On the Gopakumar–Ooguri–Vafa correspondence for Clifford–Klein 3-manifolds

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ABSTRACT. Gopakumar, Ooguri and Vafa famously proposed the existence of a correspondence between a topological gauge theory on one hand–U(N) Chern–Simons theory on the three-sphere–and a topological string theory on the other–the topological A-model on the resolved conifold. On the physics side, this duality provides a concrete instance of the large N gauge/string correspondence where exact computations can be performed in detail; mathematically, it puts forward a triangle of striking relations between quantum invariants (Reshetikhin–Turaev–Witten) of knots and 3-manifolds, curve-counting invariants (Gromov–Witten/Donaldson–Thomas) of local Calabi-Yau 3-folds, and the Eynard–Orantin recursion for a specific class of spectral curves. I here survey recent results on the most general frame of validity of this correspondence and discuss some of its implications.

CONTENTS

1. Introduction

In a series of works [**47**, **81**, **82**], Gopakumar, Ooguri and Vafa postulated and gave strong evidence for an identity between two rather different physical theories: on the one hand, U(N) Chern–Simons theory on S^3 [**94**], and on the other, topologically A-twisted string theory on the resolved conifold, $\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 2}(-1))$. From the physics side, such identification is a concrete example of 't Hooft's idea [**91**] that perturbative gauge theories with classical gauge groups in the $1/N$ expansion should be equivalent to some first-quantised string theory on a given background. The topological nature of both Chern–Simons theory and the A-model topological

string allow for detailed checks of the correspondence, which might be regarded as a simplified setting for gauge/string dualities for the type II superstring, such as the AdS/CFT correspondence [**70**].

Mathematically, the implications of the Gopakumar–Ooguri–Vafa (GOV) correspondence are perhaps even more noteworthy: the correspondence ties together, in a highly non-trivial way, two different theories of geometric invariants having *a priori* little resemblance to each other. In Witten's landmark paper [**94**] Chern–Simons observables were proposed to give rise to topological invariants of framed 3-manifolds and links therein in light of the Schwarz-type topological invariance of the quantum theory. The relation to rational conformal field theory leads in particular to the skein relations typical of knot invariants such as the HOMFLY-PT and Kaufmann invariants, and more generally, to the Reshetikhin–Turaev invariants arising in the representation theory of quantum groups and modular tensor categories [**85**, **86**]. The A-type topological string side, in turn, also enjoys a mathematical definition – of a quite different flavour. The partition function for a given Calabi–Yau target X is a formal generating function of various virtual counts of curves in X, either via stable maps [**64**] or ideal sheaves [**35**]. In particular, the GOV correspondence asserts that the asymptotic expansion of the Reshetikhin–Turaev invariant associated to the quantum group $U_q(\mathrm{sl}_N)$ at large N and fixed q^N equates the formal Gromov–Witten potential of X in the genus expansion. This has a B-model counterpart due to recent developments in higher genus toric mirror symmetry a [**19**, **42**], where the same genus expansion can be phrased in terms of the Eynard–Orantin topological recursion [**41**] on the Hori–Iqbal–Vafa mirror curve of X [**58**, **59**].

The GOV correspondence has had a profound impact for both communities involved. In Gromov–Witten / Donaldson–Thomas theory, it has laid the foundations of the use of large N dualities to solve the topological string on toric backgrounds [**4**, **6**, **77**] and to obtain all-genus results that went well beyond the existing localisation computations at the time, as well as some striking results for the intersection theory on moduli spaces of curves [**75**] and, via the relation of Chern–Simons theory to random matrices, an embryo of the remodeling proposal [**19**, **71**, **73**]. In the other direction, the integral structure of BPS invariants leads to non-trivial constraints for the structure of quantum knot invariants [**66**, **68**, **74**].

Since the original correspondence of [**47**, **81**] was confined to the case where the gauge group is the unitary group $\mathrm{U}(N)$, the base manifold is S^3, and the knot is the trivial knot, a natural question to ask is whether a similar connection could be generalised to other classical gauge groups [**17**, **18**, **89**], knots other than the unknot [**24**, **65**, **66**] (see also [**3**] for a significant generalisation, in a rather different setting), as well as categorified/refined invariants of various types [**7**, **51**]. A further natural direction would be to seek the broadest generalisation of the correspondence in its original form beyond the basic case of the three-sphere; this would require to provide a description of the string dual of Chern–Simons both in terms of Gromov–Witten theory and of the Eynard–Orantin theory on a specific spectral curve setup. This program was initiated in [**5**] (see also [**25**, **52**, **53**]) for the case of lens spaces, and what is presumably its widest frame of validity has been recently described in [**15**, **27**].

This short survey is meant as an overview of results contained in [**15**, **27**], putting them into context and highlighting their implications for quantum invariants and Calabi–Yau enumerative geometry alike. It is structured as follows: we first give a lightning review of the three-way relation between matrix models, intersection theory on $\overline{\mathcal{M}}_{g,n}$, and the Eynard–Orantin recursion. We then move on to review the uplift of these notions to the GOV correspondence for S^3. Finally, we discuss its generalisation to the setting of Clifford–Klein 3-manifolds and outline the proof of the B-model side of the correspondence for the case of the Lê–Murakami–Ohtsuki invariant [**67**].

2. A 0-dimensional aperçu: Matrix models, the topological recursion and enumerative geometry

As a warm-up for the description of the triangle of relations linking 3D TQFT, curve-counting invariants and the topological recursion, let us consider the much more basic setting of 0D QFTs – namely, matrix models. For any $N, r \in \mathbb{Z}^+$, let $U(N)$ denote the rank-N unitary group, H_N be its Lie algebra of hermitian matrices, and for x in a formal neighbourhood of the origin of $\mathbb{C}^r$ we let $\mathrm{d}\mu_N(x)$ be a family of formally finite Ad-invariant measures, absolutely continuous with respect to the standard gaussian measure $\mathrm{d}\mu_N^{(0)} = \mathrm{d}\mu_N(0)$ on $H_N \simeq \mathbb{R}^{N^2}$. The typical setup here is

$$(1) \qquad \mathrm{d}\mu_N(x) = \mathcal{N}_N \prod_{i=1}^N \mathrm{d}M_{ii} \prod_{i<j}^N \mathrm{d}\mathfrak{Re}M_{ij}\mathrm{d}\mathfrak{Im}M_{ij}\mathrm{e}^{-\frac{N}{2t}\mathrm{Tr}M^2 + \sum_j x_j \mathrm{Tr}M^j}$$
$$= \mathcal{N}_N \mathrm{d}\mu_N^{(0)} \mathrm{e}^{\sum_j x_j \mathrm{Tr}M^j}$$

where $\mathcal{N}_N \in \mathbb{C}^\star$, and the exponential deformation in the last equality should be treated as formal. For fixed t, we will denote by $Z_N(x)$ the formal total mass of $\mathrm{d}\mu_N(x)$,

$$(2) \qquad Z_N(x) = \int_{H_N} \mathrm{d}\mu_N(x) \in \mathbb{C}[[x]],$$

and for any collection of positive integers $k_1, \ldots, k_h$, $h \geq 1$, we write

$$(3) \qquad \mathsf{W}_{k_1,\ldots,k_h;N}(x) \triangleq \partial^{\sum_i k_i}_{x_{k_1},\ldots,x_{k_h}} \log Z_N(x) \in \mathbb{C}[[x]]$$

for the cumulants of the measure. Eqs. (1) and (2) define respectively a *formal hermitian 1-matrix model* and its partition function, and we are particularly interested in their formal behaviour as $N \to \infty$. It is a matter of book-keeping in the

application of Wick's theorem to Eqs. (2) and (3) to show that both $\ln Z_N$ and $W_{k_1,\ldots,k_m}$ have a formal connected ribbon graph (*fat-graph*) expansion [20, 91]

$$\ln Z_N(x) = \sum_{g \geq 0} N^{2-2g} \mathcal{F}_g(x), \tag{4}$$

$$\mathsf{W}_{k_1,\ldots,k_h;N}(x) = \sum_{g \geq 0} N^{2-2g-h} \mathcal{W}_{g,h;k_1,\ldots,k_h}(x). \tag{5}$$

The $1/N$ expansion of Eqs. (4) and (5) lends itself to two interpretations.

- The first one stems from the observation that, as the dual of a ribbon graph is a polygonulation of a connected, oriented topological 2-manifold, this $1/N$ expansion can be regarded as a genus expansion in a sum over random polygonulations of a Riemann surface. The $1/N$ expansion gives then an answer to an *enumeration problem*: it is easy to show in particular, when $\mathcal{N}_N = (t/N)^{N(N+1)/2}(\pi/N)^{-N^2/2}2^{-N}$, that the coefficients of the formal expansion in x have an enumerative meaning as a count of topologically inequivalent dual graphs. One consequence is that, as (metric) ribbon graphs can be used as a means to describe simplicial decompositions of the moduli space $\mathcal{M}_{g,n}$ of pointed Riemann surfaces via the Strebel correspondence, the enumerative content of Eq. (4) provides a combinatorial answer to topological questions on this moduli space [34, 56, 83].
- The second (physical) interpretation comes from a special case of the observation, due to 't Hooft, that the perturbative $1/N$ expansion of a U(N) gauge theory with only adjoint fields takes the shape of the expansion in the string coupling of a first quantised closed string theory, possibly with probe branes insertions [91]; in a sense the observation above that the enumeration problem of fatgraphs is related to intersection-theoretic problems on moduli spaces of curves is the most basic example of this phenomenon. Indeed, 1-matrix models are possibly the most basic avatars of such QFTs, and it is only natural to ask *what is the string dual of the formal 1-matrix model?* This is a sharp, and already non-trivial question in the context of *gauged* matrix models, as can be seen in the following example.

