms p on E which generate the topology and are such that $p(x + y) \geqq p(x)$ for all $x, y \in K$. The principal motivating example of such an A is the algebra of all continuous endomorphisms of a Hilbert space, where K is the cone of positive Hermitian operators and the topology is that of either bounded or pointwise convergence. (There are other important examples also.) The paper contains much interesting material on such algebras A, its principal results showing that the spectral behavior of certain members of K is quite analogous to that in the finite-dimensional case. In particular, the members of K whose spectrum is bounded have spectral behavior like that of positive matrices, while those in the unit interval of K (i.e., those $a \in A$ for which $0 \leqq a \leqq e$—diagonal positive matrices in the classical case) behave spectrally like positive Hermitian operators.

FAN's paper is motivated by the Krein-Milman extreme point theorem. He establishes a general lemma which is purely set-theoretical in character, involving neither topological nor vector space concepts, from which the Krein-Milman theorem follows. (Another lemma, in a sense dual to the first, is shown to imply theorems on filters due to Wallman and Stone.) He then considers a set Φ of real-valued functions on a set S, calling a set $X \subset S$ convex provided X is an intersection of sets of the form $\{x \in S: f(x) \geqq \alpha\}$. Since the family of Φ-convex sets is intersectional, the Φ-hull can be defined in the natural way. The notion of Φ-betweenness is defined for points of S and in terms of this the Φ-extreme points of subsets of X are defined. These notions appear in several theorems which generalize known results on extreme points and are related to the abstract minimum principal of Bauer.

HAMMER's second paper is motivated by his notion of a semispace at a point p in a linear space L, this being a maximal convex subset of $L \sim \{p\}$. He reviews some of the known results on semispaces, including their connection with extreme points and the fact that the semispaces form a minimal intersection base for the convex subsets of L. He then describes his system of extended topology which arose from an attempt to consider certain processes and concepts associated with convexity (and especially with semispaces) as topological in character. Many new notions are introduced, complications arising mainly from the fact that in place of the usual topological closure operation he considers an arbitrary expansive function g — i.e., one associating with each set some superset thereof. After discussing the extended topology, he interprets the various notions in terms of convexity, where gX is the union of X with all the line segments determined by points of X. Several unsolved problems are mentioned.

V.K.

UNSOLVED PROBLEMS

Like elementary number theory, the subject of convexity lends itself readily to the statement of interesting unsolved problems. Many of these can be appreciated on an intuitive level and may be accessible to anyone with a bright idea, for the subject (on the whole) is one of many ideas and specific approaches but little machinery. The discussion of unsolved problems was an important part of the Symposium, both informally and in a special session devoted to them; several of the papers published here originated in such discussion. Unsolved problems are found in many of the papers, and the list below contains other problems stated during the Symposium or sent later to the editor.

V. K.

W. CHENEY

Let $\| \ \|_p$ denote the pth-power norm in R^n, M a linear subspace of R^n, and for each $x \in R^n$ let $\pi_p x$ be the (unique) point of M which is nearest to x with respect to $\| \ \|_p$. What can be said about the behavior of $\pi_p x$ (M, x fixed) as $p \to \infty$?

H. S. M. COXETER

If the edges of a convex polyhedron *all touch* a sphere of unit radius so as to form a "crate" from which the sphere cannot escape, prove or disprove that their total length is at least $9\sqrt{3}$. (Cf. [1] for discussion of the related problem in which the requirement of touching is omitted.)

[1] A. S. Besicovitch, *A cage to hold a unit-sphere*, these Proceedings, pp. 19–20.

L. DANZER

Given a *convex body* (i.e., a convex, compact point-set with nonempty interior) C in R^n. Say its (euclidean) *width* (minimal distance between two parallel supporting hyperplanes) is $d(C)$. Define its *k-dimensional width* $d_k(C)$ to be the *maximal* width attained by any *intersection* of C with a k-dimensional flat. Clearly $d_1(C) = \operatorname{diam}(C) \geqq d_n(C) = d(C)$.

I ask for the numbers

$$q(k; n) := \inf \{d_k(C)/d(C) \colon C \text{ a convex body in } R^n \ (1 \leqq k \leqq n)\},$$

in particular, for $q(2; 3)$.

It is trivial that

$$q(k+1; n+1) \leqq q(k; n),$$

but is it true, that also

$$q(k; n+1) \leqq q(k; n)\,?$$

Note that $q(2; 3) < 1$, as shown in [1]. The example there could be simplified (using a regular tetrahedron instead of a cube), but certainly that method is not good to prove $q(2; 3) < .995$ nor will it yield *lower* bounds for $q(k; n)$.

