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Recent Developments in Lie Algebras, Groups and Representation Theory

2009–2011 Southeastern Lie Theory Workshop Series

Combinatorial Lie Theory and Applications
October 9–11, 2009, North Carolina State University

Homological Methods in Representation Theory
May 22–24, 2010, University of Georgia

Finite and Algebraic Groups
June 1–4, 2011, University of Virginia

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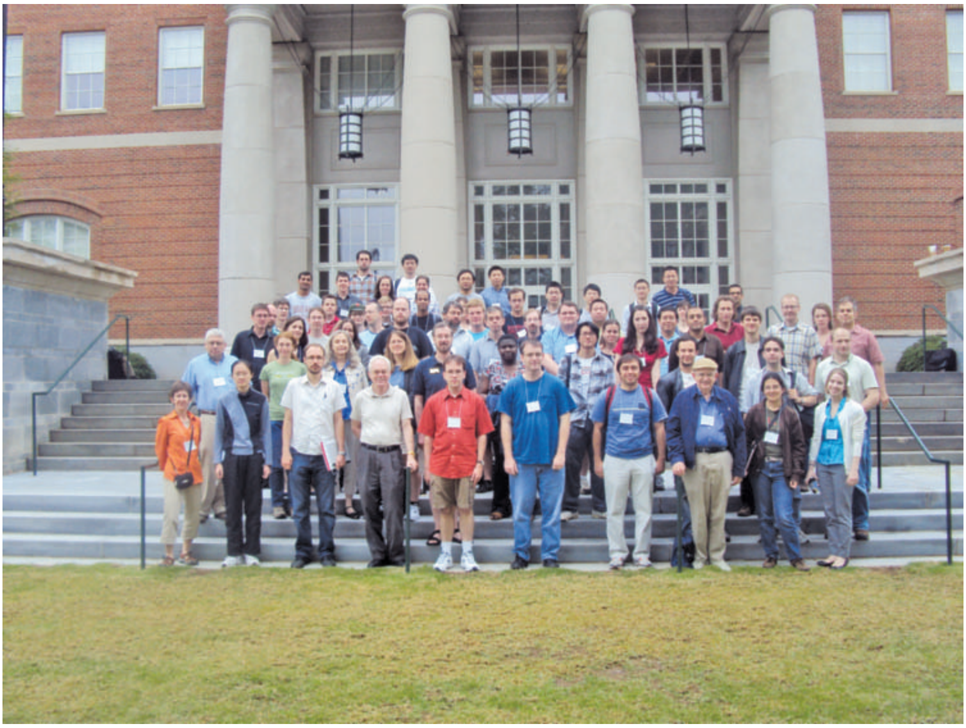

University of Georgia, May 2010

University of Virginia, June 2011

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Preface

Lie theory represents a major area of mathematical research. Besides its increasing importance within mathematics (to geometry, combinatorics, finite and infinite groups, etc.), it has important applications outside of mathematics (to physics, computer science, etc.).

During the twentieth century, the theory of Lie algebras, both finite and infinite dimensional, has been a major area of mathematical research with numerous applications. In particular, during the late 1970s and early 1980s, the representation theory of Kac-Moody Lie algebras (analogs of finite dimensional semisimple Lie algebras) generated intense interest. In part, the subject was driven by its interesting connections with such topics as combinatorics, group theory, number theory, partial differential equations, topology and with areas of physics such as conformal field theory, statistical mechanics, and integrable systems. The representation theory of an important class of infinite dimensional Lie algebras known as affine Lie algebras led to the discovery of Vertex Operator Algebras (VOAs) in the 1980s. VOAs are precise algebraic counterparts to "chiral algebras" in two-dimensional conformal field theory as formalized by Belavin, Polyakov, and Zamolodchikov. These algebras and their representations play important roles in a number of areas, including the representation theory of the Fischer-Griess Monster finite simple group and the connection with the phenomena of "monstrous moonshine," the representation theory of the Virasoro algebra and affine Lie algebras, and two-dimensional conformal field theory.

In 1985, the interaction of affine Lie algebras with integrable systems led Drinfeld and Jimbo to introduce a new class of algebraic objects known as quantized universal enveloping algebras (also called quantum groups) associated with symmetrizable Kac-Moody Lie algebras. These are q-deformations of the universal enveloping algebras of the corresponding Kac-Moody Lie algebras, and, like universal enveloping algebras, they carry an important Hopf algebra structure. The abstract theory of integrable representations of quantum groups, developed by Lusztig, illustrates the similarity between quantum groups and Kac-Moody Lie algebras. The quantum groups associated with finite dimensional simple Lie algebras also have strong connections with the representations of affine Lie algebras. The theory of canonical bases for quantum groups has provided deep insights into the representation theory of quantum groups. More recently, the theory of geometric crystals introduced by Berenstein and Kazhdan has opened new doors in representation theory. In particular, canonical bases at $q = 0$ (crystal bases) provide a beautiful combinatorial tool for studying the representations of quantum groups. The quantized universal enveloping algebra associated with an affine Lie algebra is called a quantum affine algebra. Quantum affine algebras quickly became an interesting and important topic of research, the representation theory of which parallels that of the corresponding affine Lie algebras. But the theory is much deeper and richer than its classical counterpart, providing a clearer picture of connections with the other areas mentioned above.