EXAMPLE 2.1. Take the simplest example of a gaussian measure $d\mu_N(x) = d\mu_N^{(0)}$, with a normalisation factor given by

$$\mathcal{N}_N \triangleq [\mathrm{Vol}_g(\mathrm{U}(N))]^{-1} \tag{6}$$

where the metric g is the Ad-invariant metric induced by the Killing form on H_N. The gaussian integration is trivial, and yields

$$\int_{H_N} d\mu_N^{(0)} = \left(\frac{2\pi t}{N}\right)^{N^2/2} \tag{7}$$

The normalisation factor in Eq. (6) is less trivially computed as a product of volumes of spheres [69] and it takes the form of a Barnes double gamma function:

$$\mathrm{Vol}_g(\mathrm{U}(N)) = \frac{(2\pi)^{N(N+1)/2}}{G_2(N+1)} \tag{8}$$

where $G_2(x + 1) = \Gamma(x)G_2(x)$, $G_2(1) = 1$. Computing the large N asymptotics of $\ln Z_N$ is an exercise in the use of the all-order Stirling formula for the Gamma function; the result is

$$(9) \quad \ln Z_N(t) = \frac{N^2}{2}\left(\log(t) - \frac{3}{2}\right) - \frac{1}{12}\log N + \zeta'(-1) + \sum_{g \geq 2}\frac{B_{2g}}{2g(2g-2)}N^{2-2g}.$$

We point out three features of this example.

   i. In this case we can give a full answer to the question of finding a string dual for the matrix model. Indeed, Eq. (9) coincides with the perturbative expansion of the $c = 1$ string theory at the self-dual radius [46, 49, 84], upon identifying the string coupling constant g_s and the cosmological constant μ as $g_s = t/N$, $t = i\mu$.
   ii. The coefficients of N^{2-2g} in the genus expansion of Eq. (9) have a particular geometrical significance of their own: by the Harer–Zagier formula [55], they coincide with the Euler characteristic of the moduli space of genus g-curves with no marked points and $g > 1$,

$$(10) \quad \chi(\mathcal{M}_{g,0}) = \frac{B_{2g}}{2g(2g-2)} = \frac{(-1)^{g-1}2^{2g-1}(2g-1)(2g-3)!}{2^{2g-1}-1}\int_{\overline{\mathcal{M}}_{g,1}}\psi_1^{2g-2}\lambda_g$$

   This is perhaps the simplest setting where an answer to a topological problem on the moduli space of Riemann surfaces (respectively, in the second equality, an intersection theoretic problem on $\overline{\mathcal{M}}_{g,n}$) is encoded into a formal 1-matrix model[1]. This is not an accident, and the rest of this paper will be devoted to non-trivial extensions of this phenomenon.
   iii. A specular (in a precise sense) point of view to the previous statement is given by the study of the large N expansion in Eq. (9) using *loop equations* [20, 79]; an infinite set of differential constraints on the cumulants of Eq. (3) (see [40, 72] for reviews). Its upshot in our case is that the planar free energy, $\mathcal{F}_0$, in Eq. (4) can be recovered as an integral transform of Wigner's semi-circle function,

$$\mathcal{F}_0(t) = -t\int_0^{2\sqrt{t}}z^2\rho(z)\mathrm{d}z,$$

$$(11) \qquad \rho(z) \triangleq \frac{1}{2\pi i}\sqrt{z^2 - 4t}.$$

   Equivalently, for $t \in \mathbb{C}^\star$ consider the *spectral curve* $(C_t, \mathrm{d}\lambda_t)$ given by the smooth, genus zero affine plane curve $C_t = \{(z, y) \in \mathbb{C}^2 | y^2 = z^2 - 4t\}$ endowed with the meromorphic differential $\mathrm{d}\lambda_t = y\mathrm{d}z$. Then $F_0(t)$ is recovered from the rigid special Kähler geometry relations [90]

$$(12) \qquad t = \frac{1}{2\pi i}\oint_A \mathrm{d}\lambda, \qquad \frac{\partial\mathcal{F}_0}{\partial t} = \frac{1}{2}\oint_B \mathrm{d}\lambda,$$

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[1]The distinction between *gauged* and *ungauged* matrix models is somewhat in the eye of the beholder: the same result would be obtained via the large N study of the (ungauged) Penner matrix model in a double-scaling limit; see [34].

where the integrals over $A \in H_1(C_t, \mathbb{Z})$, $B \in H_1^{\mathrm{BM}}(C_t, \mathbb{Z})$ denote respectively a contour integral encircling the cut $[-2\sqrt{t}, 2\sqrt{t}]$ of $y(z)$ in the z plane, and the sum of principal value integrals from $-\infty^-$ to $-2\sqrt{t}$ and from $-2\sqrt{t}$ to $-\infty^+$; here $-\infty^{\pm}$ refers to the two pre-images of $z = -\infty$ under the branched covering map $z : C_t \to \mathbb{C}$.

By the looks of it, Point ii) above relates the formal $1/N$ expansion of the gauged gaussian matrix model to the computation of a particular class of intersection numbers on the moduli space of curves; in physics language, these are a subset of observables in a simple example of an *A-type* topologically twisted string theory[2]. Point iii) does the same thing (at the leading order) for something superficially rather different: special geometry governs the dependence on complex (vector) moduli of the prepotential in type IIB compactification, and is precisely what is captured by the planar limit of a *B-type* topologically twisted string theory. This can actually be made more precise: it was shown in [46] that the $c = 1$ string at the self-dual radius reproduces the genus expansion of the topological A-model on the singular conifold, which is its own self-mirror on the B-side.

2.1. A-model: intersection theory on $\overline{\mathcal{M}}_{g,n}$. The triangle of relations we found in this example between formal matrix models, intersection theory on $\overline{\mathcal{M}}_{g,n}$ and special geometry on a family of curves is part of a more general story. For the general formally deformed measure $\mathrm{d}\mu_N(x)$, we write

$$(13) \qquad W_{g,h}(z_1, \ldots, z_h; x) \triangleq \prod_{i=1}^{h} z_i^{-1} \sum_{k_1, \ldots, k_h} \mathcal{W}_{g,h;k_1,\ldots,k_h}(x) \prod_{i=1}^{h} z_i^{-k_i}$$

for the generating function of the cumulants of Eq. (4) at the gth order in the $1/N$ expansion of the formal 1-matrix model of Eq. (1). At the leading (planar) order in $1/N$, the first loop equation reduces to

$$(14) \qquad W_{0,1}(z) = \frac{1}{2t} P'(z) - \frac{1}{2t} M(z) \sqrt{\sigma(z)},$$

where $\sigma(z) = (z - b_1(x,t))(z - b_2(x,t))$ and the *moment function* $M(z; x)$ as well as the branchpoints $b_i(x,t)$ are entirely determined by $P(z)$ and t. The second planar loop equation fixes the two-point function to take the form [9]
$$(15)$$
$$W_{0,2}(z_1, z_2) = -\frac{1}{2(z_1 - z_2)^2} + \frac{\sqrt{\sigma(z_1)}}{2(z_1 - z_2)^2 \sqrt{\sigma(z_1)} \sqrt{\sigma(z_2)}} - \frac{\sigma'(z_1)}{4(z_1 - z_2) \sqrt{\sigma(z_1)\sigma(z_2)}}.$$

In two remarkable papers [38, 39] (see also [36]), Eynard established the following

THEOREM 2.1. *There exist explicit generating functions of tautological classes* $\Lambda_{g,n}(x) \in \mathbb{C}[[x]] \otimes R^{\bullet}(\overline{\mathcal{M}}_{g,n})$, $\mathcal{B}_{g,n}(z_1, \ldots, z_n; x) \in \mathbb{C}[[x; z_1, \ldots z_n]] \otimes R^{\bullet}(\overline{\mathcal{M}}_{g,n})$ *such that*

$$W_{g,n}(z_1, \ldots, z_n; x) = \int_{\overline{\mathcal{M}}_{g,n}} \Lambda_{g,n}(x) \mathcal{B}_{g,n}(x, z_1, \ldots, z_n),$$

$$(16) \qquad \mathcal{F}_g(x) = \int_{\overline{\mathcal{M}}_{g,0}} \Lambda_{g,0}(x).$$

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[2]In particular, these numbers are closely related to the equivariant Gromov–Witten theory of the affine line – a deformation of topological gravity by linear insertions of Chern classes of the Hodge bundle.

In Theorem 2.1, the classes $\Lambda_{g,n}(x)$ and $\mathcal{B}_{g,n}(x, z_1, \ldots, z_n)$ are entirely determined by knowledge of $W_{0,1}$ and $W_{0,2}$ in Eqs. (14) and (15).

Theorem 2.1 encodes the solution of an intersection theoretic problem on $\overline{\mathcal{M}}_{g,n}$ into the knowledge of the correlators $W_{g,n}(z_1, \ldots, z_n; x)$ of an associated formal 1-matrix model. In fact, the applicability of Theorem 2.1 goes far beyond the realm of (multi-)matrix models: it applies to any spectral curve setup and higher order correlators defined through the Eynard–Orantin topological recursion as in the following section.

2.2. B-model: the Eynard–Orantin topological recursion. Theorem 2.1 gives a means to compute the $1/N$ expansion of the correlators of the 1-matrix model in terms of intersection numbers on $\overline{\mathcal{M}}_{g,n}$, which in general is hardly a simplification from the computational point of view. An independent way to compute them was found in 2004 by Eynard, and further developed in subsequent works with Chekhov and Orantin, in terms of a recursive solution of the loop equations, as in the following

THEOREM 2.2 ([**29**, **37**, **41**]). *Let $\overline{C}_t$ be the normalisation of the projective closure of the plane curve given by the zero locus of $y^2 - \sigma(z)$ in $\mathbb{C}^2$ with coordinates (z, y), and write $\mathrm{d}E_w(z) \in \Omega^1(\overline{C}_t \setminus \{w = z\})$ for the logarithmic derivative of the prime form on $\overline{C}_t$, normalised as $\oint_A \mathrm{d}E_w = 0$; here A is an oriented loop winding once counter-clockwise around the segment $[b_1(x,t), b_2(x,t)]$. Then, for $2g - 2 + h > 0$ the following recursions hold:*

$$W_{g,h+1}(z_0, z_1 \ldots, z_h) = \sum_{i=1,2} \operatorname*{Res}_{z=b_i} \frac{\mathrm{d}E_{z_0}(z)}{2M(z)\sqrt{\sigma(z)}} \Big(W_{g-1,h+2}(z, \overline{z}, z_1, \ldots, z_h)$$

$$(17) \qquad\qquad + \sum_{l=0}^{g} {\sum_{J \subset H}}' W_{|J|+1}^{(g-l)}(z, z_J) W_{|H|-|J|+1}^{(l)}(\overline{z}, z_{H \setminus J}) \Big),$$

$$(18) \qquad\qquad \mathcal{F}_g = \sum_{i=1,2} \operatorname*{Res}_{z=b_i} \frac{\mathrm{d}E_{z_0}(z)}{2M(z)\sqrt{\sigma(z)}} \int^z \mathrm{d}z' W_{g,1}(z')$$

Here $I \cup J = \{z_1, \ldots, z_h\}$, $I \cap J = \emptyset$, $\sum'$ denotes omission of the terms $(h, I) = (0, \emptyset)$ and (g, J), and the primitive in Eq. (18) is independent of its base point.

