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**PURE MATHEMATICS**

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Volume 63

**Group Representations:  
Cohomology, Group  
Actions and Topology**

Summer Research Institute on  
Cohomology, Representations,  
and Actions of Finite Groups  
July 7–27, 1996  
University of Washington, Seattle

Alejandro Adem  
Jon Carlson  
Stewart Priddy  
Peter Webb  
Editors



**American Mathematical Society**

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- 63 **Alejandro Adem, Jon Carlson, Stewart Priddy, and Peter Webb, Editors**, Group representations: Cohomology, group actions and topology (University of Washington, Seattle, 1996)
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*(Continued in the back of this publication)*

# Group Representations: Cohomology, Group Actions and Topology

# Proceedings of Symposia in PURE MATHEMATICS

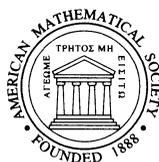
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**American Mathematical Society**  
Providence, Rhode Island

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COHOMOLOGY, REPRESENTATIONS, AND ACTIONS  
OF FINITE GROUPS HELD AT THE  
UNIVERSITY OF WASHINGTON, SEATTLE  
JULY 7–26, 1996

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## Preface

The American Mathematical Society held its 39th Summer Research Institute at the University of Washington in Seattle from July 8 to July 26, 1996. The topic of the meeting was “Cohomology, Representations and Actions of Finite Groups.”

The organizing committee consisted of Alejandro Adem, Jonathan Alperin, Jon Carlson (chair), R. James Milgram, Stewart Priddy and Peter Webb.

There were thirty-three one hour plenary/expository lectures, titles and speakers follow this preface. There were also three or four parallel sessions during the afternoons, covering multiple topics in topology and representation theory. There was also a problem session organized by David J. Benson. The written problems are enclosed as the last entry in this volume.

These proceedings consist mostly of research papers as well as a few select survey articles.

The institute was sponsored by the National Science Foundation, we are grateful for their support.

The organizing committee would like to thank Wayne Drady as well as other staff provided by the American Mathematical Society for their help in running and organizing the conference. We also appreciate Diane Reppert’s expert help in preparing this volume and the support provided by the University of Wisconsin-Madison for doing this.

Alejandro Adem

Jon Carlson

Stewart Priddy

Peter Webb

# 1996 Summer Research Institute

## Schedule of Expository Lectures

### WEEK 1

#### Monday, July 8.

Bill Browder, Princeton University

*“Group Actions”*

Jon Alperin, University of Chicago

*“Problems in representation theory”*

#### Tuesday, July 9.

Len Evens, Northwestern University

*“The May spectral sequence in cohomology of groups”*

Gunnar Carlsson, Stanford University

*“Stable homotopy and classifying spaces”*

#### Wednesday, July 10.

Alejandro Adem, University of Wisconsin

*“Cohomology of finite groups: calculations and detection”*

Geoff Robinson, University of Leicester

*“Alperin’s conjecture”*

#### Thursday, July 11.

Dave Benson, University of Georgia

*“Complexity and varieties for infinitely generated modules”*

Mark Feshbach, University of Minnesota

*“How to stably split a classifying space of a finite group”*

#### Friday, July 12.

Bill Dwyer, Notre Dame University

*“Homology decompositions”*

Jim Milgram, Stanford University

*“Some relations between finite group theory and homotopy theory”*

#### Saturday, July 13.

Stefan Jackowski, Warsaw University

*“Homotopy approximations of classifying spaces via transformation groups”*

Jeremy Rickard, University of Bristol

*“Splendid equivalences of derived categories”*

**WEEK 2****Monday, July 15.**

Geoff Mason, University of California at Santa Cruz

*"Representation-theoretic and cohomological aspects of rational conformal field theory"*

Jim McClure, Purdue University

*"Maps between classifying spaces"*

**Tuesday, July 16.**

Haynes Miller, M.I.T.

*"Elliptic moduli in algebraic topology"*

Jeremy Rickard, University of Bristol

*"Idempotent Modules: Bousfield localization in modular representation theory"*

**Wednesday, July 17.**

Michel Broue, Paris VII

*"The abelian defect groups conjectures: where have they led us"*

Guido Mislin, ETH

*"On the cohomology of the group of automorphisms of a free group"*

**Thursday, July 18.**

Stewart Priddy, Northwestern University

*"Applications of classifying spaces to group cohomology"*

Michael Aschbacher, Cal. Tech.

*"Topological methods in the study of the structure of finite groups"*

**Friday, July 19.**

Peter Webb, University of Minnesota

*"Mackey functors and their applications"*

Clarence Wilkerson, Purdue University

*"Using diagrams to handcraft classifying spaces"*

**Saturday, July 20.**

Hans-Werner Henn, University of Heidelberg

*"On the Landweber Stong Conjecture (after Bourguiba and Zarati)"*

Dan Nakano, Utah State University

*"Complexity and support varieties for finite-dimensional algebras"*

**WEEK 3****Monday, July 22.**

Ronnie Lee, Yale University

*"Spherical space form problem"*

Nick Kuhn, University of Virginia

*"Computations in generic representation theory: maps from symmetric powers to composite functors"*

**Tuesday, July 23.**

Brian Parshall, University of Virginia

*"On Specht-Weyl Correspondences"*

Paul Goerss, University of Washington

*“Hopf algebras, Dieudonné theory, and the homology of homotopy inverse limits”*

**Wednesday, July 24.**

Hans-Werner Henn, University of Heidelberg

*“Unstable modules over the Steenrod algebra and cohomology of groups”*

Karin Erdmann, Oxford University

*“Representations of symmetric groups and quasi-hereditary algebras”*

**Thursday, July 25.**

Steve Mitchell, University of Washington

*“General linear group cohomology for rings of algebraic integers”*

Dave Benson, Convener

*Problem Session*

**Friday, July 26.**

Bob Oliver, Université de Paris Nord

*“Vector bundles over classifying spaces of compact Lie groups”*

Henning Andersen, Aarhus University

*“Tilting modules for algebraic and quantum groups”*

## Problem Session, Seattle 1996

Compiled by Dave Benson

### Introduction

On one of the last days of the Seattle conference on Cohomology, Representations and Actions of Finite Groups, I hosted a problem session in which speakers were invited to present problems related to the area of the conference. I got a good response, and so I decided to collect these problems and invite further contributions from participants who were not able, for one reason or another, to contribute in person to that problem session.

