Proceedings of Symposia in Pure Mathematics
Volume **93**, 2016
http://dx.doi.org/10.1090/pspum/093/01571

Logarithmic invariants of links

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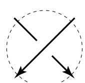
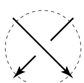
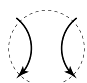
Abstract. We generalize the logarithmic knot invariant for links with several components. We also investigate its relation to the hyperbolic volume of the corresponding cone manifold for the figure-eight knot, the 5_2 knot, the Whitehead link and the Borromean rings.

Introduction

In this report, the logarithmic invariant of knots is generalized to links, and its relation to the hyperbolic volume is also investigated. This relation between the colored Jones invariant and the hyperbolic volume is known as the volume conjecture.

The Jones polynomial is the invariant of knots discovered by V. Jones [**4**], which is defined by the skein relation

$$q^{-1}\, V_{K_+}(q) - q\, V_{K_-}(q) = (q^{1/2} - q^{-1/2})\, V_{K_0}(q),$$

K_+ : , K_- : , K_0 : .

where K_+, K_-, K_0 are the knots which are equal except in the disc where they are as above. This invariant is related to a solvable lattice model and quantum R matrix of the quantum group $\mathcal{U}_q(sl_2)$. By using such relations, the Jones polynomial is generalized to various invariants, which are now called quantum invariants. The colored Jones invariant is one of these invariants corresponding to a finite dimensional irreducible representation of sl_2.

In the mid 1990's, R. Kashaev constructed a new quantum invariant of knots from the quantum dilogarithm, and he observed a relation between his invariant and the hyperbolic volume of the complement of the knot [**5**]. Let N be a positive integer greater than 1, K be a hyperbolic knot and $\langle K \rangle_N$ be Kashaev's invariant. The relation he found is

$$\lim_{N \to \infty} \frac{2\,\pi\, \log |\langle K \rangle_N|}{N} = \mathrm{Vol}_{\mathrm{hyp}}(S^3 \setminus K).$$

Here $S^3 \setminus K$ has a canonical hyperbolic structure and $\mathrm{Vol}_{\mathrm{hyp}}(S^3 \setminus K)$ is the volume with respect to this structure. He observed this relation for the simplest three

2010 *Mathematics Subject Classification.* Primary 57M27; Secondary 51M10, 58J28.
Key words and phrases. Knots, 3-manidfolds, Chern-Simons invariants.
This work was partially supported by JSPS KAKENHI Grant Numbers 90157751, 25610022.

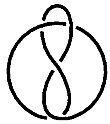
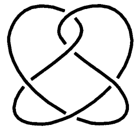
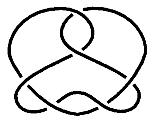
knot		hyperbolic volume	CS invariant
4_1		$2.0298\ldots$	0
5_2		$2.8281\ldots$	$-0.1532\ldots$
6_1		$3.1639\ldots$	$0.1559\ldots$

FIGURE 1. The hyperbolic volumes and the Chern-Simons invariants of the three simplest hyperbolic knots

hyperbolic knots in Figure 1. Kashaev's invariant $\langle K\rangle_N$ turned out to be equal to the colored Jones invariant corresponding to the N dimensional irreducible representation at N-th root of unity, and Kashaev's conjecture is generalized in [**8**] as follows.

CONJECTURE 1. (Complexified volume conjecture.)

$$J_N(K)_{q=\xi_N} \underset{N\to\infty}{\sim} \exp\left(\frac{N}{2\,\pi}\left(\mathrm{Vol}_{\mathrm{hyp}}(S^3\setminus K)+\sqrt{-1}\ \mathrm{CS}(S^3\setminus K)\right)\right),$$

where $J_d(K)$ be the colored Jones invariant corresponding to the d dimensional irreducible representation of sl_2 and $\xi_N=\exp(\pi\sqrt{-1}/N)$. Here J_d is normalized as $J_d(\bigcirc)=1$ for a trivial knot $\bigcirc$.

