

Proceedings of Symposia in PURE MATHEMATICS

Volume 74

Problems on Mapping Class Groups and Related Topics

Benson Farb
Editor



American Mathematical Society

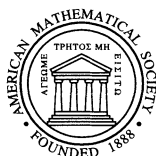
Problems on Mapping Class Groups and Related Topics

Proceedings of Symposia in PURE MATHEMATICS

Volume 74

Problems on Mapping Class Groups and Related Topics

Benson Farb
Editor



American Mathematical Society
Providence, Rhode Island

2000 *Mathematics Subject Classification*. Primary 58D29, 20F38, 30F60, 14D22, 57M99, 20F34, 20F36.

Library of Congress Cataloging-in-Publication Data

Problems on mapping class groups and related topics / Benson Farb, editor.

p. cm. — (Proceedings of symposia in pure mathematics ; v. 74)

Includes bibliographical references.

ISBN-13: 978-0-8218-3838-9 (alk. paper)

ISBN-10: 0-8218-3838-5 (alk. paper)

1. Mappings (Mathematics)—Congresses. 2. Class groups (Mathematics)—Congresses.

I. Farb, Benson.

QA360.P76 2006

511.3'3—dc22

2006048369

Copying and reprinting. Material in this book may be reproduced by any means for educational and scientific purposes without fee or permission with the exception of reproduction by services that collect fees for delivery of documents and provided that the customary acknowledgment of the source is given. This consent does not extend to other kinds of copying for general distribution, for advertising or promotional purposes, or for resale. Requests for permission for commercial use of material should be addressed to the Acquisitions Department, American Mathematical Society, 201 Charles Street, Providence, Rhode Island 02904-2294, USA. Requests can also be made by e-mail to reprint-permission@ams.org.

Excluded from these provisions is material in articles for which the author holds copyright. In such cases, requests for permission to use or reprint should be addressed directly to the author(s). (Copyright ownership is indicated in the notice in the lower right-hand corner of the first page of each article.)

© 2006 by the American Mathematical Society. All rights reserved.

The American Mathematical Society retains all rights
except those granted to the United States Government.

Copyright of individual articles may revert to the public domain 28 years
after publication. Contact the AMS for copyright status of individual articles.

Printed in the United States of America.

∞ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.

Visit the AMS home page at <http://www.ams.org/>

10 9 8 7 6 5 4 3 2 1 11 10 09 08 07 06

Dye mon, gen mon.

Behind the mountains,
more mountains.

– **Haitian proverb**

Contents

Preface	ix
---------	----

I. Cohomological, Combinatorial and Algebraic Structure

Four Questions about Mapping Class Groups M. BESTVINA	3
Some Problems on Mapping Class Groups and Moduli Space B. FARB	11
Finiteness and Torelli Spaces R. HAIN	57
Fifteen Problems about the Mapping Class Groups N. IVANOV	71
Problems on Homomorphisms of Mapping Class Groups M. KORKMAZ	81
The Mapping Class Group and Homotopy Theory I. MADSEN	91
Probing Mapping Class Groups Using Arcs R. PENNER	97
Relations in the Mapping Class Group B. WAJNRYB	115

II. Connections with 3-manifolds, Symplectic Geometry and Algebraic Geometry

Mapping Class Group Factorizations and Symplectic 4-manifolds: Some Open Problems D. AUROUX	123
The Topology of 3-manifolds, Heegaard Distance and the Mapping Class Group of a 2-manifold J. BIRMAN	133
Lefschetz Pencils and Mapping Class Groups S.K. DONALDSON	151

Open Problems in Grothendieck-Teichmüller Theory P. LOCHAK AND L. SCHNEPS	165
--	-----

III. Geometric and Dynamical Aspects

Mapping Class Group Dynamics on Surface Group Representations W. GOLDMAN	189
Geometric Properties of the Mapping Class Group U. HAMENSTÄDT	215
Problems on Billiards, Flat Surfaces and Translation Surfaces P. HUBERT, H. MASUR, T. SCHMIDT AND A. ZORICH	233
Problems in the Geometry of Surface Group Extensions L. MOSHER	245
Surface Subgroups of Mapping Class Groups A.W. REID	257
Weil-Petersson Perspectives S. WOLPERT	269