The problems are presented here in alphabetical order of contributor. They have been lightly edited for uniformity of style.

### Group actions (A. ADEM)

Let  $X$  denote a finite complex. Determining which groups can act freely on  $X$  even for familiar examples can be quite difficult. The following is a basic question first raised by Conner:

1. If  $(\mathbb{Z}/p)^r$  acts freely on  $S^{n_1} \times \cdots \times S^{n_k}$ , does this imply that  $r \leq k$ ?

This has been settled in the equidimensional case in work by Carlsson [2] and Adem–Browder [1], except when  $p = 2$  and the dimension of the spheres is 1, 3 or 7.

The following is a general homological conjecture due to G. Carlsson which would imply the above:

2. Let  $k$  be a field of characteristic  $p$  and  $C_*$  a connected finite chain complex of finitely generated free  $k(\mathbb{Z}/p)^r$ -modules. Is it true that

$$\sum_i \dim H_i(C) \geq 2^r?$$

The one dimensional case has recently been established by Adem and Swan. Conversely we have the following question first raised by Benson and Carlson:

3. Let  $G$  is a finite group of rank  $r$  (i.e.,  $r$  is the maximum of the  $p$ -ranks as  $p$  runs over the primes dividing  $|G|$ ). Does  $G$  act freely on a finite complex

$$X \simeq S^{n_1} \times \cdots \times S^{n_r}?$$

This looks like a difficult problem, even in the case where  $G = P$  is a  $p$ -group. However, one can show that if every element of order  $p$  in  $P$  is central, then it in fact acts freely on a product of  $r$  equidimensional spheres. The version of this problem using actual spheres arising from representations has been studied by U. Ray [3].

4. Does there exist a fixed integer  $N$  such that if  $G$  is a finite group with

$$\bigoplus_{i=1}^N H_i(G, \mathbb{Z}) = 0$$

then  $G = \{1\}$ ?

This was conjectured by Loday for  $N = 3$  but Milgram recently showed that the sporadic group  $M_{23}$  is a counterexample.

5. Describe the finite groups that can act freely and homologically trivially on a closed 4-manifold.

6. If  $\mathcal{S}_p(G)$  is the Brown complex (see [5] for example) of nontrivial  $p$ -subgroups of  $G$ , calculate

$$K_G^*(|\mathcal{S}_p(G)|)$$

in terms of algebraic invariants.

This is a generalized topological version of Alperin's conjecture (which can be attributed to Thévenaz [4]).

[1] A. Adem and W. Browder. *The free rank of symmetry of  $(S^n)^k$* . Invent. Math. 92 (1988), 431–440.

[2] G. Carlsson. *On the rank of abelian groups acting freely on  $(S^n)^k$* . Invent. Math. 69 (1982), 393–408.

[3] U. Ray. *Free linear actions of finite groups on products of spheres*. J. Algebra 147 (1992), 456–490.

[4] J. Thévenaz. *Equivariant  $K$ -theory and Alperin's conjecture*. J. Pure & Applied Algebra 85 (1993), 185–202.

[5] P. J. Webb. *Subgroup complexes*. Arcata Conf. on Representations of Finite Groups, Proc. Symp. Pure Maths 47 (Part 1), A.M.S. 1987, 349–365.

### The $B$ conjecture (J. L. ALPERIN)

The study of representations of finite groups and the study of group actions and cohomology of finite groups both deal with  $p$ -local subgroups. Many of the great insights and fundamental discoveries about  $p$ -local subgroups discovered in the course of the classification of finite simple groups have yet to play a role in these areas. The  $B$  conjecture is such a result; it is actually a theorem, a deep one, and its proof, for odd primes, depends on the entire classification.

7. Give a direct proof of the  $B$  conjecture.

The motivation here is to publicize the result and create curiosity about the classification.

If  $H$  is any finite group then  $E(H)$  is the largest normal subgroup with the following two properties:  $E(H)$  is perfect (so  $E(H) = E(H)'$ );  $E(H)/Z(E(H))$  is the direct product of (non-abelian) simple groups. Groups with these two properties

should be thought of as analogs of reductive groups so  $E(H)$  is the largest normal “reductive” subgroup of  $H$ , the “reductive part” of  $H$ .

Assume now that  $G$  is a finite group which has no non-identity normal subgroup of order prime to  $p$  and let  $L$  be a  $p$ -local subgroup of  $G$  (so  $L = N(Q)$  for a non-identity  $p$ -subgroup  $Q$  of  $G$ ). Let  $L^*$  be the quotient of  $L$  by its largest normal subgroup of order prime to  $p$ . In general, our hypothesis on  $G$  implies that  $L^*$  is very close to  $L$ , even equal to  $L$ , and that the largest normal subgroup of  $L$  of order prime to  $p$  is insignificant. The  $B$  conjecture makes explicit one aspect of this.

**Theorem** ( $B$ -conjecture)  $E(L^*) = E(L)^*$

That is, the “reductive part” of  $L^*$  is nothing more than the image  $E(L)^*$  of  $E(L)$  in  $L^*$ , the “reductive part” of  $L$ ! As easy as this is to state, it seems so hard to attack. A direct proof would also completely revolutionize the classification proof.

**A Yoneda description of the Steenrod operations** (R. R. BRUNER)

Let  $k = \mathbb{F}_p$ , the prime field of characteristic  $p > 0$ , and let  $A$  be a cocommutative Hopf algebra over  $k$ . Products can be defined in  $\text{Ext}_A(k, k)$  in two fundamentally different ways.

First, if  $\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow k \rightarrow 0$  is a projective resolution of  $k$ , then, given cocycles  $x : P_n \rightarrow k$  and  $y : P_m \rightarrow k$ , we can lift  $x$  to a chain map  $\tilde{x} : P \rightarrow P$  and define  $xy$  to be represented by the *Yoneda composite*  $y\tilde{x}_m : P_{n+m} \rightarrow P_m \rightarrow k$ . This method applies to any augmented algebra  $A$  over  $k$ , and, in general, yields a non-commutative product.

Second, the Hopf algebra structure on  $A$  allows us to consider  $P \otimes P$  as an  $A$ -module by pullback along the diagonal  $\Delta : A \rightarrow A \otimes A$ . The isomorphism  $k \rightarrow k \otimes k$  lifts to a chain map  $\Delta : P \rightarrow P \otimes P$  and, given cocycles  $x$  and  $y$  as above, we may define  $xy$  to be represented by the cocycle

$$(x \otimes y)\pi_{n,m}\Delta : P_{n+m} \rightarrow (P \otimes P)_{n+m} \rightarrow P_n \otimes P_m \rightarrow k.$$

It is easily shown that this produces the same product as the first definition.