The above conjecture implies that $J_N(K)_{q=\xi_N}$ grows exponentially with respect to N, while $J_{\lfloor\alpha N\rfloor}(K)$ does not have such a property for $0<\alpha<1$. However, by taking a certain derivative of $J_{\lfloor\alpha N\rfloor}(K)$, we get an invariant which grows exponentially and relates to the hyperbolic volume of certain cone manifold along K. This is the logarithmic invariant we would like to explain.

We first review the logarithmic invariant of knots in Section 1, and then extend it to links in Section 2.

Acknowledgemement. I would like to thank Y. Kabaya for a useful discussion. Mathematica and SnapPy were used for numerical computations.

1. Logarithmic invariants of knots

The logarithmic invariant of knots is introduced in [**10**], and its relation to the colored Jones invariant is given in [**9**]. For a positive integer k, let $V_k(K)$ be the colored Jones invariant without normalization satisfying $V_k(\bigcirc)=d_k$, where d_k is the quantum dimension

$$d_k=(-1)^{k-1}\,\frac{q^k-q^{-k}}{q-q^{-1}}.$$

Note that $V_k(K)=d_k\,J_k(K)$. Sometimes k is called the color of the invariant.

Definition 1.1. Let N be an integer greater than 1 and k be an integer with $0 \le k \le N$. The logarithmic invariant $\gamma_{N,k}(K)$ is defined by

$$\gamma_{N,0}(K) = \left.\frac{V_{2N}(K)}{[2N]}\right|_{q=\xi_N},$$

$$\gamma_{N,N}(K) = \left.\frac{V_{N}(K)}{[N]}\right|_{q=\xi_N},$$

$$\gamma_{N,k}(K) = \left.\frac{\xi_N}{2N}\,\frac{d}{dq}\{1\}\left(V_k(K) + V_{2N-k}(K)\right)\right|_{q=\xi_N}, \quad (1 \le k \le N-1)$$

where $\{n\} = q^n - q^{-n}$, $[n] = \frac{\{n\}}{\{1\}}$ and $\xi_N = \exp(\pi\sqrt{-1}/N)$.

The colored Jones invariant $V_k(K)$ is expressed by Habiro [**2**] as follows.

$$V_m(K) = \sum_{j=0}^{\infty} a_j(K)\,\frac{\{m+j\}\{m+j-1\}\cdots\{m-j\}}{\{1\}}. \tag{1.1}$$

Note that $a_j(K)$ does not depend on k, and the sum is finite since the terms for $j \ge m$ vanish. From (1.1), we have

$$V_k(K)|_{q=\xi_N} + V_{2N-k}(K)|_{q=\xi_N} = 0$$

for $1 \le k \le N$. Moreover, we have the following.

Theorem 1.2. *The logarithmic invariant $\gamma_{N,k}(K)$ is given by a derivative of the colored Jones invariant with respect to the color k as follows*

$$\gamma_{N,0}(K) = \left.\frac{N\,\{1\}}{2\,\pi\,\sqrt{-1}}\,\frac{d}{dm}V_m(K)\right|_{\substack{m=2N\\ q=\xi_N}},$$

$$\gamma_{N,k}(K) = \left.\frac{N\,\{1\}}{\pi\,\sqrt{-1}}\,\frac{d}{dm}V_m(K)\right|_{\substack{m=k\\ q=\xi_N}},$$

$$\gamma_{N,N}(K) = \left.\frac{N\,\{1\}}{2\,\pi\,\sqrt{-1}}\,\frac{d}{dm}V_m(K)\right|_{\substack{m=N\\ q=\xi_N}}.$$

Here we consider (1.1) *as a formal infinite sum with respect to the parameter m, differentiate each terms and substitute $m = 0$, k or N, and $q = \xi_N$. Then all but a finite number of terms vanish.*

We propose the following conjecture for the logarithmic invariant.

