Theta Functions Bowdoin 1987

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Robert C. Gunning, Editors

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Preface

Theta functions apparently first appeared in the forms $\sum_{n=0}^{\infty} m^{n^2}$, $\sum_{n=0}^{\infty} m^{1/2n(n+1)}$, $\sum_{n=0}^{\infty} m^{1/2n(n+3)}$ in the work of Jakob Bernoulli. In his work on partition theory, Euler introduced a second variable ζ and studied functions of the form $\prod_{n=1}^{\infty}(1 - q^n\zeta)^{-1}$. For Euler, the primary objects were partition functions such as $\prod(1 - q^n)$, but the function $\prod(1 - q^n\zeta)^{-1}$ was considered as a function of ζ with q occurring as a parameter; after deriving identities for the function of ζ he then set $\zeta = 1$.

Jacobi made two important notational changes that turned out to be crucial for the modern development. He replaced q by $e^{\pi i\tau}$ and ζ by e^{2iz}; thus was born the theta function in its present form

$$\theta(\tau, z) = \sum e^{\pi i n^2\tau + 2inz}.$$

The change from q to τ allowed him to formulate the "imaginary transformation" $\tau \to -1/\tau$, which together with the obvious transformation $\tau \to \tau + 2$ leads to the modular group and eventually to the modern theory of modular forms and their ramifications. (The formulation of the modular group in the variable q is complicated; see the paper by Ehrenpreis in this volume.)

In addition, Jacobi studied $\theta(\tau, z)$ as a function of z in its own right. The quasi double periodicity under $z \to z + \pi$ and $z \to z + \pi\tau$ enabled him to relate theta functions as functions of z to elliptic function theory. For Jacobi as for Euler the primary working variable was z. Of course, this theory has had far reaching generalizations to higher genera Riemann surfaces, abelian varieties, etc.

Surprisingly, theta functions made their appearance in another case of nineteenth century mathematics, namely mechanics. It was discovered by Carl Neumann and Jacobi that certain mechanical (Hamiltonian) systems could be explicitly integrated by means of theta functions. These ideas could have formed the foundation of some of the modern ideas on KdV, KP, and integrable systems in general, but the modern viewpoint seems to have been discovered without knowledge of the eighteenth century results.

When the organizing committee met to discuss the possibility of a conference on theta functions, we saw how perfectly the notation $\theta(\tau, z)$ fit into a three week conference: one week for τ, one week for z, and one week for the comma. (This conforms to the above described three aspects of theta functions that appeared in the nineteenth century.) The conference was thereby organized accordingly. The first week was devoted to the comma, that is, to the interplay of τ and z. The sections on infinite analysis, integrable systems, Kac-Moody algebras, lattice models, and physics are, roughly speaking, devoted to this interplay; the sections on Jacobi varieties, Prym varieties, and algebraic geometry emphasize the z variable. These sections form Part 1 of Volume 49. The sections on modular forms, number theory, and combinatorics emphasize the τ variable. They comprise Part 2 of Volume 49.

It was our hope in organizing the conference that the presentation of a cross section of modern work on theta functions would enable mathematicians to see where we stand now and in what directions we should go in the future.

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