When  $A$  is cocommutative the chain map  $\Delta : P \rightarrow P \otimes P$  is chain homotopy cocommutative and the chain homotopies give rise to Steenrod operations denoted

$$Sq^i : \text{Ext}_A^{s,t}(k, k) \rightarrow \text{Ext}_A^{s+i,2t}(k, k)$$

when  $p = 2$ , and similarly for odd  $p$ .

- 8. Find a description of the Steenrod operations in terms of Yoneda composites, avoiding any use of  $\Delta : P \rightarrow P \otimes P$ .

The motivation for this is computational efficiency. In mechanical calculations of  $\text{Ext}$  [6, 7], when  $P$  is large enough to strain available memory,  $P \otimes P$  is hopelessly out of reach. Worse, even when  $P$  is not overly large, the map  $\Delta$  is so redundant that it is useless. For example, in my calculations of the cohomology of the mod 2 Steenrod algebra, the calculation of  $\text{Ext}$  out to internal degree  $t = 60$  can be done in a few hours on a small workstation. However, the calculation of  $\Delta$  on the class  $h_3^3 h_5 \in \text{Ext}^{4,44}$  required 14 days and contained 25,000,000 terms. This happens because  $\Delta$  must capture all possible decompositions of an element.

In contrast, the calculation of the product structure by computing chain maps which lift cocycles is quite efficient because we need only lift those cocycles which are indecomposable, and we need only keep track of their values on an  $A$ -basis for the resolution. Using this approach, I have calculated the entire product structure

of the cohomology of the Steenrod algebra through internal degree  $t = 140$ . The largest of the vector spaces  $P_{s,t}$  involved are roughly 50,000 dimensional over  $\mathbb{F}_2$ , so that any attempt to do this calculation using  $\Delta$  would be pointless.

[6] R. R. Bruner. *Calculation of large Ext modules*. Computers in Geometry and Topology, ed. Martin C. Tangora, Marcel Dekker (1989), 79-104.

[7] R. R. Bruner. *Ext in the nineties*. Proc. 1992 Oaxtepec Conf on Alg. Top., ed. Martin C. Tangora, Contemp. Math. **146** (1993), 71-90.

### Generating thick subcategories (J. F. CARLSON)

Let  $G$  be a finite group and  $k$  a field of characteristic  $p > 0$ . Let  $B$  be a block of  $kG$  and let  $\text{stmod-}B$  be the stable category of modules in  $B$ .

Suppose that  $\mathcal{C}$  is a thick (or épaisse) subcategory of  $\text{stmod-}B$  (see Rickard [9] for background material on the stable category and thick subcategories). Let  $\mathcal{G}(\mathcal{C})$  be the Grothendieck group of  $\mathcal{C}$  and

$$\theta : \mathcal{G}(\mathcal{C}) \rightarrow \mathcal{G}(\text{stmod-}B)$$

be the natural map induced by the inclusion of  $\mathcal{C}$  in  $\text{stmod-}B$ .

9. Under what circumstances would the surjectivity of  $\theta$  imply that  $\mathcal{C} = \text{stmod-}B$ ?

The case that  $\mathcal{C}$  is the thick subcategory generated by the trivial module  $k$  was considered in [8]. In this case the surjectivity of  $\theta$  would mean that only one principal divisor of the Cartan matrix for the principal block is not a unit. Jeremy Rickard has pointed out that it is possible to construct subcategories  $\mathcal{C} \neq \text{stmod-}B$  such that  $\theta$  is surjective. But the construction seems somewhat contrived. A more natural question would be whether the subcategory  $\mathcal{C}$  generated by the Scott modules is the whole of the principal block. This question came up in discussions with Raphaël Rouquier.

[8] D. J. Benson, J. F. Carlson and G. R. Robinson. *On the vanishing of group cohomology*. J. Algebra 131 (1990), 40-73.

[9] J. Rickard. *Idempotent modules in the stable category*. J. London Math. Soc., to appear.

### Parameters in cohomology (J. F. CARLSON)

Let  $G$  be a finite group and  $k$  a field of characteristic  $p > 0$ . A sequence  $\zeta_1, \dots, \zeta_r \in H^*(G, k)$  is said to be quasi-regular (see [10]) if

- (1)  $\zeta_1$  is a regular element,
- (2) for each  $i > 1$  the map

$$\zeta_i : H^a(G, k)/(\zeta_1, \dots, \zeta_{i-1}) \rightarrow H^{a-n_i}(G, k)/(\zeta_1, \dots, \zeta_{i-1})$$

given by multiplication by  $\zeta_i$ , is an injection provided  $a \geq \sum_{j=1}^{i-1} n_j$ , and

- (3)  $H^*(G, k)$  is finitely generated as a module over  $k[\zeta_1, \dots, \zeta_r]$ .

10. (A) Does  $H^*(G, k)$  always have a quasi-regular sequence?

(B) Given a homogeneous system of parameters (satisfying (3) with  $r = p\text{-rank}(G)$ )  $\zeta_1, \dots, \zeta_r$ , is some ordering of these elements a quasi-regular sequence?

(C) Given a homogeneous system of parameters  $\zeta_1, \dots, \zeta_r$  with  $\zeta_1, \dots, \zeta_d$  a regular sequence for  $d = \text{Depth}(H^*(G, k))$ , is this a quasi-regular sequence?

These questions are of interest in computations of cohomology since, among other things, an affirmative answer would imply that all of the generators of  $H^*(G, k)$  lie in degrees at most  $\sum_i n_i$ .

[10] D. J. Benson and J. F. Carlson. *Projective resolutions and Poincaré duality complexes*. Trans. Amer. Math. Soc. 342 (1994), 447–488.

**Abelian subgroups of discrete groups** (J. CORNICK)

Recall that  $G$  is of type  $\text{FP}_\infty$  if the trivial  $\mathbb{Z}G$ -module  $\mathbb{Z}$  has a projective resolution of finite type (i.e. each projective in the resolution is finitely generated). If, in addition, the group is finitely presented this condition is equivalent to the existence of a  $K(G, 1)$  of finite type (i.e. finitely many cells in each dimension).