Conjecture 2. Let K be a hyperbolic knot in S^3. For $0 < \theta < \pi$, let K_θ be the cone manifold along K with the cone angle θ, and $k_N = \lfloor \frac{2\pi-\theta}{2\pi} N \rfloor$, the largest integer less than or equal to $\frac{2\pi-\theta}{2\pi}N$. Then the following holds

$$\gamma_{N,k_N}(K) \underset{N\to\infty}{\sim} \exp\left(\frac{N}{2\,\pi}\left(\mathrm{Vol}_{\mathrm{hyp}}(K_\theta) + \sqrt{-1}\,\mathrm{CS}(K_\theta)\right)\right).$$

Here $\mathrm{CS}(K_\theta)$ is the classical Chern-Simons invariant in [**1**]. If the cone angle θ is equal to 0, then K_0 is equal to the complement of K, and $\mathrm{CS}(K)$ is equal to the topological Chern-Simons invariant in [**8**] given by $CS(K_\theta) = -2\pi^2\mathrm{cs}(K_\theta)$.

REMARK 1.3. Since $\gamma_{N,N}(K) = J_N(K)|_{q=\xi_N}$, this conjecture is a generalization of Conjecture 1. An analogous generalization is proposed by H. Murakami in [**7**] for the colored Jones invariant, which is called the parametrized volume conjecture.

REMARK 1.4. For a hyperbolic three manifold M, $\mathrm{Vol}_{\mathrm{hyp}}(M) + \sqrt{-1}\,\mathrm{CS}(M)$ is called the complex volume of M. For the complement $S^3 \setminus K$ of a hyperbolic knot K, its complete hyperbolic structure is uniquely defined. However, if we consider an incomplete hyperbolic structure, the hyperbolic structure has a natural deformation space which is parametrized by one complex parameter ([**11**], [**13**]), and K_θ is contained in this deformation space. The complex volume can be considered as an analytic function of the deformation parameter, and the Chern-Simons invariant is naturally extended to the cone manifold K_θ.

REMARK 1.5. The logarithmic invariants of knots can be generalized to invariants of a framed knot K. Let f be the framing of K and K_0 be the 0 framed knot which is isotopic to K as a non-framed knot, then the framed version $\gamma^{(f)}_{N,k}(K)$ is defined by

$$\gamma^{(f)}_{N,k}(K) = q^{f(k^2-1)/2}\,\gamma_{N,k}(K_0).$$

Then Conjecture 1 is reformulated for a framed knot as

$$\gamma^{(f)}_{N,k_N}(K) \underset{N\to\infty}{\sim} \exp\left(\frac{N}{2\,\pi}\left(\mathrm{Vol}_{\mathrm{hyp}}(K_\theta) + \sqrt{-1}\,\mathrm{CS}^{(f)}(K_\theta)\right)\right).$$

The invariants $\mathrm{CS}(K_\theta)$ and $\mathrm{CS}^{(f)}(K_\theta)$ are the Chern-Simons invariants coming from different Chern-Simons actions. If $\theta = \frac{2\pi}{n}$ for a positive integer n, then K_θ is an orbifold. In this case,

$$\mathrm{CS}(K_{\frac{2\pi}{n}}) \equiv \mathrm{CS}^{(f)}(K_{\frac{2\pi}{n}}) \quad (\mathrm{mod}\ \frac{1}{2n})$$

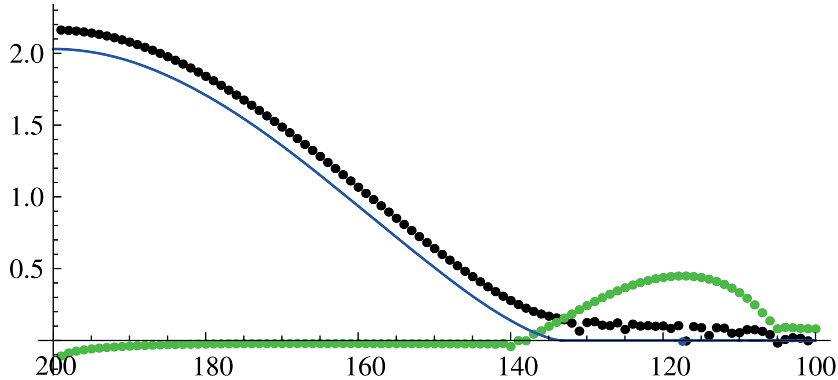
for any f, and the Chern-Simons invariant of $K_{\frac{2\pi}{n}}$ is well-defined modulo $\frac{1}{2n}$ [**3**].