Eqs. (17) and (18) together make up the *topological recursion*: they determine recursively all the higher orders of the correlators $W_{g,h}$ starting from $(g, h) = (0, 1), (0, 2)$ in Eqs. (14) and (15), and consequently the all-order expansion of the partition function. Its frame of applicability goes far beyond the formal analysis of single-cut 1-matrix models, encompassing also multi-cut solutions, multi-matrix models, and non-polynomial potentials. The overall picture that emerges can be codified by the diagram of Figure 1.

3. The GOV correspondence for S^3

How much of this story is specific to matrix models? It should be noticed that 't Hooft's original argument in no way depended on the dimensionality of the gauge theory we are doing perturbation theory on, as long as the interacting theory has a weakly coupled limit given by a free Gaussian theory, a sensible UV-regularisation scheme exists, and renormalisation does not lead to contributions to
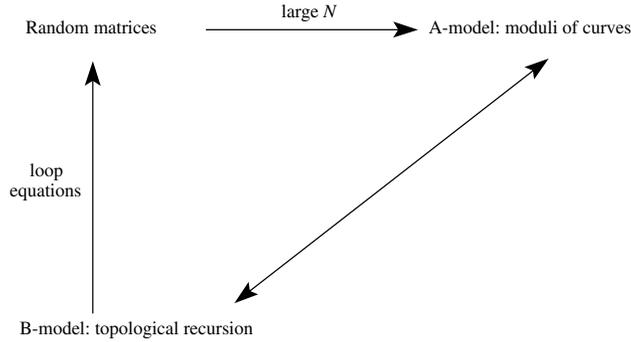
FIGURE 1. Large N duality and mirror symmetry in $d = 0$

the perturbation series which are individually factorially divergent at each order in the topological expansion. A natural, if perhaps bold, question that could be asked is then: "Can the top left corner in Figure 1 be replaced by a higher dimensional (topological) gauge theory?".

Evidence that this is not a mere speculation comes from a non-trivial example, found by Gopakumar, Ooguri and Vafa in [47, 81], where the *entire* setup of Figure 1 carries forth verbatim to higher dimension. Let M be a smooth, closed, oriented real 3-manifold and $\mathcal{K}$ be a link in it. For a smooth U(N) gauge connection $\mathcal{A}$ on M let

$$\text{(19)} \qquad \text{CS}[\mathcal{A}] \triangleq \int_M \text{Tr}_\square \left(\mathcal{A} \wedge \text{d}\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right)$$

be the Chern–Simons functional of $\mathcal{A}$. It was Witten's realisation that the formal functional integrals [94]

$$Z_N^{\text{CS}}(M, k) \quad \triangleq \quad \int D[\mathcal{A}] \exp\left[\frac{\text{i}k}{4\pi} \text{CS}[\mathcal{A}] \right],$$

$$\text{(20)} \qquad W_{N,\rho}^{\text{CS}}(M, \mathcal{K}, k) \quad \triangleq \quad Z_N^{-1} \int D[\mathcal{A}] \, \text{Tr}_\rho \left(\text{Hol}_\mathcal{K}(\mathcal{A}) \right) \exp\left[\frac{\text{i}k}{4\pi} \text{CS}[\mathcal{A}] \right]$$

lead to (framed) topological invariants of M and $\mathcal{K}$ in the form of the Reshethikin and Turaev's sl$_N$-quantum invariants; here $k \in \mathbb{Z}^*$ is the Chern–Simons level, and the coloring ρ is an irreducible representation of U(N). Their large N expansion can be worked out entirely explicitly in different ways, for example by resorting to the surgery techiniques of [94]: for the partition function we find

$$\text{(21)} \qquad \ln Z_N^{\text{CS}}(M, k) = \sum_{g \geq 0} g_s^{2g-2} \mathcal{F}_g^{\text{CS}}(t)$$

where $g_s = 2\pi\text{i}(k + N)^{-1}$, $t = g_s N$, and

$$\text{(22)} \qquad \mathcal{F}_g^{\text{CS}}(t) = \frac{|B_{2g}|}{2g(2g-2)!} \text{Li}_{3-2g}(e^{-t}) + \frac{(-1)^g B_{2g} B_{2g-2}}{2g(2g-2)(2g-2)!}$$

for $g \geq 2$, with similar formulas for $g = 0, 1$ [47]. If we trust that a higher dimensional analogue of the discussion of the previous section exists, the two questions

we are tasked to answer are:

(1) characterise the large N A-model dual of Chern–Simons theory in terms of a precise intersection theoretic problem on $\overline{\mathcal{M}}_{g,n}$;

(2) characterise its mirror, large N B-model dual in terms of the topological recursion on a specific spectral curve setup.

3.1. A-model. The A-model dual found in [**47**] is a sort of q-deformation of the setup of Example 2.1, and it has an explicit presentation in terms of an intersection theory problem on a moduli space of stable *maps*. First off, it was found in [**95**] that the partition function of U(N) Chern–Simons theory on any closed three manifold M is equal to the partition function of the *open* topological A-model on the total space of the cotangent bundle T^*M, with N Lagrangian A-branes wrapping its zero section. The proposal of [**47**] is to relate the latter at large N to the *ordinary, closed* A-model/Gromov–Witten theory on a target space obtained from T^*S^3 via a complex deformation to a normal singular variety and a bimeromorphic resolution of its singularities (the *conifold transition*), as follows. When S^3 has radius one in the canonical metric, we have an obvious isometry $T^*S^3 \simeq \mathbb{R}^3 \times S^3 \simeq \mathrm{SL}(2,\mathbb{C})$ given by the decomposition of a special linear matrix A into radial (positive definite) and polar (unitary) part:

$$(23) \qquad A = U\mathrm{e}^H, \qquad U \in \mathrm{SU}(2), \quad H \in \mathcal{H}_0(2,\mathbb{C}).$$

In particular this endows T^*S^3 with a complex structure given by its presentation as a quadric hypersurface $\det A = 1$ in $\mathrm{Mat}(2,\mathbb{C}) \simeq \mathbb{C}^4$.

We now perform the following two operations on this A-model target space:

(1) First we vary the radius of the base unit sphere to give a flat family $\psi : X = \mathrm{GL}(2,\mathbb{C}) \to \mathbb{C}^*$ via the determinant map, whose fiber $X_{[\mu]}$ at a point μ with $\mathrm{Im}\,\mu = 0$ and $\mathrm{Re}\,\mu > 0$ is isomorphic to the cotangent bundle $T^*S^3_{[\mu]}$. Notice that we are doing *nothing here* either from the point of view of Chern–Simons theory or the open topological A-model on the cotangent bundle, as a homeomorphism of the base (resp. a complex deformation of the total space) leave invariant the CS partition function (resp. the A-model partition function).

(2) The second is to add the locus of non-invertible matrices to form:

$$(24) \qquad \tilde{\psi} : \mathrm{Mat}(2,\mathbb{C}) \longrightarrow \mathbb{C}.$$

The singular fiber $X_{[0]}$ above $\mu = 0$ is the singular quadric $\det A = 0$: this is the singular conifold of Example 2.1. The latter admits a canonical toric minimal resolution

$$(25) \qquad \pi : \widehat{X} \longrightarrow X_{[0]}, \qquad \widehat{X} \triangleq \big\{ (\rho(A), v) \in X_{[0]} \times \mathbb{P}^1, \quad \rho(A)v = 0 \big\},$$

where π is the projection to the first factor. The point $A = 0$ is singular in $X_{[0]}$, and its fiber is a complex projective line with $[v_1 : v_2]$ as homogeneous coordinates. Using coordinate charts on $\mathbb{P}^1$ exhibits $\widehat{X}$ as the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$, i.e. the *resolved conifold*.

The proposal of [**47**] is that

$$(26) \qquad \mathcal{F}_g^{\mathrm{CS}}(t) = \mathrm{GW}_g^{\widehat{X}}(t)$$

where we have identified the 't Hooft parameter on Chern–Simons theory with the A-model Kähler parameter dual to the curve class of the base $\mathcal{O}(-1)_{\mathbb{P}^1}^{\oplus 2}$. In Eq. (26), $\mathrm{GW}_g^{\widehat{X}}$ denotes the genus-g primary Gromov–Witten potential of $\widehat{X}$,

$$(27) \qquad \mathrm{GW}_g^{\widehat{X}}(t) = \sum_{d \geq 0} \mathrm{e}^{td} \int_{[\overline{\mathcal{M}}_{g,0}(\widehat{X},d)]^{\mathrm{vir}}} 1$$

where $\overline{\mathcal{M}}_g(\widehat{X}, d)$ is the stack of degree d stable maps from genus g curves to $\widehat{X}$ and $[\overline{\mathcal{M}}_g(\widehat{X}, d)]^{\mathrm{vir}}$ denotes its virtual fundamental cycle.

There are a number of reasons, at various degrees of rigour, to take this proposal seriously:

(1) Firstly, it holds true asymptotically in the $t \to 0$ limit: indeed on the CS side, the partition function is known to reduce in this limit to the gauged gaussian matrix model of Example 2.1 [**84**]; and on the GW side, the $t \to 0$ regime corresponds to the small volume limit (the *singular conifold*). The claim then follows from the conifold/$c = 1$ duality we discussed in Example 2.1;

(2) There are also heuristic physics arguments in favour of the duality. One hinges on viewing Eq. (26) as an open/closed duality in the A-model, the equality can be interpreted as a brane/flux duality in the physical type IIA superstring: in principle, both sides compute superpotential observables in effective $\mathcal{N} = 1$ theories, obtained either as the world-volume theory of wrapped $D6$-branes in T^*S^3 of via turning on internal RR field strengths on the resolved side [**13**, **92**]. These string configurations can then be related by lifting them to M-theory [**10**].

