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Inner Completely Positive Maps on von Neumann Algebras

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Let M_n denote the $n \times n$ matrices over the complex numbers. Given two $n \times n$ matrices a and b consider the column vector $\xi = \binom{a}{b} \in M_n \oplus M_n$. Regarding $M_n \oplus M_n$ as a bimodule over M_n we can ask: what is the submodule H_ξ generated by ξ? That is, what are the elements of the subspace $M_n \xi M_n = \{\sum x_i \xi y_i \mid x_i, y_i \in M_n\}$? Since M_n is simple as an algebra it is clear that if $a \neq 0$ then we can obtain any $c \in M_n$ as the first entry of $\sum x_i \xi y_i$. But to do so it appears that we must forfeit control over what happens to the second entry. Conversely, we may make the second entry of $\sum x_i \xi y_i$ whatever we please but again without any control over the first.

A more refined way to determine what submodule is generated by ξ is to use Dixmier's approach to the trace. Let τ be the normalized trace on M_n. Then there are unitaries $u_1, \ldots, u_m \in M_n$ and $\alpha_1, \ldots, \alpha_m \geq 0$, $\sum \alpha_i = 1$, such that $\tau(a) = \sum \alpha_i u_i^* a u_i$. If we also choose unitaries $v_1, \ldots, v_p \in M_n$ and $\beta_1, \ldots, \beta_p \geq 0$, $\sum \beta_i = 1$, such that $\tau(b) = \sum \beta_i v_i^* b v_i$. Then $\sum \alpha_i \beta_j u_i^* v_j^* \xi v_j u_i = \binom{\tau(a)}{\tau(b)}$. We may then do the same for $a^*\xi = \binom{a^*a}{a^*b}$ and $b^*\xi = \binom{b^*a}{b^*b}$, and obtain that $\eta_1 = \binom{\tau(a^*a)}{\tau(a^*b)}$, and $\eta_2 = \binom{\tau(b^*a)}{\tau(b^*b)}$ are in H_ξ. We are now ready to prove the following theorem.

THEOREM 1. *If $\{a, b\}$ is an independent set in M_n then $H_\xi = M_n \oplus M_n$. If $b = \lambda a$ then $H_\xi = \{x \oplus \lambda x \mid x \in M_n\}$.*

PROOF. If the second condition holds then the conclusion is clear. So let us suppose that $\{a, b\}$ is independent. Then $\{\eta_1, \eta_2\} \subseteq H_\xi$ and is independent, so $\binom{1}{0}$ and $\binom{0}{1}$ are in the linear span of $\{\eta_1, \eta_2\}$. Hence $M_n \oplus M_n \subseteq H_\xi \subseteq M_n \oplus M_n$. $\square$

By a similar argument we obtain the following generalization.

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Theorem 2. *Let $a_1, \ldots, a_m \in M_n$ and $\xi = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \in M_n \oplus \cdots \oplus M_n$ (m copies). Then the submodule $M_n \xi M_n$ of $M_n \oplus \cdots \oplus M_n$ generated by ξ is isomorphic to $M_n \oplus \cdots \oplus M_n$ (k copies) where k is the dimension of the span of $\{a_1, \ldots, a_m\}$.*

A further generalization can be obtained by considering the finite dimensional algebra $M = M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_k}$, and $a_1, \ldots, a_m \in M$. Let $\xi = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \in M \oplus \cdots \oplus M$ (m copies) and consider the submodule $M\xi M$. Instead of considering the dimension of the span of $\{a_1, \ldots, a_m\}$, the relevant notion is the dimension function on the centre of M. Writing the centre of M, $Z(M)$, as $C(X)$ for $X = \{\chi_1, \ldots, \chi_k\}$, we have $d(\chi) = \dim(\operatorname{span}\{a_i(\chi)\})$. A convenient way of keeping track of this information is in terms of a decreasing sequence of central projections $d_1 \geq d_2 \geq \cdots \geq d_m$. Here d_n is the largest central projection such that $\{za_i\}$ has dimension n for each central projection $z \leq d_n$. So on $d_n - d_{n+1}$ the dimension of $\operatorname{span}\{a_i\}$ is n. Putting together fibrewise our factor result we obtain the following theorem.