In the rest of this section, we show some examples of Conjecture 2. The first example is the figure eight knot 4_1 in Figure 1. For 4_1, the coefficients $a_j(4_1)$ ($j = 0, 1, 2, \dots$) in (1.1) are all equal to 1, and the colored Jones invariants $V_k(4_1)|_{q=\xi_N}$ ($k = 1, 2, \dots$) are all real numbers. Moreover, the logarithmic invariant $\gamma_{N,k}(4_1)$ ($k = 1, 2, \dots$) are also real numbers. The values $\frac{2\pi}{N}\log|V_k(4_1)|$, $\frac{2\,\pi}{N}\log|\gamma_{N,k}(4_1)|$ and the volume of the cone manifolds along 4_1 with cone angle $2\pi\frac{N-k}{N}$ are plotted in Figure 2 for $N = 200$ and $100 \le k < 200$. The volume is computed by a formula in [**6**].

The second example is the knot 5_2 in Figure 1. For 5_2, $a_j(5_2)$ in (1.1) is given as follows

$$a_j(5_2) = (-1)^j\,q^{3j^2+5j}\sum_{l=0}^{j} q^{l^2-2l-3jl}\,\frac{\{j\}\,\{j-1\}\,\dots\,\{j-l+1\}}{\{l\}\,\{l-1\}\,\dots\,\{1\}}.$$

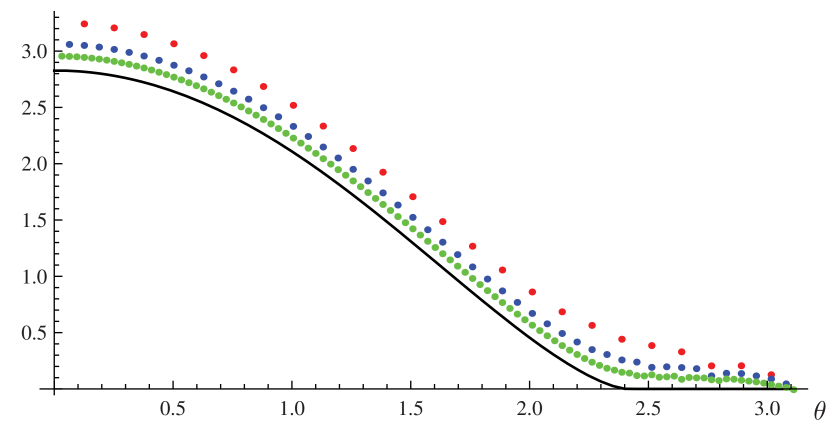
The moduli of the logarithmic invariants and the hyperbolic volume of the corresponding cone manifolds are plotted in Figure 3. The volume is computed by SnapPy, which is a computer software to investigate the hyperbolic structure of knot complements.



$\bullet$: $\frac{2\pi}{N} \log |\gamma_{N,k}(4_1)|$, $\quad \bullet$: $\frac{2\pi}{N} \log |\, V_k(4_1)|_{q=\xi_N}$,

$-$: $\mathrm{Vol}_{\mathrm{hyp}}\left((4_1)_{2\pi\frac{N-k}{N}}\right)$.

FIGURE 2. The colored Jones invariants and the logarithmic invariants for $N = 200$, and the hyperbolic volumes of the corresponding cone manifolds