Of course one may ask the same questions for any *Minkowskian* metric over R^n instead of the euclidean one.

[1] L. Danzer, *Über die maximale Dicke der ebenen Schnitte eines konvexen Körpers*, Arch. Math. 8 (1957), 314–316.

C. DAVIS

In n-space ($n > 2$) it is natural to consider, along with the diameter D and width d of a convex body K, intermediate measures. In particular, let $D_k(K) = \min D(P_k K)$, where P_k means projection to an $(n-k+1)$-flat and the minimum is over all P_k; and $d_k(K) = \max d(s_{n-k+1}K)$, where s_l means section by an l-flat and the maximum is over all s_{n-k+1}. For ellipsoids, one proves from the Fischer-Courant principle that $D_{n-k+1} = d_k$, the length of the kth principal axis. In general, of course, $d_n = D_1 = D$ and $D_n = d_1 = d$; however, the extension of the ellipsoid case can not go far, even for centrally symmetric K.

PROBLEM. In 3-space, find the possible range of variation of d_2/D_2. Perhaps even the dependence of this range on d/D could be found; see Besicovitch's problem. It is clear that $d_2/D_2 \leqq 1$, and it is equal to 1 not only for ellipsoids but for a variety of other bodies including all those with rotational symmetry. It is equal to $\sqrt{15}/4$ for the regular tetrahedron. The smallest value I know is $\sqrt{3}/2$, attained for a class of centrally symmetric octahedra including the regular, and including others of arbitrarily small d/D.

For $n > 3$ other possibilities arise: let $D_{k_1k_2}(K) = \max D_{k_1}(s_{n-k_2+1}K)$, provided $k_1 + k_2 \leqq n$; similarly $d_{k_1k_2}$, $D_{k_1k_2k_3}, \cdots$. I do not claim to see hope of proving anything about these compound quantities for general K.

(Cf. Danzer's problem.)

A. DVORETZKY

For a Minkowski space E and positive integer k, define

$$v_k(E) = \max_{\|x_i\|=1} \min_{\pm} \|\pm x_1 \pm x_2 \pm \cdots \pm x_k\|,$$

where the minimum is over all 2^k possible choices of $+$ and $-$ signs and the maximum is over all k-tuples $x_1, \cdots, x_k$ of unit vectors. What can be said about the numbers $v_k(E)$?

K. FAN

PROBLEM. Are the following two rotundity properties of a Banach space X equivalent?

(1) Every sequence $\{x_n\}$ in X with $\lim_{n,m\to\infty} \|(x_n + x_m)/2\| = 1$ is convergent.

(2) Every sequence $\{x_n\}$ in X with $\lim_{n,m\to\infty} \|(x_n + x_m)/2\| = 1$ and having no weak cluster point of norm <1 is convergent.

PROBLEM. What can be said of the structure of the lattice of all closed bounded convex sets in a normed linear space?

(For results of this nature, see [**1**; **2**].)

[**1**] E. E. Floyd and V. L. Klee, *A characterization of reflexivity by the lattice of closed subspaces*, Proc. Amer. Math. Soc. **5** (1954), 655–661.

[**2**] G. W. Mackey, *Isomorphisms of normed linear spaces*, Ann. of Math. (2) **43** (1942), 244–260.

B. GRÜNBAUM

Characterizations of circles and spheres

Besicovitch was the first to establish [**1**] the conjecture of V. Mizel that the circle is the only closed convex curve C in the plane with the property

(i) Whenever three vertices of a rectangle belong to C, the fourth vertex also belongs to C.

A simpler proof of Besicovitch's result was found by Danzer [**2**]. In view of this result, the following questions seem to arise rather naturally.

1. Is the circle the only closed convex curve (resp. simple closed curve) C with the property:

(ii) Whenever three edges of a rhombus R support C, the fourth edge of R also supports C?

2. Is the circle the only convex curve of constant width with the property:

(iii) Each point of C is the vertex of a square, all of whose vertices belong to C?

It is well known that the circle is not the only convex curve with property (iii), even if all the squares are required to be of the same size.

3. Is the $(n-1)$-dimensional sphere the only surface S of constant width in E^n with the property:

(iv) Every point of S is the vertex of a regular n-dimensional octahedron, all of whose vertices belong to S?

For $n \geqq 3$ it is even conceivable that property (iv) alone characterizes spheres among all convex surfaces.

[**1**] A. S. Besicovitch, *A problem on a circle*, J. London Math. Soc. **36** (1961), 241–244.

[**2**] L. W. Danzer, *A characterization of the circle*, these Proceedings, pp. 99–100.