- 11. What are the possible abelian subgroups  $A$  of a group  $G$  of type  $\text{FP}_\infty$ ?
- 12. Same question where  $G$  has the additional property of being in the (large) class  $\mathbf{H}\mathfrak{F}$  introduced by Kropholler [11]?

In the second case we know that there is a bound on the cohomological dimensions of the torsion free subgroups, and on the orders of the finite subgroups. It follows that  $A \cong F \oplus V$  where  $F$  is finite and  $V$  is a subgroup of a finite dimensional rational vector space.

- 13. Can every possibility occur?

It would be enough to show that the additive rationals  $\mathbb{Q}$  can be embedded in such a  $G$  (this is the case where infinitely many primes are inverted). It is well known that  $\mathbb{Z}[\frac{1}{n}]$  can be embedded in  $\langle x, y | x^y = x^n \rangle$  which is a group of type  $\text{FP}_\infty$  (this is the case where finitely many primes are inverted).

The  $\mathbf{H}\mathfrak{F}$  condition certainly makes a difference. For example, the Thompson group  $\langle x_1, x_2, \dots | x_n^{x_i} = x_{n+1}, i < n \rangle$  is of type  $\text{FP}_\infty$ , but not in  $\mathbf{H}\mathfrak{F}$ , and contains a free abelian subgroup of infinite rank.

- 14. Can one embed  $\mathbb{Q}/\mathbb{Z}$  in an  $\text{FP}_\infty$ -group? Such a group can not be in  $\mathbf{H}\mathfrak{F}$  because there is no bound on the orders of the finite subgroups.

[11] P. H. Kropholler. *On groups of type  $(FP)_\infty$* . Journal of Pure & Applied Algebra 90 (1993), 55–67.

**Constructing classifying spaces** (W. G. DWYER)

Let  $G$  be a connected compact Lie group and  $T \subseteq G$  a maximal torus. It is known that  $G$  is determined up to isomorphism by the normalizer  $NT$  of  $T$ .

- 15. Find a direct *homotopy theoretic* way to construct  $BG$  (or a  $p$ -completion of  $BG$ ) from data visible in  $NT$ .

Known methods of building  $BG$  from  $NT$  are not homotopy theoretic: they involve first building the group  $G$ , either by generators and relations or by some Lie algebra technique.

Let  $G$  be a finite group and  $p$  a prime. Let  $F_G$  be the functor from the category of finite  $p$ -groups to the category of finite sets which assigns to each finite  $p$ -group  $P$  the set of conjugacy classes of monomorphisms  $P \rightarrow G$ . This functor captures the

$p$ -fusion in  $G$ . The problem is to find an explicit way to construct the  $p$ -completion of  $BG$  from the functor  $F_G$ .

### Centralizers and cohomology (H.-W. HENN)

The following problem is concerned with the question to what extent centralizers  $C_G(E)$  of elementary abelian  $p$ -subgroups  $E$  of suitable discrete groups  $G$  can be used to study cohomology with coefficients in nontrivial  $\mathbb{F}_p[G]$ -modules.

For example assume  $G$  admits a cocompact action on a finite dimensional mod- $p$  acyclic space  $G$ -CW-complex  $X$  such that all isotropy groups are finite, e.g.  $G$  itself is finite. Let  $\mathcal{A}_p(G)$  be the Quillen category of elementary abelian  $p$ -subgroups of  $G$  and let  $\mathcal{O}$  be a full subcategory of  $\mathcal{A}_p(G)$  with the following properties:

- If  $E$  is an object of  $\mathcal{O}$  and  $E'$  is conjugate to  $E$  in  $G$  then  $E'$  is also an object of  $\mathcal{O}$ .
- If  $E$  is an object of  $\mathcal{O}$  and  $E \subset E'$  then  $E'$  is also an object of  $\mathcal{O}$ .

**Theorem** [12] *Under these assumptions the restriction maps induce a natural map*

$$H^*(G, \mathbb{F}_p) \longrightarrow \lim_{\mathcal{O}} H^*(C_G(E); \mathbb{F}_p)$$

whose kernel and cokernel are torsion with respect to the ideal

$$\mathfrak{a} := \bigcap_{E \notin \mathcal{O}} \text{Ker}(H^*(G; \mathbb{F}_p) \longrightarrow H^*(E; \mathbb{F}_p)).$$

Furthermore the higher derived functors  $\lim_{\mathcal{O}}^i H^*(C_G(E); \mathbb{F}_p)$  are also torsion with respect to  $\mathfrak{a}$ .

**Example:**  $\mathcal{O} = \mathcal{A}_p^*(G) := \{E \in \mathcal{A}_p(G) \mid E \neq \{1\}\}$ . In this case we have  $\mathfrak{a} = \tilde{H}^*(G; \mathbb{F}_p)$  and the kernel, cokernel and the higher derived functors are actually finite. In the case  $G$  is finite they are even trivial by a result of Jackowski and McClure [13], but this might be a red herring for the following question.

**16.** Find an extension of the Theorem which holds for  $H^*(G; M)$  if  $M$  runs through a suitable class of  $k[G]$ -modules. If  $G$  is finite, one would like to be able to take any finitely generated  $k[G]$ -module  $M$ .

[12] H.-W. Henn. *Commutative algebra of unstable  $K$ -modules, Lannes'  $T$ -functor and equivariant mod- $p$ -cohomology*. Heft Nr. **126**, Forschergruppe Topologie und nichtkommutative Geometrie Univ. Heidelberg, September 1995, to appear in *Journal für die reine und angewandte Mathematik*.

[13] S. Jackowski and J. McClure. *Homotopy decomposition of classifying spaces via elementary abelian subgroups*. *Topology* **31** (1992), 113-132.

### Conjugacy and Hall subgroups (S. JACKOWSKI)

**17.** Let  $H$  be a Hall  $\pi$ -subgroup of a finite group  $G$  for some set of primes  $\pi$ . Assume that for all  $h_1$  and  $h_2$  of prime power order in  $H$ , if  $h_1$  and  $h_2$  are conjugate in  $G$  then they are conjugate in  $H$ . Is the same then true for all  $h_1$  and  $h_2$  of composite order?

**Homomorphisms preserving conjugacy** (S. JACKOWSKI)

18. Let  $G$  be a finite  $p$ -group and  $R = \mathbb{Z}$  or  $\mathbb{Z}/p$ . Let  $\phi : G \rightarrow G$  be an automorphism inducing the identity map on  $H^*(G; R)$ . Does  $\phi$  preserve conjugacy classes of elements of  $G$ ? Is  $\phi$  inner?