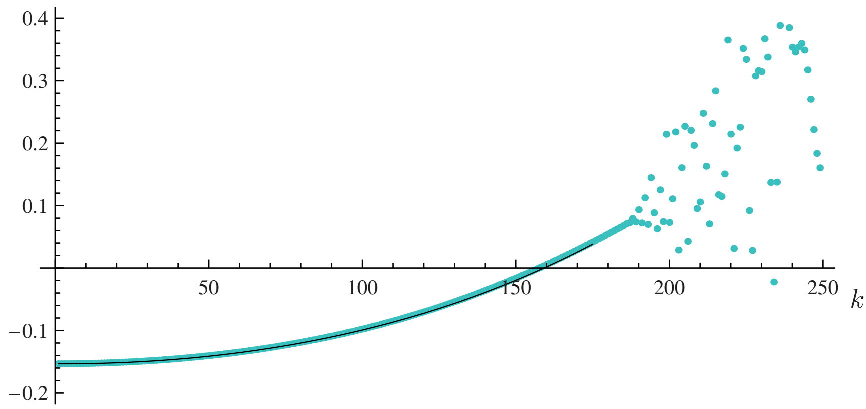


$\bullet$: $\frac{2\pi}{N} |\gamma_{N,\lfloor N(1-\frac{\theta}{2\pi})\rfloor}|$ for $N = 50$, $\quad \bullet$: $\frac{2\pi}{N} |\gamma_{N,\lfloor N(1-\frac{\theta}{2\pi})\rfloor}|$ for $N = 100$,

$\bullet$: $\frac{2\pi}{N} |\gamma_{N,\lfloor N(1-\frac{\theta}{2\pi})\rfloor}|$ for $N = 200$, $\quad -$: Hyperbolic volume of $(5_2)_\theta$

FIGURE 3. Moduli of the logarithmic invariants of the knot 5_2 for $N = 50, 100, 200$ and the hyperbolic volumes of the corresponding cone manifolds

The arguments of the ratios $\frac{\gamma_{N+1,k+1}(5_2)}{\gamma_{N,k}(5_2)}$ are plotted in Figure 4. Since $\frac{k}{N}$ and $\frac{k+1}{N+1}$ are different, we interpolate $\arg \frac{\gamma_{N+1,k}(5_2)}{\gamma_{N,k}(5_2)}$ and $\arg \frac{\gamma_{N+1,k+1}(5_2)}{\gamma_{N,k}(5_2)}$ linearly to get the value corresponding to $\frac{k}{N}$. Note that the Chern-Simons invariant is defined only for a hyperbolic manifold, and the conemanifod $(5_2)_\theta$ for a large θ is not



● : $\frac{1}{\pi}\left(\left(\frac{k}{N}-\frac{k}{N+1}\right)\arg\frac{\gamma_{N+1,k+1}(5_2)}{\gamma_{N,k}(5_2)}+\left(\frac{k+1}{N+1}-\frac{k}{N}\right)\arg\frac{\gamma_{N+1,k}(5_2)}{\gamma_{N,k}(5_2)}\right)$ mod 1 at $N=500$

– : The Cherm-Simons invariants of $(5_2)_\theta$ $\left(\theta=2\pi(1-\frac{k}{N})\right)$.

FIGURE 4. Arguments of the logarithmic invariants of the knot 5_2 for $N=500$ and the Chern-Simons invariant of the corresponding cone manifolds

hyperbolic. The black line indicating the Chern-Simons invariants overlaps the dots corresponding to the logarithmic invariants.

2. Logarithmic invariants of links

Now we generalize the logarithmic invariant to links having several components.

DEFINITION 2.1. Let L be an l component link with components L_1, L_2, $\cdots$, L_l. Let N be an integer greater than 1 and k_1, k_2, $\cdots$, k_l be l integers with $0\le k_j\le N$. Then the logarithmic invariant $\gamma_{N,k_1,k_2,\dots,k_l}(L)$ is defined by

$$\gamma_{N,k_1,k_2,\dots,k_l}(L)=$$
$$\frac{d}{dq}\{1\}\sum_{\varepsilon_1,\cdots,\varepsilon_l=\pm1}\left(\prod_{j=1}^{l}(-1)^{\varepsilon_j}\,2^{-2\delta_{k_j,N}-4\delta_{k_j,0}}\right)V_{f_N(k_1,\varepsilon_1),\cdots,f_N(k_l,\varepsilon_j)}(L),$$

where

$$f_N(k_j,\varepsilon_n)=\begin{cases}2\,N, & (k_j=0)\\ N-\varepsilon_j(N-k_j), & (1\le k_j\le N-1)\\ N, & (k_j=N)\end{cases}$$

and $\delta_{i,j}=1$ $(i=j)$, $0(i\neq j)$.