(3) A further microscopic derivation of the duality can also be given [**82**] upon relating the open partition function on the deformed conifold and the closed partition function on the resolved conifold by their UV completion in terms of a gauged linear σ-model: here the $t \to 0$ limit leads to a coexistence of Coulomb/Higgs phases on the closed string world-sheet, with the holes of the open theory arising from integrating out the contribution of Coulomb regions.

(4) What is more, the equality of Eq. (26) is a sharp mathematical statement about RTW invariants and GW invariants that can be rigorously proven, or disproven, by an explicit calculation. The l.h.s. can be rigorously shown to lead to the expansion of Eq. (22) from an explicit MacLaurin expansion of the finite sums appearing in the surgery formula for the Reshetikhin–Turaev invariant [**47**]; and on the other hand the r.h.s. is amenable to a direct localisation analysis in Gromov–Witten theory, first performed in [**43**, **48**]. The end result is an exact agreement of the two generating functions.

(5) Finally, a natural question is how the story generalises when we incorporate links in Chern–Simons theory. It was first proposed in [**81**] that when $\mathcal{K} = \bigcirc$ is the trivial knot, Eq. (26) should generalise to an equality between the colored HOMFLY invariants of Section 3 and *open* GW potentials for a suitable choice of Lagrangian A-branes L:

$$(28) \qquad W_{g,h,\vec{d}}^{\mathrm{CS}}\big(S^3, t, \bigcirc\big) = \mathrm{GW}_{g,h,\vec{d}}^{\widehat{X},L}(t)$$
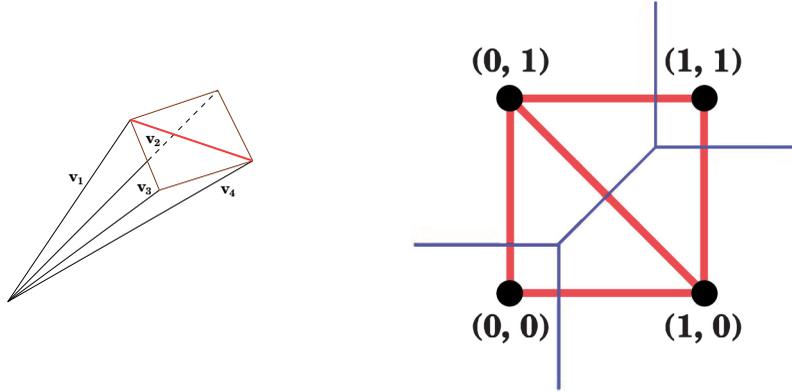
FIGURE 2. The fan (left) of the resolved conifold: its skeleton is given by four rays, labelled v_i in the picture, whose tips lie on an affine hyperplane at unit distance from the origin of $\mathbb{Z}^3$. The intersection of the fan with the hyperplane is shown on the right (the *toric diagram*); superimposed is the pq-web diagram.

where in the l.h.s. we consider a knot invariant obtained as a power sum (instead of Schur functions, as in Section 3) holonomy invariant specified by a vector of integers $d_1, \ldots, d_h$, we take its connected part, and we expand it at large N as in Eq. (5), and in the r.h.s. we take the generating function of open GW invariants with boundary on the fixed locus of a real involution (a real bundle on the equator of the base $\mathbb{P}^1$), with fixed genus g and number of holes h for the source of curves and winding numbers $d_1, \ldots, d_h$ around $S^1 \subset \mathbb{P}^1$. Indeed, the line of reasoning of [**92**] carries through to this setting, and also in this case a localisation definition/computation of open GW invariants can be performed [**63**], confirming the prediction of [**81**].

3.2. B-model. Since the A-model target space is a (non-compact) Calabi–Yau threefold, we would expect a geometric mirror picture in terms of rigid special Kähler geometry of a family of local CY3s, akin to what happened in Example 2.1 for the case of matrix models. Not only is this case, but the analogy with the spectral curve setup of random matrices is even more poignant – the special geometry relations on the local CY3 mirror (which, in this particular case, is just the deformed conifold geometry of the previous section) can be shown to reduce to Seiberg–Witten-type relations on a family of genus zero spectral curves, given by the Hori–Iqbal–Vafa mirror of $\widehat{X}$:

$$(29) \qquad t = \frac{1}{2\pi\mathrm{i}} \oint_A \ln y \, d\ln x, \qquad \frac{\partial \mathrm{GW}_0^{\widehat{X}}}{\partial t} = \frac{1}{2} \oint_B \ln y \, d\ln x$$

where A, B are homology 1-cycles relative to the principal divisors $x = 0$, $y = 0$ of the plane curve defined by $P_{\widehat{X}}(x, y) = 1 + x + y + \mathrm{e}^t = 0$; here $P_{\widehat{X}}$ is the Newton polynomial of the diagram in Figure 2. What is more, motivated by the study of the chiral boson theory of [**2**], it was first suggested in [**73**], and then postulated in full detail in [**19**], that the higher genus B-model potentials (and, by mirror symmetry,

the higher genus open GW invariants of $\widehat{X}$) should be computed by the topological recursion of Eqs. (17) and (18) upon identifying $ydx = W_{0,1}(x)$. This prediction has been subsequently proved in full rigour in [42] for all toric Calabi–Yau manifolds that are symplectic quotients[3].

4. The GOV correspondence for Clifford–Klein 3-manifolds

We have seen how the picture of Figure 1 generalises verbatim to the higher dimensional setting of Chern–Simons theory on S^3: the top-right corner has a concrete A-model picture in terms of the Gromov–Witten theory of the resolved conifold, and the bottom-left corner is encoded by the topological recursion on its Hori–Iqbal–Vafa mirror curve. The emerging picture is not only beautiful and unexpected; from a practical point of view, it has deeply affected the relation between quantum invariants, curve-counting invariants, and the topological recursion. Appealing and impactful as it has been, this is however just *one* example – placed in the classical setting of U(N) theories, encompassing the simplest closed 3-manifold, and with the simplest knot therein. It is then natural to seek a strict generalisation of the GOV correspondence to other[4]

(1) **knots**: in its strictest sense, the GOV correspondence has been shown to carry through to the case of torus knots both in terms of an enumerative theory of open GW invariants on the A-side [32] and the topological recursion on a specific spectral curve setup on the B-side [24]. For general knots, it would seem that substantially new ideas are needed both on the A- and the B-side of topological string theory [3]; the role of the topological recursion in particular is however less clear in this setting [50].

(2) **(classical) gauge groups**: SO(N) and Sp(N) Chern–Simons theory at large N, which compute in particular the Kauffmann invariant of links, can also fit in the picture of the previous section by considering suitable orientifolds of the resolved conifold [89], for which an operative definition of unoriented invariants can be given either by localisation or via the topological vertex [17, 18].

(3) **3-manifolds**: this is possibly the boldest generalisation – replace altogether S^3 by an arbitrary closed 3-manifold M. At face value this boils down to solving the problem for arbitrary knots on S^3: the partition function of a link $\mathcal{L}$ in M would then be recovered by the surgery formula from the partition function of a 2-component link $\mathcal{L} \sqcup \mathcal{L}_M$ in S^3, where $\mathcal{L}_M$ is a framed link M can be obtained from (this always exists by Lickorish's theorem). However, both the surgery formula in Reshetikhin–Turaev theory [54, 94], and functional localisation in Chern–Simons theory [12, 14, 61] lead to an expression of the partition function as a sum over contribution labelled by flat connections (classical vacua) on M:

$$(30) \qquad Z_N^{\mathrm{CS}}(M,k) = \sum_{\mathsf{v} \in \mathrm{Hom}(\pi_1(M), \mathrm{U}(N))/\mathrm{U}(N)} Z_{N,\mathsf{v}}^{\mathrm{CS}}(M,k).$$

---

[3]See [44] for a recent generalisation of this result to the case of orbifolds.

[4]The following list leaves out one important generalisation, tying together categorification on the knot theory side [51], refined BPS counting on the A-model side [30, 60, 62], and some version of the β-deformation on the B-model side [1, 7, 26].

It was first proposed in [**5**] that the finer invariants given by the *individual sum-mands* $Z_{N,\mathsf{v}}^{\mathrm{CS}}$ may also be interpreted as the A/B-model partition function on a background specified by v; notice that this is a more refined object to deal with than the partition function of the link $\mathcal{L}_M$, where all these contributions are summed over. That a dual curve counting theory exists is encouraged by the successful test of this proposal for the case of $L(p,1)$ lens spaces in [**5**, **53**]. The case of more general 3-manifolds was considered in [**15**, **16**, **25**, **27**], and we review it below.

4.1. CS theory on Clifford–Klein 3-manifolds. We start by recalling the following

DEFINITION 4.1. A Clifford–Klein 3-manifold (M, g) is a closed oriented smooth 3-manifold M admitting a smooth metric g of everywhere strictly positive Ricci curvature.

Equivalently, by Hamilton's theorem, it is a spherical space form, $M = S^3/\Gamma$ for Γ a freely acting finite isometry group of S^3 w.r.t. its canonical metric; and by Perelman's elliptisation theorem, it follows that these coincide with the orientable 3-manifolds having finite fundamental group. The classification of the possible Γ goes back to Hopf [**57**]: these are central extensions of the left-acting finite subgroups of $\mathrm{SL}_2(\mathbb{C})$, which admit an ADE classification; see [**15**, Appendix A] for more details. We restrict henceforth for simplicity of exposition to the case where the central extension is trivial and Γ is one such $\mathrm{SL}_2(\mathbb{C})$-subgroup; most of our arguments to follow will be unaffected by this.