**Varieties and exponents of cohomology** (I. J. LEARY)

19. Let  $I_n(G)$  be the annihilator of  $p^n$  in the  $p$ -local cohomology,  $H^*(G; \mathbb{Z}_{(p)})$ , of a finite group  $G$ . Let  $J_n = J_n(G)$  be the radical of the image of  $I_n$  in  $H^*(G; \mathbb{F}_p)$ . Is  $J_n$  closed under the Steenrod power operations?

Let  $V_G(k)$  be the variety of homomorphisms from  $H^*(G; \mathbb{F}_p)$  to an algebraically closed field  $k$  and let  $V_n(G, k)$  be the subvariety of  $V_G(k)$  corresponding to  $J_n$ . The question is equivalent to: Is  $V_n(G, k)$  equal to a union of the images of  $V_E(k)$ , where  $E$  ranges over some family of elementary abelian subgroups of  $G$ ?

The  $V_n(G, k)$  were introduced by J. F. Carlson in [14]. In [15] it is shown that  $J_n(G \wr C_p)$  is closed under the Steenrod powers provided that  $J_n(G)$  and  $J_{n-1}(G)$  are.

[14] J. F. Carlson. *Exponents of modules and maps*. Invent. Math. **95** (1989) 13–24.

[15] I. J. Leary. *On the integral cohomology of wreath products*. Preprint, 1996.

**Quantum Galois Theory** (G. MASON)

Let  $V$  be a simple vertex operator algebra (VOA) and let  $G$  be a finite group of automorphisms of  $V$ .

20. Is there a Galois correspondence between subgroups of  $G$  and subVOAs of  $V$  containing  $V^G$  given by sending  $H \leq G$  to  $V^H$ .

The answer is known to be yes if  $G$  is dihedral or nilpotent. There is a similar conjecture for *all* groups, which should be endowed with a suitable (Krull) topology.

**Classifying spaces of wreath products** (R. J. MILGRAM)

The wreath product

$$\mathbb{Z}/2 \wr \mathbb{Z}/2 \wr \cdots \wr \mathbb{Z}/2 = \text{Wr}_n(\mathbb{Z}/2)$$

has the property that  $\text{Out}(\text{Wr}_n(\mathbb{Z}/2))$  is a finite 2-group for each  $n$ . Consequently it has a single dominant summand (see for example [16] or [17] for background material on stable splittings and dominant summands)

$$X_{\text{Wr}_n(\mathbb{Z}/2), \mathbb{F}_2} \subseteq \Sigma^\infty B\text{Wr}_n(\mathbb{Z}/2).$$

21. Find the structure of this summand.

**Remark:** One can see that  $\text{Out}(\text{Wr}_n(\mathbb{Z}/2))$  is a 2-group by induction. There exists a pair of subgroups  $(\mathbb{Z}/2)^{2^{n-1}} \subseteq \text{Wr}_n(\mathbb{Z}/2)$  which are permuted by any outer automorphism. Hence an index two subgroup fixes one and is determined up to an obvious 2-group by its action on the quotient  $\text{Wr}_n(\mathbb{Z}/2)/(\mathbb{Z}/2)^{2^n} \cong \text{Wr}_{n-1}(\mathbb{Z}/2)$ .

**Example:** For  $D_8 = \mathbb{Z}/2 \wr \mathbb{Z}/2$  the dominant summand is  $X_{D_8, \mathbb{F}_2} = \bar{B}L_3(2)$ .

22. Is it true that  $X_{\text{Wr}_3(\mathbb{Z}/2), \mathbb{F}_2} = BL_3(2) \wedge BL_3(2) \rtimes_{\mathbb{Z}/2} E\mathbb{Z}/2 \cap BS_8/\mathbb{R}\mathbb{P}^\infty$ ? Here, what is meant by this is that the intersection is taken in  $BW_3(\mathbb{Z}/2)$ , and the obvious

splitting summand  $\mathbb{R}\mathbb{P}^\infty$  is removed from  $BS_8$  before the intersection is taken.

**23.** Find the complete splitting at the prime two of  $B\tilde{A}_8$ .

**Remark:**  $\text{Syl}_2(\tilde{A}_8) = \text{Syl}_2(M_{22}) = \text{Syl}_2(M_{23}) = \text{Syl}_2(\text{McL})$ .

**24.** Is it true that the dominant summand  $X_{\text{Syl}_2(\tilde{A}_8), \mathbb{F}_2}$  is  $\bar{B}\text{McL}$  (at the prime 2), that is to say, that  $\bar{B}\text{McL}$  is indecomposable at 2?

[16] D. J. Benson. *Stably splitting BG*. Bull. of the American Math. Society 33 (1996), 189–198.

[17] J. Martino. *Classifying spaces and their maps*. Contemp. Math. 188 (1995), 161–198.

### New Doomsday Conjecture (N. MINAMI)

**25.** For each  $s$ , does there exist some integer  $n(s)$  such that no nontrivial element in the image of

$$(\mathcal{P}^0)^{n(s)}(\text{Ext}_{\mathcal{A}_*}^{s,*}(\mathbb{Z}/p, \mathbb{Z}/p)) \subseteq (\text{Ext}_{\mathcal{A}_*}^{s,p^{n(s)}*}(\mathbb{Z}/p, \mathbb{Z}/p))$$

is a nontrivial permanent cycle?

**Remarks:** (1) This is true for  $s = 1$  by the Adams Hopf invariant one theorem and its odd primary analogue by Liulevicius and Shimada–Yamanoshita.

(2) This is true for  $s = 2$  when  $p \geq 5$ , by Miller–Ravenel–Wilson and Ravenel.

(3) An iterated transfer analogue of this conjecture holds (Minami).

(4) No counterexample is known!

(5) The first unsolved case is when  $s = 2$  and  $p = 2, 3$ . From the viewpoint of homotopy theory, the case  $p = 3$  appears to be more tractable. On the other hand, the case  $p = 2$  appears to be more difficult and it essentially claims that there are just finitely many Kervaire invariant one elements. Recall that Milgram has also predicted the same claim, from his philosophy: maximal subgroups of symmetric groups should play essential roles in the stable homotopy groups of the sphere (via the Barratt–Priddy–Quillen theorem  $B\Sigma_\infty^+ \simeq Q_0S^0$ ). He then speculated that there are just finitely many Kervaire invariant one elements from the finiteness of the finite sporadic simple groups. Conversely, we may ask the following:

**26.** Can the new doomsday conjecture offer any interesting speculations about finite group theory?

**27.** Study the odd primary Mahowald  $\eta_j$ -elements, using the Hopkins–Miller spectrum  $EO_{p-1}$ .

**Remarks:** (1) Unlike the original 2 primary case due to Mahowald, the existence of the odd primary  $\eta_j$ , which is detected by  $h_0h_j$  in the Adams  $E_2$  term, is still unknown.

