

Double quantization of Seiberg–Witten geometry and W-algebras

Taro Kimura

ABSTRACT. We show that the double quantization of Seiberg–Witten spectral curve for Γ -quiver gauge theory defines the generating current of $W(\Gamma)$ -algebra in the free field realization. We also show that the partition function is given as a correlator of the corresponding $W(\Gamma)$ -algebra, which is equivalent to the AGT relation under the gauge/quiver (spectral) duality.

CONTENTS

1. Introduction and summary
 2. Seiberg–Witten spectral curve and 1st quantization
 3. Operator formalism and 2nd quantization
 4. From double quantization to Virasoro/W-algebra
 5. Z -state
 6. Quiver W-algebra
 7. Affine quiver W-algebra
 8. Quiver elliptic W-algebra
- Acknowledgments
References

1. Introduction and summary

Recent progress on non-perturbative aspects of supersymmetric gauge theory exhibits interesting connections with various concepts of mathematical physics. In particular, Seiberg–Witten theory for four-dimensional $\mathcal{N} = 2$ gauge theory [1, 2] provides a geometrical interpretation for the Coulomb branch moduli space as a base of the algebraic integrable system [3–6]. As a result, the Seiberg–Witten curve can be identified as a classical spectral curve of the affine Toda chain for the pure SYM theory, the $SU(2)$ rational spin chain for $\mathcal{N} = 2$ SQCD with the fundamental hypermultiplets [7].

2010 *Mathematics Subject Classification*. Primary 81R10; Secondary 81T60, 14D21, 81R50.

This work was supported in part by Keio Gijuku Academic Development Funds, JSPS Grant-in-Aid for Scientific Research (No. JP17K18090), the MEXT-Supported Program for the Strategic Research Foundation at Private Universities “Topological Science” (No. S1511006), JSPS Grant-in-Aid for Scientific Research on Innovative Areas “Topological Materials Science” (No. JP15H05855), and “Discrete Geometric Analysis for Materials Design” (No. JP17H06462).

TABLE 1. Gauge theory geometry and IIA/M/F theory realization associated with rational/trigonometric/elliptic integrable systems. See, for example, [16].

Gauge theory geometry	Geometric realization	Integrable system
\mathbb{R}^4	IIA string on CY3	rational
$\mathbb{R}^4 \times S^1$	M theory on CY3 $\times S^1$	trigonometric
$\mathbb{R}^4 \times T^2$	F theory on CY3 $\times T^2$	elliptic

The relation between gauge theory and integrable system has various generalizations:

Dimensional hierarchy

Replacing the spacetime on which the gauge theory is defined with five and six dimensional manifolds compactified on S^1 and T^2 , it corresponds to trigonometric and elliptic deformations of rational integrable system [8, 9], as shown in Table 1. The Seiberg–Witten curve for 4d/5d/6d theory coincides with the spectral curve of rational/trigonometric/elliptic integrable system.

Quiver gauge theory

In the relation to integrable system, change of the gauge symmetry doesn't lead to change of underlying symmetry on the integrable system side. For this purpose, we need quiver gauge theory labeled by a quiver graph Γ . The corresponding symmetry group for integrable system turns out to be Lie group G_Γ whose Dynkin diagram is given by Γ , and the Seiberg–Witten geometry is described by the fundamental characters of G_Γ group [9].

Ω -background and quantization

The four-dimensional spacetime for gauge theory allows an equivariant deformation with Ω -background [10, 11], and its specific limit $(\epsilon_1, \epsilon_2) \rightarrow (\hbar, 0)$, called the Nekrasov–Shatashvili (NS) limit, corresponds to quantization of integrable system [12, 13]. Then the Seiberg–Witten curve characterized by the algebraic relation is promoted to difference equation, and this quantum spectral curve is identified as Baxter's TQ-relation for G_Γ -spin chain, which is also equivalent to the corresponding Bethe ansatz equation [14] described by the polynomial equation called the q -character [15] (Table 2).

From this point of view, we still have another deformation parameter: We have used only one of equivariant parameters to quantize the integrable system. Thus it is natural to ask *what is the underlying algebraic structure for generic Ω -background parametrization*. It has been recently shown that the polynomial equation still holds for generic (ϵ_1, ϵ_2) just by replacing the q -character with another algebraic structure, called the qq -character [17–19]. See also [20–28] for its realization in gauge theory and topological string theory.