Now since $|\Gamma| < \infty$, the sum in Eq. (30) truncates at finite N. Denote by $\mathfrak{V}_{\Gamma,N}$ the finite set of gauge-equivalent flat connections and by $\mathfrak{V}_\Gamma = \lim_{N \to \infty} \mathfrak{V}_{\Gamma,N}$ be its direct limit with respect to the composition of morphisms given by the embedding $\mathrm{U}(N) \hookrightarrow \mathrm{U}(N+1)$. Then

$$(31) \qquad \mathfrak{V}_\Gamma = \mathbb{N}^{R+1}, \qquad \mathfrak{V}_{\Gamma,N} = \left\{ (N_0, \ldots, N_R) \in \mathbb{N}^{R+1}, \quad N_0 + \sum_{i=1}^R D_i N_i = N \right\}.$$

where R is the number of nodes in the simply-laced Dynkin diagram associated to Γ, and D_i are the respective Dynkin indices. When $N \to \infty$, we thus consider a CS vacuum $[\mathcal{A}]_t$ parametrized by $t \triangleq N_i g_s$ for $i \in \{0, \ldots, R\}$, and in particular the rank is encoded in the 't Hooft parameter $t = N g_s$. The resulting partition functions at $N = \infty$ have a standard $1/N$ expansion

$$(32) \qquad \ln Z_{N,\mathsf{v}(t)}^{\mathrm{CS}}(g_s) = \sum_{g \geq 0} g_s^{2g-2} \mathcal{F}_g^{\mathrm{CS}}(t_0, \ldots, t_R)$$

with free energies depending now on $R + 1$ 't Hooft parameters.

These manifolds also carry a natural class of knots with them. For S^3, the standard GOV correspondence focused on the unknot – which could be regarded as the fibre knot of the Hopf fibration on S^2. Now Clifford–Klein manifolds are also Seifert-fibred, and in the ADE case they can be regarded as Seifert fibrations over an ADE $\mathbb{P}^1$-orbifold with one (resp. three) exceptional fibres for Dynkin type A (resp. D and E). As for S^3, we will similarly be interested in the $1/N$ expansion of the RTW invariant for the knots $\mathcal{K}_f$ running around these exceptional fibres:

$$(33) \qquad W_{N,\mathsf{v}(t)}^{\mathrm{CS}}(\mathcal{K}_f, g_s, \vec{d}) = \sum_{g \geq 0} g_s^{2g-2} \mathcal{F}_{g,h}(t_0, \ldots, t_R; \vec{d})$$

The quest is to find now a Calabi–Yau threefold geometry $\widehat{X}_\Gamma$ with special Lagrangians L_Γ, as well as a spectral curve setup Σ_Γ for each such Γ, such that open/closed GW theory on $(\widehat{X}_\Gamma, L_\Gamma)$ and the topological recursion on Σ_Γ lead to Eqs. (32) and (33). This program was completed in [**15**, **27**], generalising ideas on the lens space case in [**5**].

4.2. A-model from geometric transition. On the A-model the idea is fairly simple: we take seriously the geometric transition argument of Section 3.2 and apply it to the setting at hand. To this aim, notice that the Γ-action on the resolved conifold $\widehat{X}$ only acts on the first factor of Eq. (25), giving a fibrewise action on $p : \widehat{X} \to \mathbb{P}^1$ (the second factor in Eq. (25)); the fibre over a point $z \in \mathbb{P}^1$ is isomorphic to a Gorenstein surface singularity of the same ADE type of Γ. The topological A-string on this target geometry can be studied in several phases, two of which are distinguished:

- in the *orbifold* chamber, we are looking at a theory of twisted stable maps on the Calabi–Yau stack $\widehat{X}^\Gamma_{\mathrm{orb}} \triangleq [\widehat{X}/\Gamma]$. This is the maximally singular phase containing the Γ-orbifold of the conifold point, which is the natural point of expansion for the dual Chern–Simons theory;
- in the *large radius* chamber, we take a crepant resolution $\widehat{X}^\Gamma_{\mathrm{res}}$ of the singularities of (the coarse moduli space of) $\widehat{X}^\Gamma_{\mathrm{orb}}$ obtained by canonically resolving the surface singularity $\mathbb{C}^2/\Gamma$ fibrewise, and we are looking at the ordinary GW theory of Y^Γ_{res}.

We have four remarks about the resulting target space geometry.

R1: in either case the target supports at least a $(\mathbb{C}^\star)^2$ torus action; this is the product of the 1-torus action that rotates the base $\mathbb{P}^1$ and the fibrewise 1-torus action inherited from the scalar action on the $\mathbb{C}^2$-fibre of $\widehat{X}$. In particular, as for the usual toric case, we will specialise to a resonant 1-subtorus by imposing that its action is trivial on the canonical bundle of the target; this acts with compact fixed 0- and 1- dimensional fixed loci. This in turn allows to define equivariant GW invariants by localisation on either $\widehat{X}^\Gamma_{\mathrm{res}}$ or $\widehat{X}^\Gamma_{\mathrm{orb}}$.

R2: As we already mentioned, the Γ-action is fibrewise and it covers the trivial action on the base $\mathbb{P}^1$; this circumstance will be important in a moment.

R3: For all Γ, there are natural anti-holomorphic involutions whose fixed loci define Lagrangians for both $\widehat{X}^\Gamma_{\mathrm{res}}$ and $\widehat{X}^\Gamma_{\mathrm{orb}}$. These generalise the toric Lagrangian branes of [**8**, **22**, **81**], and the residual Calabi–Yau $\mathbb{C}^\star$ action of **R1** above allows to define/compute open GW invariants by localisation. In particular, the methods of [**22**] carry through verbatim to the setting at hand.

R4: The anti-holomorphic involution of [**89**] turns out to commute with the Γ-action, for all ADE types, and an orientifold theory can be defined in the same way.

4.3. B-model from geometric engineering. R1-R2 above define completely what would be the top-right (A-model) corner of a diagram like Figure 1 for the case at hand. Now notice that in **R1** above, for type D and E the torus action we highlighted does not extend to a full three-torus action on $\widehat{X}^\Gamma$ since Γ is

non-abelian. In particular, we cannot resort to the toric mirror symmetry methods of [**58**, **59**] to find a spectral curve setup for these cases, unlike for type A. A way out on physical grounds is however pointed at by **R2**: since the action is fibrewise, the topological A-string on these backgrounds is known to geometrically engineer in a suitable limit a 4-dimensional $\mathcal{N} = 2$ gauge theory with simply-laced compact gauge group corresponding to the Dynkin type of Γ (and no adjoint hypermultiplets, since $\mathbb{P}^1$ has genus zero) [**62**]; so it is expected in this degenerate limit, which corresponds to a divisor at infinity in the stringy Kähler moduli space of $\widehat{X}^\Gamma$, to have a mirror picture in terms of spectral curves of Seiberg–Witten type [**62**, **88**]. As a matter of fact, even away from this limit the A-model is still expected to give rise to a gauge theory with eight supercharges, albeit in one dimension higher – namely $\mathcal{N} = 1$ pure super Yang–Mills theory on $\mathbb{R}^4 \times S^1$, with the "field-theory limit" of [**62**] corresponding to the fifth-dimensional circle shrinking to zero-size.

Now for type A_N, the five-dimensional theory also enjoys a description in terms of spectral curves of Seiberg–Witten/Hori–Iqbal–Vafa type: these are the spectral curves of the periodic Ruijsenaars system (relativistic Toda chain) with N sites [**80**]; the 4d limit corresponds to the non-relativistic limit. Furthermore, it has long been known that for all ADE types the relevant SW curves should coincide with the spectral curves of the Lie-theoretic generalisation of the non-relativistic Toda chain [**76**] specialised to ADE Lie algebras. Putting all this together, it is then natural to look for a relativistic deformation of [**76**] to supply the spectral curves defining a candidate B-model mirror for the non-toric geometries of the previous section, and these can be computed from the setup of [**45**, **93**]. We refer the reader to [**27**, Section 2] for a more detailed review; the upshot is that the spectral curves can be computed in the following two steps:

(1) fix ρ to be a minimal irreducible $\mathcal{G}_\Gamma$-module, where $\mathcal{G}_\Gamma$ is the simply-connected simple Lie group over $\mathbb{C}$ of ADE type specified by Γ, and consider the characteristic polynomial of a group element g in the representation ρ

$$(34) \qquad \mathfrak{p}_\rho^\Gamma(g) \triangleq \det_\rho(\mathrm{id}\, x - g) : \mathcal{T}_\Gamma / \mathrm{Weyl}(\mathcal{G}_\Gamma) \to \mathbb{C}[x]$$

Here $\mathcal{T}_\Gamma$ is the Cartan torus of $\mathcal{G}_\Gamma$. We can decompose this on the Weyl character ring upon expanding the determinant in a polynomial with coefficients given by anti-symmetric characters of the representation ρ, and then write the latter as polynomials of the fundamental characters u_i:

$$(35) \qquad \mathfrak{p}_\rho^\Gamma(g) = \sum_{n=0}^{\dim \rho} (-1)^{\dim \rho - n} \chi_{\wedge^{\dim \rho - n} \rho}(g) x^n, \qquad \chi_{\wedge^n} \in \mathbb{Z}[u_1, \ldots, u_R]$$
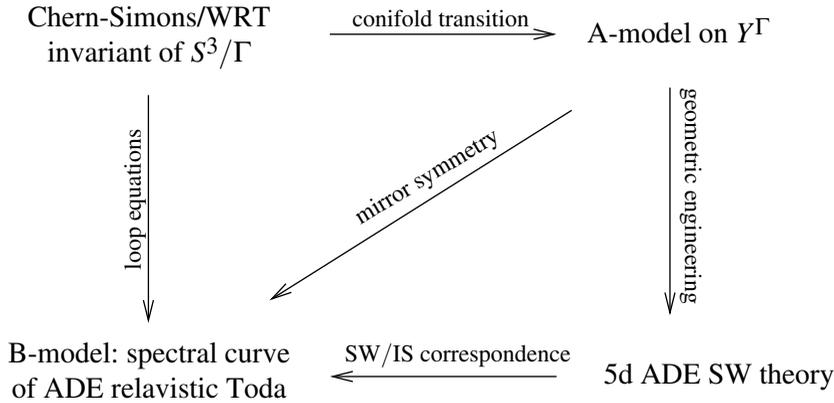
(2) Now let u_k be the character of the maximally dimensional fundamental representation[5]. Then the spectral curve is defined by the family of non-compact Riemann surfaces given by the polynomial equation

$$(36) \qquad \mathfrak{p}_\rho^\Gamma \left(x; u_i + \delta_{i,\mathsf{k}}(y + u_0 y^{-1}) \right) = 0$$

along with a choice of recursion differential given by the Poincaré one form $W_{0,1} = \ln y\, \mathrm{d} \ln x$, and higher order generating functions obtained by the topological recursion on Eq. (36).