(2) When  $p = 2$ , Mahowald used  $bo$  to study his  $\eta_j$  family. On the other hand,  $EO_{p-1}$  has been constructed by Hopkins–Miller so as to be a  $p$  primary periodic analogue of  $bo$ .

(3) We hope such a contribution would provide us with a better idea how to prove (or disprove) the existence of odd primary  $\eta_j$ 's. In fact, there is some relationship between  $\eta_j$  and the Kervaire invariant one element (and its odd primary analogue). Furthermore, Hopkins–Miller  $EO_{p-1}$  at least contains the information about Ravenel's odd primary Kervaire invariant one theorem for  $p \geq 5$  and Toda's

differential (which was a starting point in Ravenel’s proof) for  $p \geq 3$ . In this way, this problem might provide us with some clue how to generalize Ravenel’s result to the case  $p = 3$ .

**Cohomology for restricted simple Lie algebras (D. K. NAKANO)**

Let  $\mathfrak{g}$  be a restricted Lie algebra and  $u(\mathfrak{g})$  be the restricted enveloping algebra corresponding to  $\mathfrak{g}$ . Block and Wilson [21] have proven that any restricted simple Lie algebra over an algebraically closed field  $k$  of characteristic larger than 7 is either *classical* or of *Cartan type*. The classical Lie algebras arise as the Lie algebra of a connected reductive algebraic group scheme. For a reductive group scheme  $G$  with  $\text{Lie}(G) = \mathfrak{g}$ , let  $G_1$  be the kernel of the Frobenius morphism. It is well-known that representations for  $G_1$  are the same as looking at representations for  $u(\mathfrak{g})$ . The cohomology ring  $H^\bullet(u(\mathfrak{g}), k)$  has been computed by Friedlander and Parshall [22], and Andersen and Jantzen [18]. They prove that if  $p$  is larger than the Coxeter number of the Weyl group of  $G$ , then  $H^{\text{odd}}(u(\mathfrak{g}), k) = 0$  and  $H^{2\bullet}(G_1, k) \cong H^{2\bullet}(u(\mathfrak{g}), k) \cong k[\mathcal{N}]$ , where  $\mathcal{N}$  is the well-studied variety of nilpotent elements in  $\mathfrak{g}$ . The variety  $\mathcal{N}$  is often referred to as the nullcone.

Almost nothing is known about the cohomology ring  $H^\bullet(u(\mathfrak{g}), k)$  when  $\mathfrak{g}$  is a Lie algebra of Cartan type. The Lie algebras of Cartan type can be divided into four infinite families of algebras denoted by  $W$ ,  $S$ ,  $H$  and  $K$ . Let  $R = k[x]/(x^p = 0)$ . The derivations of  $R$  form a restricted Lie algebra of dimension  $p$  called  $W(1, 1)$ . A basis for  $W(1, 1)$  is given by  $\langle x^i \frac{d}{dx} : i = 0, 1, \dots, p - 1 \rangle$  with Lie multiplication

$$\left[ x^i \frac{d}{dx}, x^j \frac{d}{dx} \right] = (i - j)x^{i+j-1} \frac{d}{dx}.$$

In general the restricted Lie algebras of type  $W$  are denoted by  $W(m, 1)$ , and constructed by taking derivations of  $k[x_1, x_2, \dots, x_m]/(x_j^p = 0 : j = 1, 2, \dots, m)$ . The other families of Cartan type Lie algebras arise as subalgebras of  $W(m, 1)$  which stabilize a specific differential form.

Friedlander and Parshall [23] have shown there exists a finite map  $\Phi^\bullet : S^\bullet(\mathfrak{g}^\#) \rightarrow H^{2\bullet}(u(\mathfrak{g}), k)$  where  $\mathfrak{g}^\#$  is the dual of the Lie algebra  $\mathfrak{g}$ . Let  $|g|_k$  be the maximal ideal spectrum of  $H^{2\bullet}(u(\mathfrak{g}), k)$ . The map  $\Phi^\bullet$  induces a map on varieties:  $\Phi : |g|_k \rightarrow g \cong \mathbb{A}^{\dim_k \mathfrak{g}}$ . We propose the following fundamental problems. After the statement of the problem remarks are given. From this point on,  $\mathfrak{g}$  refers to a Lie algebra of Cartan type.

**28.** Calculate the cohomology ring  $H^\bullet(u(\mathfrak{g}), k)$ . for  $\mathfrak{g}$ .

For  $\mathfrak{g} = W(1, 1)$ , the dimensions of the graded components of  $H^\bullet(u(\mathfrak{g}), k)$  were computed for  $p = 5, 7$  [26]. In particular the Poincaré series for  $H^\bullet(u(\mathfrak{g}), k)$  for  $p = 5$  is given by the formula

$$P(t) = \sum_{n=0}^{\infty} \frac{1}{6}(2n + 2)(4n^2 + 2n + 3) t^{2n}.$$

The obstruction in generalizing this result for larger primes is in the calculation of  $H^\bullet(u(\mathfrak{b}^+), k)$  where  $\mathfrak{b}^+ = \langle x^i \frac{d}{dx} : i = 1, \dots, p - 1 \rangle$ . Evidence via calculations indicates that this computation will become increasingly difficult as  $p$  gets large. For classical Lie algebras, the computation of the cohomology ring depends heavily on using results about the ambient algebraic group. A formal setting using algebraic groups is developed for Lie algebras of Cartan type in [24], [25]. This might prove

to be useful in attacking this problem.