After the classification of the finite simple groups (now complete), a full understanding of the representation theory of finite simple groups over fields k of arbitrary characteristic provides a major problem for the 21st century. The sporadic Fischer-Griess monster (mentioned above) gives one important example of a finite simple group closely related to Lie theory. Apart from the alternating groups and the 26 sporadic simple groups, the finite simple groups come in infinite families closely related to the finite groups of rational points $G(q)$ of simple algebraic groups G over algebraically closed fields k of positive characteristic $p > 0$. (The finite Ree and Suzuki groups are variations on this theme.) The representation theory of these finite groups of Lie type thus form a key area of investigation. One can consider a field F, algebraically closed for simplicity, having characteristic ℓ, and investigate the category of $FG(q)$-representations. There are three cases to consider.

First, in case $\ell = 0$, take $F = \mathbb{C}$, the complex numbers. This theory is the so-called ordinary representation theory of $G(q)$. As a result of work of Deligne, Lusztig, and many other mathematicians over the past 35 years, the ordinary theory is quite well understood in comparison to the cases in which $\ell > 0$.

Second, if $\ell = p$ (the equal characteristic case), take $F = k$. By work of Steinberg, the irreducible $kG(q)$-modules all lift to irreducible rational representations of the algebraic group G. This fact has provided strong motivation for the study of the modular representation theory of the semisimple algebraic groups G over the past 30 years. For example, a famous conjecture due to Lusztig posits the characters of the irreducible representations when the characteristic p is large (bigger than the Coxeter number). For each type, this conjecture has been proved for p "large enough" by Andersen-Jantzen-Soergel. The proof follows a path from characteristic p to quantum groups at a root of unity to affine Lie algebras and perverse sheaves. Thus, it ultimately involves the infinite dimensional Lie theory discussed above. Although this approach fails to provide effective bounds on the size of the prime p, a new avenue via a related combinatorial category has been recently investigated by Fiebig. As a result of Fiebig's work, very large effective bounds for Lusztig's conjecture are now known. In addition, the determination of the characters for small p (i.e., less than the Coxeter number) remains largely uninvestigated.

Third, when $0 < \ell \neq p$ (the cross-characteristic case), much less is known in general. When G is a general linear group $GL_n(k)$, the determination of the decomposition numbers for the finite groups $GL_n(q)$ can be determined in terms of decomposition numbers for q-Schur algebras and then for quantum groups over fields of positive characteristic. This is the so-called Dipper-James theory. There are close connections with the representation theory of Hecke algebras and symmetric groups. In other types, much less is known; for example, the classification of the irreducible representations is incomplete. A major problem for these other types would be to replace the quantum groups used for $GL_n(q)$ by some suitable structure.

The modular representation theory has provided a crucial interface with the theory of finite dimensional algebras (especially, the theory of quasi-hereditary algebras introduced by Cline, Parshall and Scott). It seems likely that this direction will continue to prove fruitful. Another significant feature of the modular representation theory of the finite groups of Lie type and the associated algebraic and quantum groups is the existence of a rich accompanying homological theory. Homological problems emerge immediately because of the failure of complete reducibility. In the equal characteristic case, the homological theory has been extensively developed, for the finite groups of Lie type, quantum enveloping algebras at roots of unity, restricted Lie algebras and infinitesimal group schemes, as well as other settings. Geometric ideas enter via the theory of support varieties, which associate to each finite-dimensional module for a restricted Lie algebra (or finite group scheme) an algebraic variety. In the cross-characteristic case, much less in known about the cohomology. In the equal characteristic case, there is a considerable body of work involving the homological algebra of the infinitesimal groups, and relations between the cohomology of G, its infinitesimal subgroups, and its finite subgroups.

Finally, we mention that the modular representation theory of general finite groups itself has a strong Lie-theoretic flavor. In part, this is due to the famous Alperin conjecture, suggesting that the irreducible modular representations of general finite group should be classified in a "weight theoretic" way, much like irreducible modules for a complex semisimple Lie algebra are classified by their highest weights. Another notable conjecture, the Broué conjecture, has been recently verified for symmetric groups by Chuang and Rouquier using a the new method of "*categorification*".

In 2009, the three editors established a network of Lie theorists in the southeastern region of the U.S. and proposed an annual regional workshop series of 3 to 4 days in Lie theory. The aim of these workshops was to bring together senior and junior researchers as well as graduate students to build and foster cohesive research groups in the region. With support from the National Science Foundation and the affiliated universities in the region, three successful workshops were held at North Carolina State University, the University of Georgia and the University of Virginia in 2009, 2010 and 2011 respectively. Each of these workshops was attended by over 70 participants. The workshops included expository talks by senior researchers and afternoon AIM style discussion sessions with a goal to educate graduate students and junior researchers in the early part of their study for research in different aspects of Lie theory. In the third workshop at the University of Virginia, Professor Leonard Scott was honored on the eve of his retirement for his lifetime contributions to many of the aforementioned topics.

The plenary speakers in the three workshops were invited to contribute to this proceedings. Most of the articles presented in this book are self-contained, and several survey articles, by Jon Carlson, Jie Du, Bob Griess, and David Hemmer are accessible to a wide audience of readers.

The editors take this opportunity to acknowledge the conference participants, the contributors, and the editorial offices of the American Mathematical Society for making this volume possible.

Kailash C. Misra
Daniel K. Nakano
Brian J. Parshall

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