B. GRÜNBAUM—T. S. MOTZKIN

A graph is *k-polyhedral* if its nodes and edges can be identified with the vertices and edges of a k-dimensional convex polyhedron. (See [1] for references and for some properties of polyhedral graphs.) For $n \geqq 5$, the complete graph with n nodes is known to be 4-polyhedral. Conjecture: For $k \geqq 4$, every k-polyhedral graph is 4-polyhedral.

[1] Branko Grünbaum and Theodore S. Motzkin, *On polyhedral graphs*, these Proceedings, pp. 285–290.

P. C. HAMMER

1. Reflection over a convex curve in the plane (B. H. Neumann). Let C be a closed convex curve in the plane with no line segments in its boundary. From a point p exterior to C choose that line of support through p which has C on its left (looking from p). Let $q = fp$ be the reflection of p through the point of contact on the line of support.

Problem. Is there a simple closed curve B (other than C) such that $fB = B$?

Remarks (Hammer). It may be shown that there are n-point sets X for each $n \geqq 3$ such that $fX = X$ and that the union of all of these is unbounded in the plane. Moreover if Y is an open "annulus" bordering C, then $X = \bigcup\{f^n Y : -\infty < n < \infty\}$ is an open set and its boundary is fixed under f. However, is X bounded and if so is its boundary a simple closed curve for some Y? That there is a large class of convex curves C with solution curves B may be seen as follows:

Let B be a closed convex curve and in each family of parallel chords of B take the two which cut a fixed smaller area α from B ($\alpha < 4/9$ area B). Then the intersection C of all closed strips between such parallel chords determine a convex curve C such that $fB = B$.

On the other hand it is easy to construct a convex curve C such that $\{f^n p\} n \geqq 1$, for certain points p converges to a point on C. Whether or not one may find a curve C such that $\{f^n p\}$ is unbounded for some p we have not settled.

Note that the transformation f is area preserving and that the problem is actually affine.

2. The X-ray problems (Hammer). Suppose there is a convex hole in an otherwise homogeneous solid and that X-ray pictures taken are so sharp that the "darkness" at each point determines the length of a chord in C along an X-ray line. (No diffusion, please.) How many pictures must be taken to permit exact reconstruction of the body if:

a. The X-rays issue from a finite point source?

b. The X-rays are assumed parallel?

For the planar counterpart, we have shown that two perpendicular directions are insufficient for (a) and we conjecture that 3 directions are sufficient, although

whether or not such directions must be strategically chosen is also open.

3. SELF-CIRCUMFERENCE (HAMMER). Let C be a closed convex curve in the plane. Then, as Minkowski first proved, to each interior point p of C there is determined an (asymmetric) metric with C as the unit circle, i.e., $d(p, q) = 1$ for $q \in C$. Hence to each p interior to C there is determined two circumferences $a_+(p)$ and $a_-(p)$ of C.

PROBLEM. What are the properties of the set of points such that $a_+(p) = a_-(p)$? In particular, if p_0 is the point such that the minimum value of the set $\min [a_+(p), a_{-1}(p)]$ is achieved at p_0 is $a_+(p_0) = a_-(p_0)$? Moreover are $\min a_+(p)$ and $\min a_-(p)$ achieved at the same point?

REMARKS. If C is symmetrical with respect to a point then $a_+(p) = a_-(p)$ for each p interior to C. The minimum value of circumference might be called the *self-circumference* of C. These circumferences are affine invariant.

4. SELF-CIRCUMFERENCE (GOLAB VIA B. GRÜNBAUM). Let the situation be as in (3). What is the maximum self-circumference for all convex curves C?

REMARKS (HAMMER). For the triangle the self-circumference 9 is achieved and p_0 is the centroid. In this case $a_+(p_0) = a_{-1}(p_0)$. Laugwitz showed that the regular hexagon has minimal self-circumference 6 and the square maximal self-circumference 8 among convex curves with a center. Presumably 6 is the absolute minimum self-circumference. This is one of the few cases in which the circle does not appear as an extreme solution.

V. KLEE

Suppose K is a compact convex subset of a Hausdorff linear space. Must the topology of K be locally convex; i.e., is it true that for each $x \in K$ and each neighborhood U of x there exists a neighborhood V of x *relative to* K such that V is convex and $V \subset U$? Must K have the fixed point property? Must K have an extreme point?

A. KOSIŃSKI

For $0 < n < m$ and $0 \leqq k \leqq n - 1$, an m-dimensional compactum in E^m will be called (n, k)-*convex* provided its intersection with each n-dimensional affine subspace is k-acyclic. Then $(1, 0)$-convexity is (by definition) equivalent to ordinary convexity, and $(n, n-1)$-convexity is also known to be equivalent to ordinary convexity. There exists a simple geometric characterization of $(2, 0)$-convex sets in E^3.