The resulting web of relations is pictured in Figure 3.

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[5]In two cases there is an ambiguity which is resolved as follows. For type E_6, pick either of the **27** or $\overline{\mathbf{27}}$. For type A_{2n}, Eq. (36) is modified by shifting $u_i \to u_i + \delta_{i,n} y + \delta_{i,n+1} u_0 y^{-1}$.

Chern-Simons/WRT
invariant of S^3/Γ

$\xrightarrow{\text{conifold transition}}$

A-model on Y^Γ

loop equations

mirror symmetry

geometric engineering

B-model: spectral curve
of ADE relativistic Toda

$\xleftarrow{\text{SW/IS correspondence}}$

5d ADE SW theory

FIGURE 3. The GOV correspondence for S^3/Γ.

4.4. The LMO invariant and the topological recursion. Our proposal fills all the vertices in the diagram of Figure 3: the sl_N-Reshetikhin–Turaev knot invariants of fibre knots in S^3/Γ in the $1/N$ expansion around a fixed flat connection would equate the open Gromov–Witten potentials of $(\widehat{X}^\Gamma, L^\Gamma)$ on one hand, and the Eynard–Orantin invariants of Eq. (36) on the other; a similar statement applies to the RT invariant of S^3/Γ itself, the closed GW potential of $\widehat{X}^\Gamma$, and the topological recursion free energies.

However this is all conjectural for the moment – so how do we prove this? One way to proceed is to resort to the relation of Chern–Simons theory on Seifert fibred spaces (such as the Clifford–Klein manifolds) and matrix models. It was shown by Mariño in [71] that Witten's surgery formula for these spaces allows to rewrite the CS partition function as well as the Wilson loops around certain classes of knots as a matrix integral; in particular, the quantum invariants Eqs. (32) and (33) of S^3/Γ and fibre knots therein around the exceptional fibres fall squarely in this category [12, 14, 71]. When $\mathsf{v} = 0$ is the trivial flat connection, the resulting matrix model turns out to be a trigonometric deformation of the gauged gaussian matrix model of Example 2.1: it is a canonical ensemble with gaussian 1-body interaction and a sum of q-deformed Vandermonde 2-body interactions, with coefficients determined by the orders of the exceptional fibres of the Seifert fibration [11, 16, 24, 71]: this restriction gives rise to the so-called LMO invariant.

This presentation is amenable to a large N analysis via loop equations, akin to that of Point iii) in Example 2.1, leading for all Γ to a singular integral equation to be solved by the input datum of the recursion – the planar disk function $W_{0,1}$. Such an analysis was performed in [15, 16, 24, 27]; the strategy, and ensuing results are as follows.

(1) It can readily be shown that these are single-cut matrix models due to the gaussian nature of the 1-body potential, and that the (exponentiated) planar resolvent $W_{0,1}(x)$ is never a log-algebraic function of its argument except when Γ is the trivial group. However, a strategy introduced in [24] for torus knots and then employed on a full scale on [16] is to considered a symmetrised version of the resolvent, which leads to the same large N eigenvalue density as the original resolvent on the physical cut and may

be such that the sheet transitions given by crossing the cuts close to a finite group.

(2) The previous step can be performed for any Seifert 3-manifold as that is the level of generality to which the ideas of [71] apply. It can be shown however that the *only* Seifert spaces for which the group of sheet transitions is finite, and whose planar resolvent thus gives rise to *algebraic* spectral curves, are precisely the Clifford–Klein 3-manifolds: there is no hope to extract an algebraic setup for the large N/B-model curve, even for the restriction to the trivial flat connection, for parabolic and hyperbolic Seifert 3-manifolds. In the elliptic case, instead, $e^{W_{0,1}}$ is a root of an algebraic equation in $\mathbb{C}^\star \times \mathbb{C}^\star$:

$$(37) \qquad P^\Gamma(x, e^{W_{0,1}}; t) = 0$$

where t is the 't Hooft parameter $g_s N$. It should be noticed that this symmetrisation is not unique; however the freedom of choice here is in bijection with the freedom of choice of an irreducible $\mathcal{G}_\Gamma$-module on the Toda side: in either case (see [15, 27] and [78] respectively) it can be shown that these choices do not affect the calculation of the partition function, so we will henceforth drop the subscript ρ from Eq. (36).

(3) For these cases, a computational tour-de-force leads to determine in full detail the spectral curve for type A [24], D [16], and E_6 [15]; substantial information can be extracted for $E_{7,8}$, with a full solution available in all cases in the limit $t \to 0$.

(4) Now, proving the B-model side of the GOV correspondence for the LMO invariant amounts to finding a restriction $u_i(t)$ of the Toda hamiltonians of the ADE relativistic Toda chain such that

$$(38) \qquad \mathfrak{p}^\Gamma\left(x; u_i + \delta_{i,\mathsf{k}}(y + u_0 y^{-1})\right)\big|_{u_i = u_i(t)} = P^\Gamma(x, y; t)$$

A detailed analysis of both spectral curves setup shows[6] that this is indeed the case for all ADE types [15, 27].

(5) Finally, having a matrix integral representation for the partition function of Chern–Simons theory on these spaces allows to rigorously derive the topological recursion of Eqs. (17) and (18) for the cumulants of the resulting distribution. On the other hand, on the B-model side, the topological recursion can be either regarded as the definition of the higher genus open/closed topological B-string on a family of curves, or, from a physics standpoint, it could be derived from the chiral boson theory on the spectral curve obtained from dimensional reduction of the BCOV Kodaira–Spencer theory of gravity [13, 33]. Since the two theories boil down to the same recursion with the same input datum, we reach the conclusion that they give rise to the same invariants to *all orders in* $1/N$ and arbitrary colourings of the invariants. This yields an all-genus proof of the B-model side of the GOV correspondence for this type of manifolds, restricted to the LMO invariant.

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[6]Technically, the polynomials almost never match on the nose, but it can always be shown that upon restriction the Chern–Simons spectral curve arises as a non-trivial reducible component of a degenerate limit of the Toda curve.

5. Conclusions

We have reviewed how a strict generalisation of the GOV correspondence that bears all the ingredients of the simplest original setting of [**47**, **81**], including a geometric A-model theory of open/closed Gromov–Witten invariants and an all-genus B-model theory governed by the topological recursion on a specific spectral curve setup, can be given for the case of Clifford–Klein 3-manifolds – and these alone, according to our remarks in the previous section. This opens severals directions for future research, including an extension to the setting of refined/categorified invariants (particularly on the B-model side), quantum integrability, and the relation to gauge theories. We single out here three more topics in particular on which we hope to report in the near future.

B-model general flat backgrounds. Since most of the analysis of the previous section was restricted to the study of the LMO invariant, it is natural to ask how to extend the GOV strings correspondence to a general Chern–Simons vacuum, thus completing the proof of the B-model version of the GOV correspondence. On the B-model side, the family of relativistic Toda spectral curves was constructed in [**15**] for type $ADE_{6,7}$, and the missing E_8 case has recently been treated in detail in [**27**]. The more difficult bit here is to provide a complete large N analysis of the matrix model, although this might be possible by suitably rewriting the finite sums expressions of [**14**, **71**] in terms of ordinary eigenvalue models on the real line. Establishing an explicit solution of the loop equations for this matrix model in terms of the topological recursion applied on the corresponding Toda spectral curve would give a full proof of the B-side of the GOV correspondence. This would also shed light on some of the difficulties encountereed in the analysis of arbitrary flat backgrounds to the case of general lens spaces and non-SU(2) abelian quotients of the resolved conifold [**25**].

Non-toric "remodeling-the-B-model". Another line of developement consists of extending the remodeling proposal of [**19**] to the non-toric setting at hand for type D and E. A first step here would be to fully spell out the computation of the disk functions as in [**21**, **22**] for the case at hand, and then derive the topological recursion from the analysis of the descendent theory. A promising route would be to derive the J- and R-calibrations for the quantum cohomology of $\widehat{X}^{\Gamma}$ from the steepest descent analysis of oscillating integrals of the Toda differential, as in [**23**], and then retrieve the topological recursion from Givental's R-action on the associated cohomological field theory [**36**, **44**]. This would lead to a proof of the remodeling conjecture on an important class of examples, beyond the toric case.

The quantised McKay correspondence. One of the more intriguing consequences of the B-model GOV correspondence is that several limits of the B-model geometries of Section 4.4 would provide a unified construction of spectral curves relevant for both non-toric mirror symmetry and the theory of Frobenius manifolds. In particular, the $u_0 \to 0$ limit, corresponding to the limit where the Kähler volume of the base $\mathbb{P}^1$ in $\widehat{X}^{\Gamma}$ is sent to infinity, gives rise to a family of 1-dimensional Landau–Ginzburg models for the stack $[\mathbb{C}^2/\Gamma]$ and its crepant resolution: this would finally grant access to a host of explicit computations on the descendent theory, which are likely to be instrumental in the proof of the quantum McKay correspondence in full generality [**28**, **31**, **87**]. This would notably include the higher genus theory, by following the arguments employed for the type A case in [**23**]. Moreover, Dubrovin's almost duality would relate these mirrors to the LG formulation of the Frobenius

manifold structure on the orbit spaces of extended affine Weyl groups and ordinary Weyl groups associated to simply-laced root systems upon considering, respectively, the relativistic and non-relativistic limit of the Toda curves of Section 4.4 [**27**]. We plan to further explore this in future work.