For this invariant, we propose the following conjecture.

CONJECTURE 3. Let L be a hyperbolic link in S^3. For $0<\theta_j<\pi$ $(j=1,\cdots,l)$, let $L_{\theta_1,\cdots,\theta_l}$ be the cone manifold along the components of L with the cone angle θ_j around L_j, and $k_{N,j}=\lfloor\frac{2\pi-\theta_j}{2\pi}N\rfloor$, the largest integer less or equal to $\frac{2\pi-\theta_j}{2\pi}N$.

Then the following holds

$$\gamma_{N,k_{N,1},\cdots,k_{N,l}}(L) \underset{N\to\infty}{\sim} e^{\frac{N}{2\pi}\left(\mathrm{Vol}_{\mathrm{hyp}}(L_{\theta_1,\cdots,\theta_l})+\sqrt{-1}\,\mathrm{CS}(L_{\theta_1,\cdots,\theta_l})\right)}.$$

We investigate the above relation for some examples. Let L_{WH} be the Whitehead link given in Figure 5. The colored Jones invariant of L_{WH} is given in [**2**] as follows

$$V_{k_1,k_2}(L_{\mathrm{WH}}) = \sum_{j=0}^{\min(k_1,k_2)} q^{-j(j+3)} \frac{\{k_1+j\}!\,\{k_2+j\}!\,\{j\}!}{\{k_1-j-1\}!\,\{k_2-j-1\}!\,\{2j+1\}!\,\{1\}},$$

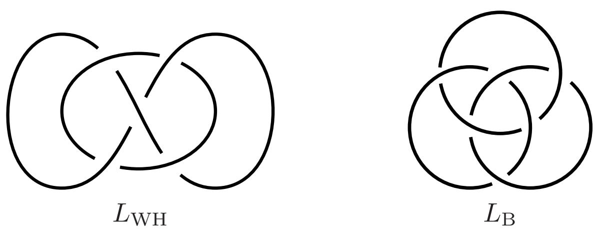


FIGURE 5. The Whitehead link L_{WH} and the Borromean rings L_{B}

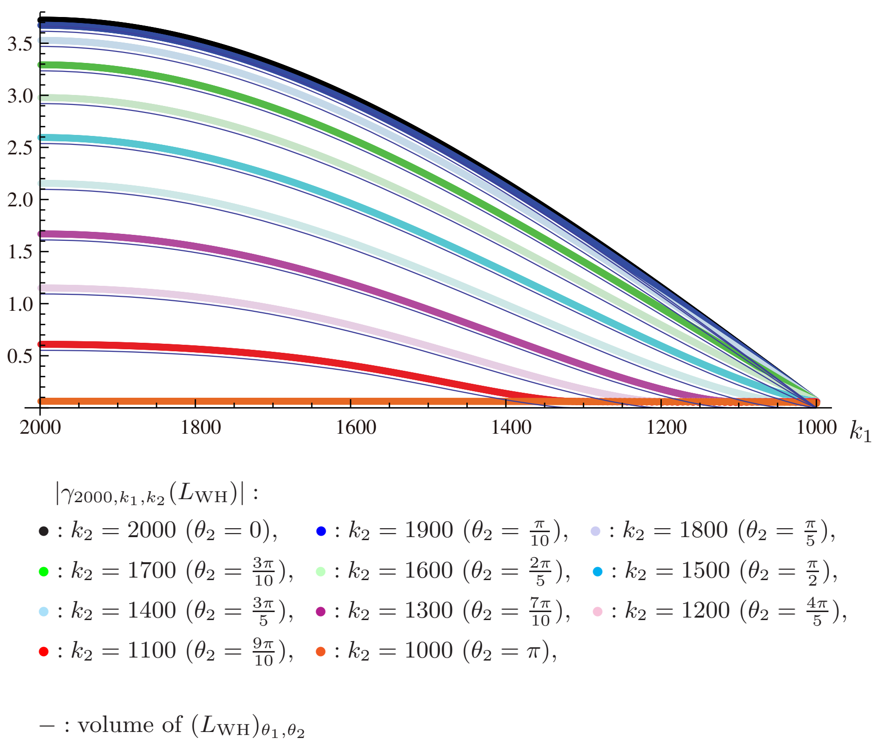


FIGURE 6. Values of $\frac{2\pi}{N}\log|\gamma_{N,k_1,k_2}(L_{\mathrm{WH}})|$ at $N = 2000$ and the volumes of the cone manifolds $(L_{\mathrm{WH}})_{\theta_1,\theta_2}$