**29.** Calculate the Krull dimension of the ring  $H^\bullet(u(\mathfrak{g}), k)$ .

**30.** Determine when cohomology ring  $H^\bullet(u(\mathfrak{g}), k)$  is Cohen–Macaulay.

In case when  $\mathfrak{g} = W(1, \mathbf{1})$ , it was shown that the Krull dimension of  $H^\bullet(u(\mathfrak{g}), k)$  is  $p - 1$ . Recall that  $\dim_k W(1, \mathbf{1}) = p$  and the dimension of a maximal torus is 1. In the classical case the dimension of  $\mathcal{N}$  is the dimension of the Lie algebra minus the dimension of a maximal torus. One is tempted to conjecture that the Krull dimension of the cohomology ring for Cartan type Lie algebras (perhaps with a lower bound on the prime) will also be equal to the dimension of the Lie algebra minus the dimension of a maximal torus. With information about the Poincaré series of the cohomology ring, one might want to look at the Cohen–Macaulay question using ideas given in [19], [20]. One should note that the cohomology ring for classical Lie algebras is Cohen–Macaulay by using the aforementioned results.

**31.** Give a nice geometric description of  $\Phi(|\mathfrak{g}|_k) \subset \mathfrak{g}$ . When is this variety irreducible?

Once again we will return to the example when  $\mathfrak{g} = W(1, \mathbf{1})$ . Let  $G = \text{Aut}(\mathfrak{g})$ . One can show that  $G$  is the semidirect product of a one-dimensional torus  $T$  and a unipotent algebraic group  $U$ . In [24], it was shown that  $\Phi(|\mathfrak{g}|_k) = G \cdot e_{-1}$  where  $e_{-1} = \frac{d}{dx}$ . Moreover, this variety is irreducible. One might wonder if the variety  $\Phi(|\mathfrak{g}|_k)$  can be expressed as the  $G$ -closure of some nilpotent element for other Lie algebras of Cartan type.

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### Counting characters (G. R. ROBINSON)

For a fixed prime  $p$ , a finite group  $G$  and a non-negative integer  $d$ , let  $k_d(G)$  denote the number of irreducible ordinary characters  $\chi$  of  $G$  such that  $p^d \chi(1)_p = |G|_p$ .

**32.** Is there a “natural” complex  $\mathcal{C}$  on which  $G$  acts such that  $\dim_F(H_n(\mathcal{C})) = k_n(G)$  for each non-negative integer  $n$ , where  $F$  is a suitable field?

**Possible Evidence:** Let  $G, H$  be finite groups. Note that

$$k_d(G \times H) = \sum_{i=0}^d k_i(G)k_{d-i}(H).$$

This suggests that if  $\mathcal{C}(G)$  and  $\mathcal{C}(H)$  exist, perhaps  $\mathcal{C}(G \times H)$  is the product  $\mathcal{C}(G) \times \mathcal{C}(H)$  (up to suitable homotopy).

### Transfers in Morava $K$ -theory (N. STRICKLAND)

Let  $\mathbb{G}$  be the formal group associated to  $K(n)$ . Formally,  $\mathbb{G}$  is the functor from algebras over  $K(n)^*$  to abelian groups defined by

$$\mathbb{G}(R) = \varinjlim_k \text{Hom}_{K(n)^*}(K(n)^*CP^k, R).$$

For many interesting spaces  $X$  (including  $BU(k), MU(k), K(\mathbb{Z}, k), \Omega S^{2k+1}$ ) there is a simple conceptual interpretation of the functor represented by  $K(n)^*X$  in terms of  $\mathbb{G}$ . In particular, with suitable definitions,  $K(n)^*BA$  represents  $\text{Hom}(A^*, \mathbb{G})$  (where  $A$  is a finite Abelian group with Pontrjagin dual  $A^* = \text{Hom}(A, S^1)$ ) and  $K(n)^*B\Sigma_{p^k}/$  (transfers from partition subgroups) represents the scheme  $\text{Sub}_k(\mathbb{G})$  of subgroups of  $\mathbb{G}$  of order  $p^k$ .

To extend this to  $K(n)^*BH$  for more general finite groups  $H$ , it seems necessary to have a conceptual understanding of transfers.

**33.** Give a purely algebraic construction of transfers between the rings representing  $\text{Hom}(A^*, \mathbb{G})$  and  $\text{Hom}(B^*, \mathbb{G})$  when  $B \leq A$ . This should be visibly independent of any choice of coordinate on  $\mathbb{G}$  or decomposition of  $A, B$  or  $B/A$  as a product of cyclic groups.

In the Greenlees–May theory of Tate spectra, it turns out that  $(t_H K(n))^H = 0$ . It follows that there is a natural isomorphism  $K(n)_*BH \cong K(n)^*BH$  of modules over  $K(n)^*BH$ , making  $K(n)^*BH$  into a Poincaré duality algebra over  $K(n)^*$ . Greenlees and May never write down explicitly what the map is, but I suspect that it is given by a slant product with  $\text{tr}_{\Delta}^{H^2}(1) \in K(n)^*BH \otimes K(n)^*BH$  (where  $\Delta \leq H^2$  is the diagonal subgroup). A good understanding of transfers should explain why the resulting map  $K(n)_*BH \rightarrow K(n)^*BH$  is an isomorphism.

### The Kriz group (N. STRICKLAND)

Igor Kriz has proved that  $K(2)^*B(U_4\mathbb{F}_3)$  is not concentrated in even degrees (where  $U_4\mathbb{F}_3$  is the group of  $4 \times 4$  upper triangular matrices over  $\mathbb{F}_3$  with ones on the diagonal, and  $K(2)$  is defined at  $p = 3$ ). However, Kriz does not compute the full answer, he merely exhibits an element in odd degree.

**34.** Compute  $K(2)^*B(U_4\mathbb{F}_3)$  completely. Can the odd degree elements be described as Massey products?

### Finite rings (N. STRICKLAND)

Let  $R$  be a finite  $p$ -local ring. The general structure theory of such rings says that  $R/\text{Rad}(R) \cong \prod_{k,l} M_k(\mathbb{F}_{p^l})^{n_{kl}}$  where  $n_{kl} = 0$  for almost all  $(k, l)$  (Here we have used Wedderburn's theorem, that a finite division ring is a field). For want of a better word, I'll say that  $R$  is unextended if  $n_{kl} = 0$  whenever  $l > 1$ , so  $R/\text{Rad}(R)$  is a product of matrix algebras over  $\mathbb{F}_p$ .

**35.** Let  $X$  be a finite  $p$ -torsion spectrum, so that  $[X, X]$  is a finite ring. Is this always unextended? Are there elementary ways to test whether a ring is unextended? Note that a commutative ring  $R$  is unextended if and only if the sequence  $\{a^{p^k}\}$  is eventually constant for all  $a \in R$ , but the fact that  $\mathbb{F}_{p^n} \subseteq M_n(\mathbb{F}_p)$  prevents any simple-minded extension of this.