Acknowledgments

References

[1] Mina Aganagic, Miranda C. N. Cheng, Robbert Dijkgraaf, Daniel Krefl, and Cumrun Vafa, *Quantum geometry of refined topological strings*, J. High Energy Phys. **11** (2012), 019, front matter + 52. MR3036500

[2] Mina Aganagic, Robbert Dijkgraaf, Albrecht Klemm, Marcos Mariño, and Cumrun Vafa, *Topological strings and integrable hierarchies*, Comm. Math. Phys. **261** (2006), no. 2, 451–516, DOI 10.1007/s00220-005-1448-9. MR2191887

[3] Mina Aganagic, Tobias Ekholm, Lenhard Ng, and Cumrun Vafa, *Topological strings, D-model, and knot contact homology*, Adv. Theor. Math. Phys. **18** (2014), no. 4, 827–956. MR3277674

[4] Mina Aganagic, Albrecht Klemm, Marcos Mariño, and Cumrun Vafa, *The topological vertex*, Comm. Math. Phys. **254** (2005), no. 2, 425–478, DOI 10.1007/s00220-004-1162-z. MR2117633

[5] Mina Aganagic, Albrecht Klemm, Marcos Mariño, and Cumrun Vafa, *Matrix model as a mirror of Chern-Simons theory*, J. High Energy Phys. **2** (2004), 010, 46, DOI 10.1088/1126-6708/2004/02/010. MR2046799

[6] Mina Aganagic, Marcos Mariño, and Cumrun Vafa, *All loop topological string amplitudes from Chern-Simons theory*, Comm. Math. Phys. **247** (2004), no. 2, 467–512, DOI 10.1007/s00220-004-1067-x. MR2063269

[7] Mina Aganagic and Shamil Shakirov, *Knot homology and refined Chern-Simons index*, Comm. Math. Phys. **333** (2015), no. 1, 187–228, DOI 10.1007/s00220-014-2197-4. MR3294947

[8] M. Aganagic and C. Vafa, *Mirror symmetry, D-branes and counting holomorphic discs* (2000), available at `hep-th/0012041`.

[9] Gernot Akemann, *Higher genus correlators for the Hermitian matrix model with multiple cuts*, Nuclear Phys. B **482** (1996), no. 1-2, 403–430, DOI 10.1016/S0550-3213(96)00542-1. MR1424003

[10] Michael Atiyah, Juan Maldacena, and Cumrun Vafa, *An M-theory flop as a large N duality*, J. Math. Phys. **42** (2001), no. 7, 3209–3220, DOI 10.1063/1.1376159. Strings, branes, and M-theory. MR1840340

[11] D. Bar-Natan and R. Lawrence, *A rational surgery formula for the LMO invariant*, Israel J. Math. **140** (2004), 29–60. MR2054838

[12] Chris Beasley and Edward Witten, *Non-abelian localization for Chern-Simons theory*, J. Differential Geom. **70** (2005), no. 2, 183–323. MR2192257

[13] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes*, Comm. Math. Phys. **165** (1994), no. 2, 311–427. MR1301851

[14] Matthias Blau and George Thompson, *Chern-Simons theory on Seifert 3-manifolds*, J. High Energy Phys. **9** (2013), 033, front matter+25. MR3144954

[15] G. Borot and A. Brini, *Chern-Simons theory on spherical Seifert manifolds, topological strings and integrable systems*, Adv. Theor. Math. Phys., to appear (2018), available at 1506. 06887.

[16] Gaëtan Borot, Bertrand Eynard, and Alexander Weisse, *Root systems, spectral curves, and analysis of a Chern-Simons matrix model for Seifert fibered spaces*, Selecta Math. (N.S.) **23** (2017), no. 2, 915–1025, DOI 10.1007/s00029-016-0266-6. MR3624903

[17] Vincent Bouchard, Bogdan Florea, and Marcos Mariño, *Counting higher genus curves with crosscaps in Calabi-Yau orientifolds*, J. High Energy Phys. **12** (2004), 035, 35 pp. (2005), DOI 10.1088/1126-6708/2004/12/035. MR2130115

[18] Vincent Bouchard, Bogdan Florea, and Marcos Mariño, *Topological open string amplitudes on orientifolds*, J. High Energy Phys. **2** (2005), 002, 35, DOI 10.1088/1126-6708/2005/02/002. MR2140507

[19] Vincent Bouchard, Albrecht Klemm, Marcos Mariño, and Sara Pasquetti, *Remodeling the B-model*, Comm. Math. Phys. **287** (2009), no. 1, 117–178, DOI 10.1007/s00220-008-0620-4. MR2480744

[20] E. Brézin, C. Itzykson, G. Parisi, and J. B. Zuber, *Planar diagrams*, Comm. Math. Phys. **59** (1978), no. 1, 35–51. MR0471676

[21] Andrea Brini, *Open topological strings and integrable hierarchies: remodeling the A-model*, Comm. Math. Phys. **312** (2012), no. 3, 735–780, DOI 10.1007/s00220-012-1489-9. MR2925133

[22] A. Brini and R. Cavalieri, *Open orbifold Gromov-Witten invariants of $[\mathbb{C}^3/\mathbb{Z}_n]$: localization and mirror symmetry*, Selecta Math. (N.S.) **17** (2011), no. 4, 879–933, available at 1007. 0934. MR2861610 (2012i:14068)

[23] A. Brini, R. Cavalieri, and D. Ross, *Crepant resolutions and open strings*, J. Reine Angew. Math. in press (2013), https://doi.org/10.1515/crelle{2017{0011, available at 1309.4438.

[24] Andrea Brini, Bertrand Eynard, and Marcos Mariño, *Torus knots and mirror symmetry*, Ann. Henri Poincaré **13** (2012), no. 8, 1873–1910, DOI 10.1007/s00023-012-0171-2. MR2994763

[25] Andrea Brini, Luca Griguolo, Domenico Seminara, and Alessandro Tanzini, *Chern-Simons theory on $L(p,q)$ lens spaces and Gopakumar-Vafa duality*, J. Geom. Phys. **60** (2010), no. 3, 417–429, DOI 10.1016/j.geomphys.2009.11.006. MR2600004

[26] Andrea Brini, Marcos Mariño, and Sébastien Stevan, *The uses of the refined matrix model recursion*, J. Math. Phys. **52** (2011), no. 5, 052305, 24, DOI 10.1063/1.3587063. MR2839073

[27] Andrea Brini, *E_8 spectral curves* (2017), available at 1711.05958.

[28] Jim Bryan and Tom Graber, *The crepant resolution conjecture*, Algebraic geometry—Seattle 2005. Part 1, Proc. Sympos. Pure Math., vol. 80, Amer. Math. Soc., Providence, RI, 2009, pp. 23–42, DOI 10.1090/pspum/080.1/2483931. MR2483931

[29] Leonid Chekhov and Bertrand Eynard, *Hermitian matrix model free energy: Feynman graph technique for all genera*, J. High Energy Phys. **3** (2006), 014, 18, DOI 10.1088/1126-6708/2006/03/014. MR2222762

[30] Jinwon Choi, Sheldon Katz, and Albrecht Klemm, *The refined BPS index from stable pair invariants*, Comm. Math. Phys. **328** (2014), no. 3, 903–954, DOI 10.1007/s00220-014-1978-0. MR3201216

[31] Tom Coates, Hiroshi Iritani, and Hsian-Hua Tseng, *Wall-crossings in toric Gromov-Witten theory. I. Crepant examples*, Geom. Topol. **13** (2009), no. 5, 2675–2744, DOI 10.2140/gt.2009.13.2675. MR2529944

[32] D.-E. Diaconescu, V. Shende, and C. Vafa, *Large N duality, Lagrangian cycles, and algebraic knots*, Comm. Math. Phys. **319** (2013), no. 3, 813–863, DOI 10.1007/s00220-012-1563-3. MR3040377

[33] R. Dijkgraaf and C. Vafa, *Two-dimensional Kodaira–Spencer theory and three-dimensional Chern–Simons gravity* (2007), available at 0711.1932.

[34] Jacques Distler and Cumrun Vafa, *A critical matrix model at $c = 1$*, Modern Phys. Lett. A **6** (1991), no. 3, 259–270, DOI 10.1142/S0217732391000221. MR1087735

[35] S. K. Donaldson and R. P. Thomas, *Gauge theory in higher dimensions*, The geometric universe (Oxford, 1996), Oxford Univ. Press, Oxford, 1998, pp. 31–47. MR1634503

[36] P. Dunin-Barkowski, N. Orantin, S. Shadrin, and L. Spitz, *Identification of the Givental formula with the spectral curve topological recursion procedure*, Comm. Math. Phys. **328** (2014), no. 2, 669–700, DOI 10.1007/s00220-014-1887-2. MR3199996

[37] Bertrand Eynard, *Topological expansion for the 1-Hermitian matrix model correlation functions*, J. High Energy Phys. **11** (2004), 031, 35 pp. (2005), DOI 10.1088/1126-6708/2004/11/031. MR2118807

[38] B. Eynard, *Intersection numbers of spectral curves* (2011), available at `1104.0176`.

[39] B. Eynard, *Invariants of spectral curves and intersection theory of moduli spaces of complex curves*, Commun. Number Theory Phys. **8** (2014), no. 3, 541–588, DOI 10.4310/CNTP.2014.v8.n3.a4. MR3282995

[40] B. Eynard, T. Kimura, and S. Ribault, *Random matrices* (2015), available at `1510.04430`.

[41] B. Eynard and N. Orantin, *Invariants of algebraic curves and topological expansion*, Commun. Number Theory Phys. **1** (2007), no. 2, 347–452, DOI 10.4310/CNTP.2007.v1.n2.a4. MR2346575

[42] B. Eynard and N. Orantin, *Computation of open Gromov-Witten invariants for toric Calabi-Yau 3-folds by topological recursion, a proof of the BKMP conjecture*, Comm. Math. Phys. **337** (2015), no. 2, 483–567, DOI 10.1007/s00220-015-2361-5. MR3339157

[43] C. Faber and R. Pandharipande, *Hodge integrals and Gromov-Witten theory*, Invent. Math. **139** (2000), no. 1, 173–199, DOI 10.1007/s002229900028. MR1728879

[44] B. Fang, C-C. M. Liu, and Z. Zong, *On the Remodeling Conjecture for Toric Calabi-Yau 3-Orbifolds* (2016), available at `1604.07123`.