Theorem 3

$$M\xi M \cong d_1 M \oplus d_2 M \oplus \cdots \oplus d_m M.$$

The main interest in this question comes from completely positive maps. We shall suppose that M is a σ-finite von Neumann algebra on a Hilbert space H such that H forms part of a standard form for M. This means that there exists a selfadjoint conjugate linear isometry J such that $JMJ = M'$ and a closed self-dual cone $P \subseteq H$ with $JP = P$. Given a von Neumann algebra M such a triple, (H, J, P), always exists, and is in a sense unique (see Haagerup [**4**]). When $M = M_n$, the $n \times n$ matrices over $\mathbf{C}$, $H = (M_n, (\cdot \mid \cdot)_\tau)$ where $(x \mid y)_\tau = \tau(xy^*)$ and τ is the normalized trace on M_n, $Jx = x^*$, and $P = M_n^+$. Observe that for any M, H is an $M - M$ bimodule under the rule $x\xi y = xJy^*J\xi$, $x, y \in M$, and $\xi \in H$.

Given a completely positive normal map Φ: $M \to M$ one may form its Stinespring dilation H_Φ. This is a Hilbert space upon which we have a pair of commuting normal representations (π, ρ), one of M and one of M', (see e.g., Takesaki [**7**, Theorem IV.3.6] or [**5**, §1]). Now H_Φ also becomes an $M - M$ bimodule under the rule $x\xi y = \pi(x)\rho(Jy^*J)\xi$, for $x, y \in M$, and $\xi \in H_\Phi$. Such bimodules have been called by Connes [**3**, p. 44] correspondences from M to M. A correspondence from M to M, in general, is the same as a binormal representation of $M \otimes M^{\mathrm{op}}$, and it is convenient to pass back and forth between a correspondence H and its associated representation π_H.

The completely positive maps we are interested in here are of the form $\Phi(x) = \sum a_i^* x a_i$ with $\{a_i\} \subseteq M$. We call such maps *inner* by analogy with automorphisms, for it turns out that an automorphism θ of M is an inner completely positive map if and only if there is a unitary $u \in M$ such that $\theta(x) = u^*xu$ and $a_i = z_i u$ for some $\{z_i\}$ in the centre of M.

The correspondence associated to an inner completely positive map $\Phi = \sum a_i^* \cdot a_i$ has a simple description using the Hilbert space H. Let $\xi_0 \in P$ be a cyclic and separating vector for M. Let $\xi_\Phi = \sum_i^\oplus a_i\xi_0 \in \sum_i^\oplus H$. Then H_Φ can be identified with $\overline{M\xi_\Phi M} \subseteq \sum_i^\oplus H$. By identify we mean that there is a unitary between the Hilbert spaces commuting with the associated representations of $M \otimes M^{\text{op}}$. When $M = M_n$, $H = M_n$ as a Hilbert space and $\xi_0 = 1$ is a cyclic and separating vector. So the submodule $\overline{M\xi_\Phi M}$ is exactly the one considered in Theorem 2.

Let π_Φ be the binormal representation associated with H_Φ, and π_H be the representation associated to H. Then π_Φ is a subrepresentation of $\pi_H \otimes 1$. Hence $H_\Phi = p(\sum^\oplus H)$ for some projection p in the commutant of $\pi_H \otimes 1(M \otimes M^{\text{op}})$. As π_H is a multiplicity free representation it turns out that such a projection can always be diagonalized, i.e., $p \sim \text{diag}(d_1, d_2, \dots)$ where $\{d_i\}$ is a decreasing sequence of central projections in M (see Anantharaman-Delaroche and Havet [**1**, Lemma 3.2]). Hence $H_\Phi \cong d_1H \oplus d_2H \oplus \dots$. From Theorem 3 it is clear that the d_i's describe the dimension of $\{a_i\}$ over the centre of M.

Let $\mathscr{E}$ be a conditional expectation of M onto its centre $Z(M)$, (it follows, in particular, from a theorem of Takesaki [**6**] that such a conditional expectation always exists). Given $a_1, \dots, a_n \in M$ let $\text{gram}\{a_1, \dots, a_n\}$ be the $n \times n$ matrix with ij entry $\mathscr{E}(a_i^* a_j)$. If z is a central projection we say that $\{a_1, \dots, a_n\}$ is independent over z if the central support of the determinant of $\text{gram}\{a_1, \dots, a_n\}$ is z. If we were to decompose M into a direct integral over its centre this would be the same thing as saying that $\{a_1(\zeta), \dots, a_n(\zeta)\}$ is independent almost everywhere for $\zeta \in z$. Heuristically the d_i's are defined as follows: d_n is the supremum of the central projections z such that there exists a set of indices $I_z = \{i_1, \dots, i_n\}$ such that $\{a_{i_1}, \dots, a_{i_n}\}$ is independent over z. More details are given in [**5**]. We can now state the main theorem.