where $\{j\}! = \{j\}\{j-1\}\cdots\{2\}\{1\}$. Hence $\gamma_{N,k_1,k_2}(L_{\mathrm{WH}})$ is given by

$$\gamma_{N,k_1,k_2}(L_{\mathrm{WH}}) = \frac{d}{dq} \sum_{\varepsilon_1,\varepsilon_2=\pm1} (-1)^{\varepsilon_1+\varepsilon_2}\, 2^{-2\delta_{k_1,N}-2\delta_{k_2,N}-4\delta_{k_1,0}-4\delta_{k_2,0}} \times \sum_{j=0}^{\min(k_1,k_2)} q^{-j(j+3)} \frac{\{f_N(k_1,\varepsilon_1)+j\}!\,\{f_N(k_2,\varepsilon_2)+j\}!\,\{j\}!}{\{f_N(k_1,\varepsilon_1)-j-1\}!\,\{f_N(k_2,\varepsilon_2)-j-1\}!\,\{2j+1\}!}.$$

The values of $|\gamma_{N,k_1,k_2}(L_{\mathrm{WH}})|$ are plotted in Figure 6. The volumes of the corresponding cone manifolds are computed by a formula in [**6**].

Let L_{B} the Borromean rings in Figure 5. The colored Jones invariant of L_{B} is given in [**2**] as follows

$$V_{k_1,k_2,k_3}(L_{\mathrm{B}}) = \sum_{j=0}^{\min(k_1,k_2,k_3)} \frac{\{k_1+j\}!\,\{k_2+j\}!\,\{k_3+j\}!\,(\{j\}!)^2}{\{k_1-j-1\}!\,\{k_2-j-1\}!\,\{k_3-j-1\}!\,(\{2j+1\}!)^2\,\{1\}}.$$

Therefore, we have

$$\gamma_{N,k_1,k_2,k_3}(L_{\mathrm{B}}) = \frac{d}{dq} \sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3=\pm1} (-1)^{\varepsilon_1+\varepsilon_2+\varepsilon_3}\, 2^{-2(\delta_{k_1,N}+\delta_{k_2,N}+\delta_{k_3,N})-4(\delta_{k_1,0}+\delta_{k_2,0}+\delta_{k_3,0})} \times \sum_{j=0}^{\min(k_1,k_2,k_3)} \frac{\{f_N(k_1,\varepsilon_1)+j\}!\,\{f_N(k_2,\varepsilon_2)+j\}!\,\{f_N(k_3,\varepsilon_3)+j\}!\,(\{j\}!)^2}{\{f_N(k_1,\varepsilon_1)-j-1\}!\,\{f_N(k_2,\varepsilon_2)-j-1\}!\,\{f_N(k_3,\varepsilon_3)-j-1\}!\,(\{2j+1\}!)^2}.$$

The values of $|\gamma_{N,k_1,k_2,k_3}(L_{\mathrm{B}})|$ and the volumes of the corresponding cone manifolds $(L_{\mathrm{B}})_{\theta_1,\theta_2,\theta_3}$ are plotted as in Figure 7. The volumes are computed by a formula in [**6**].

REMARK 2.2. For $L = L_{\mathrm{WH}}$ or L_{B}, we can check from the actual formulas of $\gamma_{N,k_1,\cdots,k_l}(L)$ that the following formula holds

$$\gamma_{N,k_1,\cdots,k_l}(L) = \text{const.}\ \frac{d}{dm_j} V_{m_1,\cdots,m_l}(L)\Big|_{m_1=k_1,\cdots,m_l=k_l}$$

for any $m_j \in \{m_1,\cdots,m_l\}$. This is a generalization of Theorem 1.

