Problems on Mapping Class Groups and Related Topics

Benson Farb
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Problems on 
Mapping Class Groups 
and Related Topics

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Editor

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Providence, Rhode Island
Dye mon, gen mon.

Behind the mountains,
more mountains.

– Haitian proverb
## Contents

Preface ix

### I. Cohomological, Combinatorial and Algebraic Structure

- **Four Questions about Mapping Class Groups**
  M. Bestvina 3

- **Some Problems on Mapping Class Groups and Moduli Space**
  B. Farb 11

- **Finiteness and Torelli Spaces**
  R. Hain 57

- **Fifteen Problems about the Mapping Class Groups**
  N. Ivanov 71

- **Problems on Homomorphisms of Mapping Class Groups**
  M. Korkmaz 81

- **The Mapping Class Group and Homotopy Theory**
  I. Madsen 91

- **Probing Mapping Class Groups Using Arcs**
  R. Penner 97

- **Relations in the Mapping Class Group**
  B. Wajnryb 115

### II. Connections with 3-manifolds, Symplectic Geometry and Algebraic Geometry

- **Mapping Class Group Factorizations and Symplectic 4-manifolds: Some Open Problems**
  D. Auroux 123

- **The Topology of 3-manifolds, Heegaard Distance and the Mapping Class Group of a 2-manifold**
  J. Birman 133

- **Lefschetz Pencils and Mapping Class Groups**
  S.K. Donaldson 151
Open Problems in Grothendieck-Teichmüller Theory
P. Lochak and L. Schneps

III. Geometric and Dynamical Aspects

Mapping Class Group Dynamics on Surface Group Representations
W. Goldman

Geometric Properties of the Mapping Class Group
U. Hamenstät

Problems on Billiards, Flat Surfaces and Translation Surfaces
P. Hubert, H. Masur, T. Schmidt and A. Zorich

Problems in the Geometry of Surface Group Extensions
L. Mosher

Surface Subgroups of Mapping Class Groups
A.W. Reid

Weil-Petersson Perspectives
S. Wolpert

IV. Braid Groups, \text{Out}(F_n) and other Related Groups

Braid Groups and Iwahori-Heck Algebras
S. Bigelow

Automorphism Groups of Free Groups, Surface Groups and Free Abelian Groups
M. Bridson and K. Vogtmann

Problems: Braid Groups, Homotopy, Cohomology and Representations
F.R. Cohen

Cohomological Structure of the Mapping Class Group and Beyond
S. Morita

From Braid Groups to Mapping Class Groups
L. Paris
Preface

The study of mapping class groups, the moduli space of Riemann surfaces, Teichmüller geometry and related areas has seen a recent influx of young mathematicians. Inspired by this, I had the idea to solicit from some of the senior people in the area papers that would focus primarily on open problems and directions. I proposed that these problems might range in scope from specific computations to broad programs. The idea was then to bring these papers together into one source, most likely a book. This book would then be a convenient location where younger (and indeed all) researchers could go in order to find problems that might inspire them to further work. I was especially interested in having problems formulated explicitly and accessibly. The result is this book.

The appearance of mapping class groups in mathematics is ubiquitous; choosing topics to cover seemed an overwhelming task. In the end I chose to solicit papers which would likely focus on those aspects of the topic most deeply connected with geometric topology, combinatorial group theory, and surrounding areas.

Content. For organizational purposes the papers here are divided into four groups. This division is by necessity somewhat arbitrary, and a number of the papers could just as easily have been grouped differently.

The problems discussed in Part I focus on the combinatorial and (co)homological group-theoretic aspects of mapping class groups, and the way in which these relate to problems in geometry and topology. The most remarkable recent success in this direction has been the proof by Madsen and Weiss of the Morita-Mumford-Miller Conjecture on the stable cohomology of mapping class groups. Further problems arising from this work are described in Madsen’s paper. Other cohomological aspects, including those related to various subgroups, most notably the Torelli group, are discussed in the papers of Bestvina and Hain. The combinatorial and geometric group theory of mapping class groups admits a rich and interesting structure. Ideas and problems coming out of this point of view are discussed in the papers of Farb, Ivanov, Korkmaz, Penner and Wajnryb.

Part II concentrates on connections between various classification problems in topology and their combinatorial reduction to (still open) problems about mapping class groups. In dimension three this reduction is classical. It arises from the fact that every 3-manifold is a union of two handlebodies glued along their boundaries. This construction and many of the problems arising from it are described in Birman’s paper. The reduction of the classification of 4-dimensional symplectic manifolds to purely combinatorial topological questions about surfaces and mapping class groups is more recent. The general idea is that (by a theorem of Donaldson) each closed symplectic 4-manifold admits a symplectic Lefschetz pencil. These are a kind of “fibration with singularities”, and the main piece of data that
determines a Lefschetz pencil is its monodromy, which is a collection of mapping classes. These ideas and a number of problems arising from them are presented in the papers of Auroux and Donaldson. Finally, connections with algebraic geometry and number theory via Grothendieck-Teichmüller theory are given in the paper of Lochak and Schneps. One can begin to see this connection, for example, in Belyi’s theorem that a complex algebraic curve is defined over $\mathbb{Q}$ if and only if it is a branched cover over $S^2$ branched only over $\{0, 1, \infty\}$.

A wide variety of problems, from understanding billiard trajectories to the classification of Kleinian groups, can be reduced to differential and synthetic geometry problems about moduli space. Such problems and connections are discussed in Part III in the papers of Hamenstädt, Mosher, Reid and Wolpert. Those with heavily dynamical flavor are discussed in the papers of Goldman and Hubert, Masur, Schmidt and Zorich.

Mapping class groups are related, both concretely and philosophically, to a number of other groups. While braid groups are technically a special example of a type of mapping class group, the study of these groups has its own special flavor, and in most instances much more is known in this case. The papers of Bigelow, Cohen and Paris concentrate on problems related to braid groups. There has also been a long-running analogy between mapping class groups, linear groups, and automorphism groups of free groups. Problems relating to this analogy are explored in the papers of Bridson and Vogtmann and Morita.

Acknowledgements. The entire content of this book is due to the authors of the individual papers. I feel priviledged to have edited a collection of papers from such experts. It is a great pleasure to thank them for their generosity and their time, not to mention their willingness to openly share their ideas and viewpoints. I must admit my surprise at how little nagging I had to do to complete this project; indeed, every single paper in this volume (except, I must admit, mine) was completed in a timely manner. I hope and believe that the visions and problems shared by these authors here will have a significant influence on the development of the field.

I would like to thank Sergei Gelfand for his continued support, and for his prodding, without which this project would not have been completed.
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