THEOREM 4. *Let M be a σ-finite von Neumann algebra and $\Phi = \sum a_i^* \cdot a_j$ an inner completely positive map. Then $H_\Phi \cong \sum \oplus d_i H$ where $\{d_i\}$ is the decreasing sequence of central projections constructed above.*

This is proved in [**5**]. Here I shall indicate the approach. Let e be the projection onto $[Z(M)\xi_0]$ and $\eta_i = ea_i^*\xi_\Phi$. As $e \in Z(M)' = \mathbf{C}(M, M')''$, by Kaplansky's density theorem e is a strong limit of a bounded net of operators $X_i \in \mathbf{C}(M, M')$, the algebra on H generated by M and M'. As each X_i is of the form $\sum x_i y_i'$, we have that $X_i a_j^* \xi_\Phi \in H_\Phi$. Hence $\eta_i \in H_\Phi$. Now $\eta_i = \sum_j^\oplus \mathscr{E}(a_i^* a_j)\xi_0$. When $M = M_n$, $\mathscr{E}(x) = \tau(x)$, so the η_i are generalizations of the ones used to prove Theorem 1.

The next step is to show that $\{\eta_i\}$ generate H_Φ as a bimodule: i.e., $H_\Phi = \overline{\sum M\eta_i M}$. For this it is enough to show that $\xi_\Phi \in \overline{\sum M\eta_i M}$. Since we are working with the closure of $\sum M\eta_i M$, it is enough to show that, for n

sufficiently large, $\xi_\Phi^0 = \sum_{i=1}^{n}{}^{\oplus} a_i\xi_0 \oplus 0 \in \overline{\sum M\eta_i M}$. For each subset I of $\{1,\dots,n\}$, let $\Delta_I = \mathrm{gram}\{a_i \mid i \in I\}$ and z_I the part of the centre over which Δ_I is nonsingular. For each I one shows that $z_I\xi_\Phi^0$ is within ε of an element of $\sum M\eta_i M$. Assembling the pieces one obtains the same conclusion for ξ_Φ^0. This computation is straightforward but complicated to write down. One is then left with the problem of determining the dimension function for $\{\eta_i\}$ and showing that it is the same as that for $\{a_i\}$. This is a problem in linear algebra over the commutative algebra $Z(M)$.

Let us conclude by returning to the case $M = M_n$. Every completely positive map from M_n to M_n is inner. Given $\Phi\colon M \to M$, Choi **[2]** gave a simple way of finding an independent set $\{a_i\} \subseteq M$ with $\Phi = \sum a_i^* \cdot a_i$. Let $\{e_{ij}\}$ be the usual set of $n \times n$ matrix units and $E = \sum_{i,j} e_{ij} \otimes e_{ij} \in M \otimes M_n$. Let $\Phi_n = \Phi\otimes\mathrm{id}$, and $X = \Phi_n(E)$. Then $X \geq 0$ and we may write $X = \sum_{i=1}^k X_i$ with each X_i a rank one operator and k the rank of X. Now write $X_i = A_iA_i^*$ with A_i a $n^2 \times 1$ matrix. Let a_i be the $n \times n$ matrix obtained by letting the first row of a_i be the first n entries of A_i, the second row of a_i be the second n entries of A_i, and so on. Then one checks that for $x \in \{e_{ij}\}$, $\Phi(x) = \sum_{i=1}^k a_i^* x a_i$. The set $\{e_{ij}\}$ is a basis for M_n so the equation holds for arbitrary x. As we may choose the A_i to be pairwise orthogonal, $\{a_i\}$ is independent. This gives the following result.

THEOREM 5. *Let $\Phi\colon M_n \to M_n$ be a completely positive map, then $H_\Phi \cong \sum_{i=1}^k \oplus M_n$ where k is the rank of $\Phi_n(E)$.*

As an application consider the following special case. Let $h \in M_n^+$, let $\Phi(x) = h \circ x$, the Schur-Hadamard product of h and x. As is well known Φ is completely positive.

COROLLARY 6. *Let $h \in M_n^+$ and $\Phi(x) = h\circ x$. Then $H_\Phi \cong \sum_{i=1}^k M_n$ where k is the rank of h.*

PROOF. We must show that h and $\Phi_n(E)$ have the same rank. We shall show that there is a unitary U, a permutation matrix in fact, with $U\Phi_n(E)U = h\otimes e_{11}$. If p is any positive integer let $e_p = e_{ii}$ where $p \equiv i \bmod n$. One checks that $U = \sum_{i,j=1}^n e_{i+j-1} \otimes e_{ij}$ does the job. For example, when $n = 3$

$$U = \left(\begin{array}{c|c|c} e_1 & e_2 & e_3 \\ \hline e_2 & e_3 & e_1 \\ \hline e_3 & e_1 & e_2 \end{array}\right).$$

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