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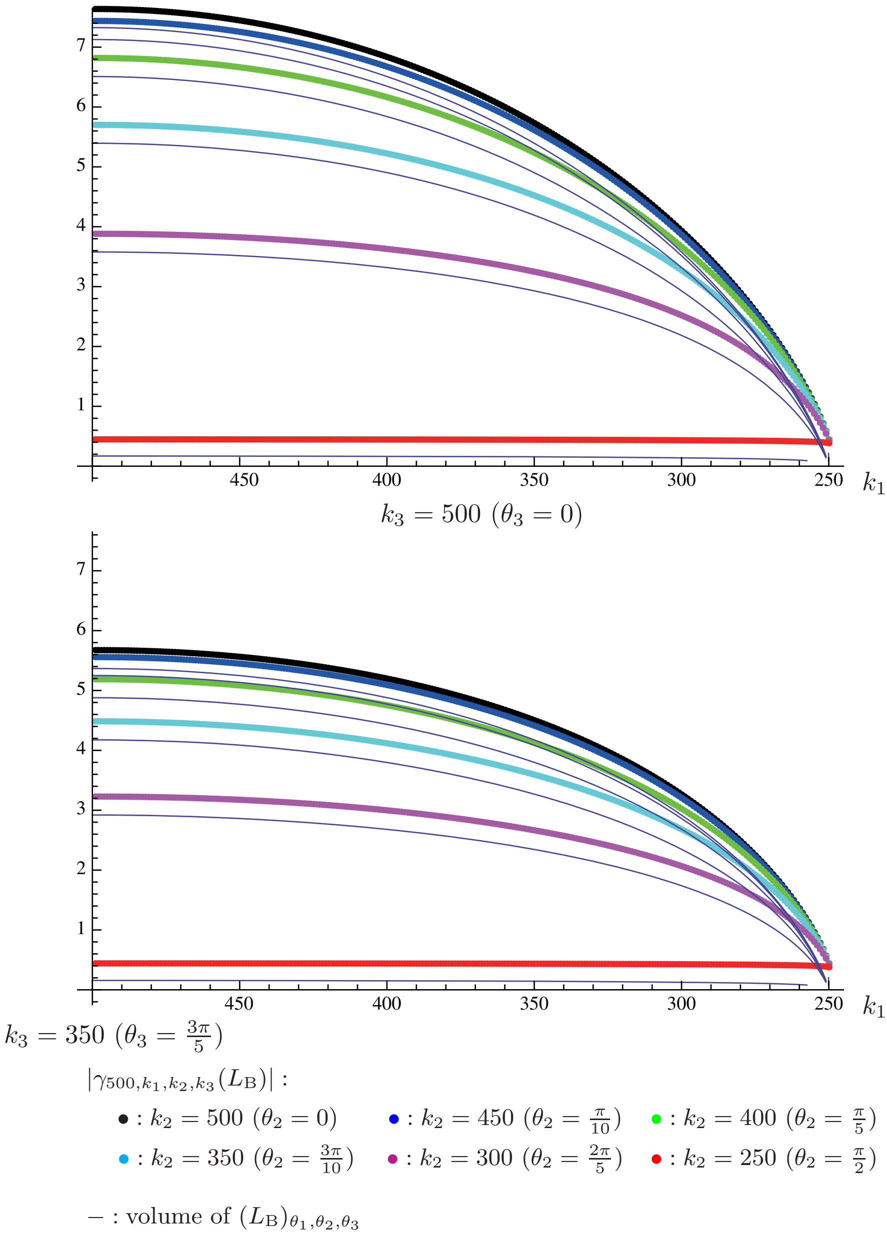

FIGURE 7. Values of $|\gamma_{N,k_1,k_2,k_3}(L_{\mathrm{B}})|$ at $N = 500$ and the volumes of the cone manifolds $(L_{\mathrm{B}})_{\theta_1,\theta_2,\theta_3}$

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