IV. Braid Groups, $\text{Out}(F_n)$ and other Related Groups

Braid Groups and Iwahori-Heck Algebras S. BIGELOW	285
Automorphism Groups of Free Groups, Surface Groups and Free Abelian Groups M. BRIDSON AND K. VOGTMANN	301
Problems: Braid Groups, Homotopy, Cohomology and Representations F.R. COHEN	317
Cohomological Structure of the Mapping Class Group and Beyond S. MORITA	329
From Braid Groups to Mapping Class Groups L. PARIS	355

Preface

The study of mapping class groups, the moduli space of Riemann surfaces, Teichmüller geometry and related areas has seen a recent influx of young mathematicians. Inspired by this, I had the idea to solicit from some of the senior people in the area papers that would focus primarily on open problems and directions. I proposed that these problems might range in scope from specific computations to broad programs. The idea was then to bring these papers together into one source, most likely a book. This book would then be a convenient location where younger (and indeed all) researchers could go in order to find problems that might inspire them to further work. I was especially interested in having problems formulated explicitly and accessibly. The result is this book.

The appearance of mapping class groups in mathematics is ubiquitous; choosing topics to cover seemed an overwhelming task. In the end I chose to solicit papers which would likely focus on those aspects of the topic most deeply connected with geometric topology, combinatorial group theory, and surrounding areas.

Content. For organizational purposes the papers here are divided into four groups. This division is by necessity somewhat arbitrary, and a number of the papers could just as easily have been grouped differently.

The problems discussed in Part I focus on the combinatorial and (co)homological group-theoretic aspects of mapping class groups, and the way in which these relate to problems in geometry and topology. The most remarkable recent success in this direction has been the proof by Madsen and Weiss of the Morita-Mumford-Miller Conjecture on the stable cohomology of mapping class groups. Further problems arising from this work are described in Madsen's paper. Other cohomological aspects, including those related to various subgroups, most notably the Torelli group, are discussed in the papers of Bestvina and Hain. The combinatorial and geometric group theory of mapping class groups admits a rich and interesting structure. Ideas and problems coming out of this point of view are discussed in the papers of Farb, Ivanov, Korkmaz, Penner and Wajnryb.

Part II concentrates on connections between various classification problems in topology and their combinatorial reduction to (still open) problems about mapping class groups. In dimension three this reduction is classical. It arises from the fact that every 3-manifold is a union of two handlebodies glued along their boundaries. This construction and many of the problems arising from it are described in Birman's paper. The reduction of the classification of 4-dimensional symplectic manifolds to purely combinatorial topological questions about surfaces and mapping class groups is more recent. The general idea is that (by a theorem of Donaldson) each closed symplectic 4-manifold admits a *symplectic Lefschetz pencil*. These are a kind of "fibration with singularities", and the main piece of data that

determines a Lefschetz pencil is its monodromy, which is a collection of mapping classes. These ideas and a number of problems arising from them are presented in the papers of Auroux and Donaldson. Finally, connections with algebraic geometry and number theory via Grothendieck-Teichmüller theory are given in the paper of Lochak and Schneps. One can begin to see this connection, for example, in Belyi's theorem that a complex algebraic curve is defined over $\bar{\mathbf{Q}}$ if and only if it is a branched cover over S^2 branched only over $\{0, 1, \infty\}$.

A wide variety of problems, from understanding billiard trajectories to the classification of Kleinian groups, can be reduced to differential and synthetic geometry problems about moduli space. Such problems and connections are discussed in Part III in the papers of Hamenstädt, Mosher, Reid and Wolpert. Those with heavily dynamical flavor are discussed in the papers of Goldman and Hubert, Masur, Schmidt and Zorich.

Mapping class groups are related, both concretely and philosophically, to a number of other groups. While braid groups are technically a special example of a type of mapping class group, the study of these groups has its own special flavor, and in most instances much more is known in this case. The papers of Bigelow, Cohen and Paris concentrate on problems related to braid groups. There has also been a long-running analogy between mapping class groups, linear groups, and automorphism groups of free groups. Problems relating to this analogy are explored in the papers of Bridson and Vogtmann and Morita.