PROBLEM. Find a geometric characterization of (n, k)-convexity.

T. S. MOTZKIN

Find an intrinsic characterization of those n-tuples $(k_0, \cdots, k_{n-1})$ such that

there exists an n-dimensional convex polyhedron having (for $0 \leqq i \leqq n-1$) exactly k_i faces of dimension i.

R. R. PHELPS

Suppose K is a compact convex set in a locally convex Hausdorff linear space. Must K include a point x such that for each $y \in K \sim \{x\}$, $\sup fK = fx > fy$ for some continuous linear functional f?

H. H. SCHAEFER

I. Let E denote a (Hausdorff) locally convex vector space over R, K a convex cone of vertex 0 in E, and K' the dual cone (of linear forms non-negative on K) in the topological dual E' of E. Does there always exist $K \subset E$ such that $E = K - K$ and $E' = K' - K'$? (In a normed space, the cone K spanned by a ball of radius $r > 0$ and with center at a distance $> r$ from the origin, answers the question affirmatively.)

II. Denote by A an algebra over R, provided with a locally convex vector space topology under which multiplication is separately continuous. A *spectral element* of A is an element contained in a subalgebra which is the continuous homomorphic image of some $\mathscr{C}(X)$ (X compact). Is a subalgebra of A, consisting entirely of spectral elements, necessarily commutative? (This question arises in connection with Part II, §3 of [1].)

[1] Helmut H. Schaefer, *Convex cones and spectral theory*, these Proceedings, pp. 451–471.

F. A. VALENTINE—E. G. STRAUS

Does there exist a nonempty compact set S in R^n such that $2 \leqq m(x) \leqq \aleph_0$ for all $x \in S$, where $m(x)$ is the number of convex subsets of S which are maximal relative to being convex, including x, and having dimension $\geqq n-1$?

INDEX OF UNSOLVED PROBLEMS

APPENDIX

See Pták, p. 450.

The author wishes to apologize for treating theorem (6.2) too casually. Upon re-examining the proof he has come to the conclusion that a part of it should be given in more detail. For this reason the following explanations are offered.

In the proof of (6.2) the sentence beginning "Indeed, this shows first..." should be replaced by the following:

First of all, $r(t)$ is continuous since it may be approximated by convex combinations $\Sigma\lambda_i a_i(t)$ of continuous functions. To show that r belongs to the $\sigma(E, E')$ closure of A, take any $y \in E'$. We may clearly assume $|y| \leqslant 1$. Suppose that $|\langle a - r, y\rangle| \geqslant \alpha$ for all $a \in A$ and some $\alpha > 0$; let $A^{(+)}$ and $A^{(-)}$ be the sets of those $a \in A$ for which the difference $\langle a - r, y\rangle$ is respectively positive or negative. Since r belongs to the closure of A in P, it is in the closure of one of them, $A^{(+)}$ say. We have $\langle a - r, y\rangle \geqslant \alpha$ for each $a \in A^{(+)}$. The same argument shows that there exists a convex mean $m = \Sigma_1^q \lambda_i a_i$ with $a_i \in A^{(+)}$ such that $|m(t) - r(t)| \leqslant \frac{1}{2}\alpha$ for each $t \in T$; it follows that

$$\frac{1}{2}\alpha \geqslant \langle m - r, y\rangle = \left\langle \sum_1^q \lambda_i (a_i - r), y \right\rangle = \sum_1^q \lambda_i \langle a_i - r, y\rangle \geqslant \alpha$$

which is a contradiction.

Thus the only thing to be done is to show the existence of the convex means. This is only done for A and ϵ but is obviously true also for $A^{(+)}$ and $\frac{1}{2}\alpha$.

The method of convex means has been developed further in the author's papers, *An extension theorem for separately continuous functions and its application to functional analysis,* Czechoslovak Math. J. **89** (1964), 562–581, and *Algebraic extensions of topological spaces,* Contributions to Extension Theory of Topological Structures (Proc. Sympos. Berlin, 1967), Deutsch. Verlag Wissensch., Berlin, 1969, pp. 179–188.

AUTHOR INDEX

Italic numbers refer to pages on which a complete reference to a work by the author is given.

Roman numbers refer to pages on which a reference is made to a work of the author. For example, under Minkowski would be the page on which a statement like the following occurs: "This theorem was proved earlier by Minkowski [7, § 2] in the following manner. . . ."

Boldface numbers indicate the first page of the articles in this volume.

SUBJECT INDEX

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