### Spectra and $Q$ -algebras (N. STRICKLAND)

Let  $\mathcal{T}$  be the category of pointed, compactly generated, weakly Hausdorff spaces. Consider the functor  $Q : \mathcal{T} \rightarrow \mathcal{T}$  defined by  $QX = \varinjlim_n \Omega^n \Sigma^n X$ . It is important here that we use  $Q$  itself (not one of the combinatorial approximations to  $Q$ ), and that we regard it as a functor on  $\mathcal{T}$  rather than the homotopy category. The functor  $Q$  is a monad, so we can consider the category  $\mathcal{T}^Q$  of  $Q$ -algebras. We can also consider the category  $\mathcal{S}$  of spectra (over the universe  $\mathbb{R}^\infty$ ) as defined by Lewis and May. It is easy to see that  $\Omega^\infty$  gives a functor  $\mathcal{S} \rightarrow \mathcal{T}^Q$ . I have explained to a number of people a proof that  $\Omega^\infty : \mathcal{S} \rightarrow \mathcal{T}^Q$  is full and faithful.

**36.** Is this functor an equivalence of categories?

### Tensor induction for Mackey functors (T. YOSHIDA)

**37.** Define tensor induction for Mackey functors. Let  $G$  be a finite group and  $H$  a subgroup of  $G$ . Then for a  $kH$ -module  $L$ , we have a tensor induced  $kG$ -module  $\text{Jnd}_H^G(L)$ . Tensor induction preserves tensor product, but not direct sum. It is natural to expect that Mackey functors have also tensor induction. The tensor induction has to preserve tensor product of Mackey functors and furthermore to satisfy Tambara's axiom for multiplicative transfer.

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### $G$ -sets and functors (T. YOSHIDA)

Let  $\mathbf{Sp}(G)$  and  $\mathbf{Binom}$  be the 2-categories of spans and bimodules, respectively [33]. Here, an object of the 2-category  $\mathbf{Sp}(G)$  is a finite  $G$ -set, a 1-morphism from  $Y$  to  $X$  is a pair of  $G$ -maps ( $X \leftarrow A \rightarrow Y$ ), and a 2-morphism from  $(X \leftarrow A \rightarrow Y)$  to  $(X \leftarrow A' \rightarrow Y)$  is a  $G$ -map from  $A$  to  $A'$  which makes two triangles commutative. We are familiar to an example similar to the above problem, that is, a 2-functor from  $\mathbf{Sp}(G)$  to  $\mathbf{CATS}$ , the 2-category of all categories. In fact, the assignment

$$\begin{aligned} G/H &\longmapsto \mathbf{Mod}_{kH}, \\ (G/H \leftarrow G/A \rightarrow G/K) &\longmapsto (\mathbf{Mod}_{kH} \longrightarrow \mathbf{Mod}_{kA} \longrightarrow \mathbf{Mod}_{kK}) \end{aligned}$$

gives a 2-functor.

**38.** Build the theory of (*non-monoidal*) 2-functors from  $\mathbf{Sp}(G)$  to  $\mathbf{Binom}$ . More generally, build (2-)representation theory of a monoidal (2-)category.

The Mackey algebra  $\mu_R(G)$  is the path algebra of  $\mathbf{Sp}(G)$ . A representation of  $\mathbf{Sp}(G)$  is nothing but a Mackey functor. Furthermore, there is a tensor product of two Mackey functors, so it is natural to expect that the representation category  $\mathbf{Sp}(G)$  has the following property.

**39.** Is the category  $\mathbf{Sp}(G)$  of spans of a finite group  $G$  (and the Mackey ring  $\mu_R(G)$ ) Morita equivalent to a Hopf algebra? (Mackey Hopf algebra!)

**40.** Let  $G$  be a finite group and  $S$  a finite  $G$ -monoid. Under what conditions is the monoidal category of crossed  $G$ -sets over  $S$  symmetric or braided?

A crossed  $G$ -set  $X$  is a finite  $G$ -set equipped with a  $G$ -map called a weight function  $X \rightarrow S; x \mapsto \|x\|$ . The tensor product  $X \otimes Y$  of crossed  $G$ -sets  $X, Y$  is the cartesian product  $X \times Y$  with weight  $\|(x, y)\| := \|x\| \cdot \|y\|$ . The monoidal category of crossed  $G$ -sets is braided if  $S$  is commutative or  $S = G^c$ , the  $G$ -monoid  $G$  with  $G$ -action defined by  $G$ -conjugation.

If the monoidal category is braided, then  $(\mathcal{C}[C_S(H)])^{N_G(H)}$ , the ring of  $N_G(H)$ -fixed points in the semigroup algebra  $\mathcal{C}[C_S(H)]$ , is commutative for any subgroup  $H$  of  $G$ .

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### The Dijkgraaf–Witten invariant (T. YOSHIDA)

Let  $M$  be a closed oriented  $d$ -manifold,  $G$  a finite group, and  $[\alpha] \in H^3(G, U(1))$ . The Dijkgraaf–Witten invariant is defined as follows:

$$Z_{G,\alpha}(M) := \frac{1}{|G|} \sum_{\gamma: \pi_1(M) \rightarrow G} \langle \gamma^*(\alpha), [M] \rangle,$$

where  $\gamma^* : H^3(G, U(1)) \rightarrow H^3(M, U(1))$  is the map induced by  $\gamma$ . In particular, when  $\alpha = 1$ ,

$$Z_{G,1}(M) = \frac{|\mathrm{Hom}(\pi_1(M), G)|}{|G|},$$

and so this is defined for any  $d$ -manifold  $M$ .

We have the *integrality conjecture*:

$$|G| Z_{G,1}(M) \equiv 0 \pmod{\mathrm{gcd}(|H_1(M)|/|G|H_1(M)|, |G|)}.$$

Perhaps, this conjecture will be solved if  $M$  is a 3-manifold because its fundamental group has some special properties.

**41.** Let  $M$  be a closed oriented 3-manifold. Find an integer valued function  $f(m, n)$  for which

$$|G| Z_{G, \alpha}(M) \equiv 0 \pmod{f(|H_1(M)|, |G|)}.$$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GEORGIA, ATHENS GA 30606, USA  
*E-mail address:* `djb@byrd.math.uga.edu`

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