Acknowledgements. The entire content of this book is due to the authors of the individual papers. I feel privileged to have edited a collection of papers from such experts. It is a great pleasure to thank them for their generosity and their time, not to mention their willingness to openly share their ideas and viewpoints. I must admit my surprise at how little nagging I had to do to complete this project; indeed, every single paper in this volume (except, I must admit, mine) was completed in a timely manner. I hope and believe that the visions and problems shared by these authors here will have a significant influence on the development of the field.

I would like to thank Sergei Gelfand for his continued support, and for his prodding, without which this project would not have been completed.

Titles in This Series

- 74 **Benson Farb, Editor**, Problems on mapping class groups and related topics, 2006
- 73 **Mikhail Lyubich and Leon Takhtajan, Editors**, Graphs and patterns in mathematics and theoretical physics (Stony Brook University, Stony Brook, NY, June 14–21, 2001)
- 72 **Michel L. Lapidus and Machiel van Frankenhuysen, Editors**, Fractal geometry and applications: A jubilee of Benoît Mandelbrot, Parts 1 and 2 (San Diego, California, 2002 and École Normale Supérieure de Lyon, 2001)
- 71 **Gordana Matić and Clint McCrory, Editors**, Topology and Geometry of Manifolds (University of Georgia, Athens, Georgia, 2001)
- 70 **Michael D. Fried and Yasutaka Ihara, Editors**, Arithmetic fundamental groups and noncommutative algebra (Mathematical Sciences Research Institute, Berkeley, California, 1999)
- 69 **Anatole Katok, Rafael de la Llave, Yakov Pesin, and Howard Weiss, Editors**, Smooth ergodic theory and its applications (University of Washington, Seattle, 1999)
- 68 **Robert S. Doran and V. S. Varadarajan, Editors**, The mathematical legacy of Harish-Chandra: A celebration of representation theory and harmonic analysis (Baltimore, Maryland, 1998)
- 67 **Wayne Raskind and Charles Weibel, Editors**, Algebraic K -theory (University of Washington, Seattle, 1997)
- 66 **Robert S. Doran, Ze-Li Dou, and George T. Gilbert, Editors**, Automorphic forms, automorphic representations, and arithmetic (Texas Christian University, Fort Worth, 1996)
- 65 **M. Giaquinta, J. Shatah, and S. R. S. Varadhan, Editors**, Differential equations: La Pietra 1996 (Villa La Pietra, Florence, Italy, 1996)
- 64 **G. Ferreyra, R. Gardner, H. Hermes, and H. Sussmann, Editors**, Differential geometry and control (University of Colorado, Boulder, 1997)
- 63 **Alejandro Adem, Jon Carlson, Stewart Priddy, and Peter Webb, Editors**, Group representations: Cohomology, group actions and topology (University of Washington, Seattle, 1996)
- 62 **János Kollár, Robert Lazarsfeld, and David R. Morrison, Editors**, Algebraic geometry—Santa Cruz 1995 (University of California, Santa Cruz, July 1995)
- 61 **T. N. Bailey and A. W. Knap, Editors**, Representation theory and automorphic forms (International Centre for Mathematical Sciences, Edinburgh, Scotland, March 1996)
- 60 **David Jerison, I. M. Singer, and Daniel W. Stroock, Editors**, The legacy of Norbert Wiener: A centennial symposium (Massachusetts Institute of Technology, Cambridge, October 1994)
- 59 **William Arveson, Thomas Branson, and Irving Segal, Editors**, Quantization, nonlinear partial differential equations, and operator algebra (Massachusetts Institute of Technology, Cambridge, June 1994)
- 58 **Bill Jacob and Alex Rosenberg, Editors**, K -theory and algebraic geometry: Connections with quadratic forms and division algebras (University of California, Santa Barbara, July 1992)
- 57 **Michael C. Cranston and Mark A. Pinsky, Editors**, Stochastic analysis (Cornell University, Ithaca, July 1993)
- 56 **William J. Haboush and Brian J. Parshall, Editors**, Algebraic groups and their generalizations (Pennsylvania State University, University Park, July 1991)
- 55 **Uwe Jannsen, Steven L. Kleiman, and Jean-Pierre Serre, Editors**, Motives (University of Washington, Seattle, July/August 1991)

For a complete list of titles in this series, visit the
AMS Bookstore at www.ams.org/bookstore/.