[45] Vladimir V. Fock and Andrey Marshakov, *Loop groups, clusters, dimers and integrable systems*, Geometry and quantization of moduli spaces, Adv. Courses Math. CRM Barcelona, Birkhäuser/Springer, Cham, 2016, pp. 1–66. MR3675462

[46] Debashis Ghoshal and Cumrun Vafa, *c = 1 string as the topological theory of the conifold*, Nuclear Phys. B **453** (1995), no. 1-2, 121–128, DOI 10.1016/0550-3213(95)00408-K. MR1358307

[47] Rajesh Gopakumar and Cumrun Vafa, *On the gauge theory/geometry correspondence*, Adv. Theor. Math. Phys. **3** (1999), no. 5, 1415–1443, DOI 10.4310/ATMP.1999.v3.n5.a5. MR1796682

[48] T. Graber and R. Pandharipande, *Localization of virtual classes*, Invent. Math. **135** (1999), no. 2, 487–518, DOI 10.1007/s002220050293. MR1666787

[49] David J. Gross and Igor Klebanov, *One-dimensional string theory on a circle*, Nuclear Phys. B **344** (1990), no. 3, 475–498, DOI 10.1016/0550-3213(90)90667-3. MR1076752

[50] Jie Gu, Hans Jockers, Albrecht Klemm, and Masoud Soroush, *Knot invariants from topological recursion on augmentation varieties*, Comm. Math. Phys. **336** (2015), no. 2, 987–1051, DOI 10.1007/s00220-014-2238-z. MR3322393

[51] Sergei Gukov, Albert Schwarz, and Cumrun Vafa, *Khovanov-Rozansky homology and topological strings*, Lett. Math. Phys. **74** (2005), no. 1, 53–74, DOI 10.1007/s11005-005-0008-8. MR2193547

[52] Nick Halmagyi, Takuya Okuda, and Vadim Yasnov, *Large-N duality, lens spaces and the Chern-Simons matrix model*, J. High Energy Phys. **4** (2004), 014, 10, DOI 10.1088/1126-6708/2004/04/014. MR2080896

[53] Nick Halmagyi and Vadim Yasnov, *The spectral curve of the lens space matrix model*, J. High Energy Phys. **11** (2009), 104, 21, DOI 10.1088/1126-6708/2009/11/104. MR2628861

[54] Søren Kold Hansen, *Reshetikhin-Turaev invariants of Seifert 3-manifolds and a rational surgery formula*, Algebr. Geom. Topol. **1** (2001), 627–686, DOI 10.2140/agt.2001.1.627. MR1875611

[55] J. Harer and D. Zagier, *The Euler characteristic of the moduli space of curves*, Invent. Math. **85** (1986), no. 3, 457–485. MR848681

[56] J. L. Harer, *The cohomology of the moduli space of curves*, Theory of moduli (Montecatini Terme, 1985), 1988, pp. 138–221. MR963064

[57] H. Hopf, *Zum Clifford-Kleinschen Raumproblem*, Math. Ann. **95** (1926), no. 1, 313–339. MR1512281

[58] K. Hori, A. Iqbal, and C. Vafa, *D-branes and mirror symmetry* (2000), available at `hep-th/0005247`.

[59] K. Hori and C. Vafa, *Mirror symmetry* (2000), available at `hep-th/0002222`.

[60] Amer Iqbal, Can Kozçaz, and Cumrun Vafa, *The refined topological vertex*, J. High Energy Phys. **10** (2009), 069, 58, DOI 10.1088/1126-6708/2009/10/069. MR2607441

[61] Johan Källén, *Cohomological localization of Chern-Simons theory*, J. High Energy Phys. **8** (2011), 008, 32, DOI 10.1007/JHEP08(2011)008. MR2876080

[62] Sheldon Katz, Albrecht Klemm, and Cumrun Vafa, *M-theory, topological strings and spinning black holes*, Adv. Theor. Math. Phys. **3** (1999), no. 5, 1445–1537, DOI 10.4310/ATMP.1999.v3.n5.a6. MR1796683

[63] Sheldon Katz and Chiu-Chu Melissa Liu, *Enumerative geometry of stable maps with Lagrangian boundary conditions and multiple covers of the disc*, The interaction of finite-type and Gromov-Witten invariants (BIRS 2003), Geom. Topol. Monogr., vol. 8, Geom. Topol. Publ., Coventry, 2006, pp. 1–47, DOI 10.2140/gtm.2006.8.1. MR2402818

[64] Maxim Kontsevich, *Enumeration of rational curves via torus actions*, The moduli space of curves (Texel Island, 1994), Progr. Math., vol. 129, Birkhäuser Boston, Boston, MA, 1995, pp. 335–368, DOI 10.1007/978-1-4612-4264-2_12. MR1363062

[65] J. M. F. Labastida and Marcos Mariño, *Polynomial invariants for torus knots and topological strings*, Comm. Math. Phys. **217** (2001), no. 2, 423–449, DOI 10.1007/s002200100374. MR1821231

[66] José M. F. Labastida, Marcos Mariño, and Cumrun Vafa, *Knots, links and branes at large N*, J. High Energy Phys. **11** (2000), Paper 7, 42, DOI 10.1088/1126-6708/2000/11/007. MR1806596

[67] Thang T. Q. Le, Jun Murakami, and Tomotada Ohtsuki, *On a universal perturbative invariant of 3-manifolds*, Topology **37** (1998), no. 3, 539–574, DOI 10.1016/S0040-9383(97)00035-9. MR1604883

[68] Kefeng Liu and Pan Peng, *Proof of the Labastida-Mariño-Ooguri-Vafa conjecture*, J. Differential Geom. **85** (2010), no. 3, 479–525. MR2739811

[69] I. G. Macdonald, *The volume of a compact Lie group*, Invent. Math. **56** (1980), no. 2, 93–95, DOI 10.1007/BF01392542. MR558859

[70] Juan Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998), no. 2, 231–252, DOI 10.4310/ATMP.1998.v2.n2.a1. MR1633016

[71] Marcos Mariño, *Chern-Simons theory, matrix integrals, and perturbative three-manifold invariants*, Comm. Math. Phys. **253** (2005), no. 1, 25–49, DOI 10.1007/s00220-004-1194-4. MR2105636

[72] Marcos Mariño, *Chern-Simons theory, matrix models and topological strings*, International Series of Monographs on Physics, vol. 131, The Clarendon Press, Oxford University Press, Oxford, 2005. MR2177747

[73] Marcos Mariño, *Open string amplitudes and large order behavior in topological string theory*, J. High Energy Phys. **3** (2008), 060, 34, DOI 10.1088/1126-6708/2008/03/060. MR2391060

[74] Marcos Mariño, *String theory and the Kauffman polynomial*, Comm. Math. Phys. **298** (2010), no. 3, 613–643, DOI 10.1007/s00220-010-1088-6. MR2670922

[75] M. Mariño and C. Vafa, *Framed knots at large N*, Contemp. Math. **310** (2002), 185–204, available at `hep-th/0108064`. MR1950947 (2004c:57024)

[76] E. J. Martinec and N. P. Warner, *Integrable systems and supersymmetric gauge theory*, Nucl. Phys. **B459** (1996), 97–112, available at `hep-th/9509161`.

[77] D. Maulik, A. Oblomkov, A. Okounkov, and R. Pandharipande, *Gromov-Witten/Donaldson-Thomas correspondence for toric 3-folds*, Invent. Math. **186** (2011), no. 2, 435–479, DOI 10.1007/s00222-011-0322-y. MR2845622

[78] A. McDaniel and L. Smolinsky, *A Lie-theoretic Galois theory for the spectral curves of an integrable system. II*, Trans. Amer. Math. Soc. **349** (1997), no. 2, 713–746. MR1401779

[79] A. A. Migdal, *Loop Equations and 1/N Expansion*, Phys. Rept. **102** (1983), 199–290.

[80] Nikita Nekrasov, *Five-dimensional gauge theories and relativistic integrable systems*, Nuclear Phys. B **531** (1998), no. 1-3, 323–344, DOI 10.1016/S0550-3213(98)00436-2. MR1657857

[81] H. Ooguri and C. Vafa, *Knot invariants and topological strings*, Nucl. Phys. **B577** (2000), 419–438, available at `hep-th/9912123`.

[82] Hirosi Ooguri and Cumrun Vafa, *Worldsheet derivation of a large N duality*, Nuclear Phys. B **641** (2002), no. 1-2, 3–34, DOI 10.1016/S0550-3213(02)00620-X. MR1928179

[83] R. C. Penner, *Perturbative series and the moduli space of Riemann surfaces*, J. Differential Geom. **27** (1988), no. 1, 35–53. MR918455

[84] Vipul Periwal, *Topological closed-string interpretation of Chern-Simons theory*, Phys. Rev. Lett. **71** (1993), no. 9, 1295–1298, DOI 10.1103/PhysRevLett.71.1295. MR1232813

[85] N. Reshetikhin and V. G. Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups*, Invent. Math. **103** (1991), no. 3, 547–597, DOI 10.1007/BF01239527. MR1091619

[86] N. Yu. Reshetikhin and V. G. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Comm. Math. Phys. **127** (1990), no. 1, 1–26. MR1036112

[87] Yongbin Ruan, *The cohomology ring of crepant resolutions of orbifolds*, Gromov-Witten theory of spin curves and orbifolds, Contemp. Math., vol. 403, Amer. Math. Soc., Providence, RI, 2006, pp. 117–126, DOI 10.1090/conm/403/07597. MR2234886

[88] N. Seiberg and E. Witten, *Erratum: "Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory"*, Nuclear Phys. B **430** (1994), no. 2, 485–486, DOI 10.1016/0550-3213(94)00449-8. MR1303306

[89] S. Sinha and C. Vafa, *SO and Sp Chern–Simons at large N* (2000), available at `hep-th/0012136`.

[90] Andrew Strominger, *Special geometry*, Comm. Math. Phys. **133** (1990), no. 1, 163–180. MR1071240

[91] G. 't Hooft, *A Planar Diagram Theory for Strong Interactions*, Nucl. Phys. **B72** (1974), 461.

[92] Cumrun Vafa, *Superstrings and topological strings at large N*, J. Math. Phys. **42** (2001), no. 7, 2798–2817, DOI 10.1063/1.1376161. Strings, branes, and M-theory. MR1840317

[93] Harold Williams, *Double Bruhat cells in Kac-Moody groups and integrable systems*, Lett. Math. Phys. **103** (2013), no. 4, 389–419, DOI 10.1007/s11005-012-0604-3. MR3029326

[94] Edward Witten, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. **121** (1989), no. 3, 351–399. MR990772

[95] E. Witten, *Chern-Simons gauge theory as a string theory*, The Floer memorial volume, Progr. Math., vol. 133, Birkhäuser, Basel, 1995, pp. 637–678. MR1362846

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