TABLE 2. Ω -background and (double) quantization of Seiberg–Witten geometry described by G_Γ -character/ q -character/ qq -character for Γ -quiver gauge theory.

Ω -background	SW geometry	Integrable system
$(\epsilon_1, \epsilon_2) = (0, 0)$	character	classical
$(\epsilon_1, \epsilon_2) = (\hbar, 0)$	q -character	quantum
$(\epsilon_1, \epsilon_2) \neq (0, 0)$ (w/ t -extention)	qq -character	doubly quantum (W-algebra)

We proceed with this program to construct W-algebras using quiver gauge theory [29–31]. In our construction, in addition to consider generic equivariant parameters (ϵ_1, ϵ_2) , we incorporate the higher time variables, which deform UV prepotential by the local operators [32]. Our claim is as follows:

The qq -character together with double quantization of Seiberg–Witten spectral curve for the t -extended Γ -quiver gauge theory defines the generating current of $W(\Gamma)$ -algebra.

The doubly quantum spectral curve is an operator relation, in contrast to the classical and quantum curves which are relations between correlation functions. This double quantization scheme will be discussed in Sec. 3 in more detail.

Another claim about the partition function is:

The t -extended Γ -quiver partition function is a state (Z -state) in the Hilbert space, and the non- t -extended partition function is given by a correlation function of $W(\Gamma)$ -algebra.

This statement is analogous, but slightly different from the AGT relation, which claims that the partition function of gauge theory enjoying gauge symmetry G is given by a correlation function of $W(G)$ -algebra [33, 34].¹ This reflects that our construction of $W(\Gamma)$ is equivalent to AGT for $W(G)$ by the gauge/quiver duality $G \leftrightarrow \Gamma$ [38–40], which is confirmed for A -type quiver (Fig. 1). In the classical limit, this duality becomes the Nahm–Fourier–Mukai duality between the phase space of Hitchin system on a cylinder and the phase space of periodic monopoles [9]. We remark a similar realization of W-algebra correlators associated with the quiver structure [41–43].

Let us summarize our construction of quiver W-algebras [29, 30]. We start with the classical Seiberg–Witten curve given in the limit $(\epsilon_1, \epsilon_2) \rightarrow (0, 0)$. For Γ -quiver gauge theory, this spectral curve is given by the fundamental characters of G_Γ group, if the quiver diagram Γ coincides with a Dynkin diagram for finite simple Lie algebras. This construction is also applicable to generic quiver which

¹Our construction naturally gives rise to the q -deformation of $W(\Gamma)$ -algebra from 5d quiver gauge theory, while the ordinary AGT relation claims a connection between 4d gauge theory and $W(G)$ -algebra without q -deformation. The counterpart of our construction is thus the q -AGT relation for 5d gauge theory and the q -deformed $W(G)$ -algebra [35]. A similar idea to realize the Virasoro/W-algebra through quantization of geometry is found in [36, 37] for non- q -deformed case.

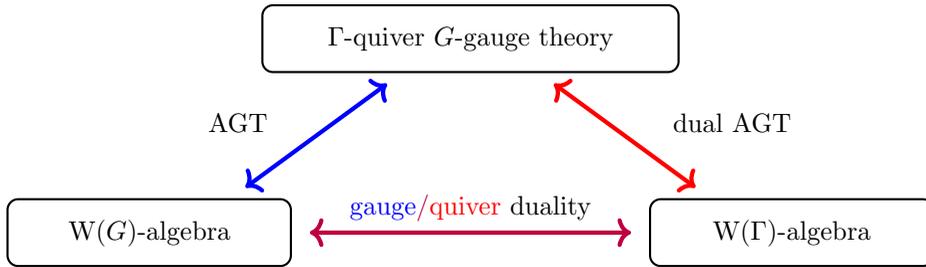


FIGURE 1. AGT and dual AGT

doesn't correspond to any finite Lie algebra. Tuning on the equivariant parameter as $(\epsilon_1, \epsilon_2) \rightarrow (\hbar, 0)$, we obtain the quantum Seiberg–Witten curve, which is identified as the TQ-relation of the integrable G_Γ -spin chain. We then consider the double quantization of Seiberg–Witten curve by taking into account the higher time variables with the fully generic equivariant parameters (ϵ_1, ϵ_2) . We obtain an operator relation, which is identified as the quantum Sugawara construction of the generating current of $W(\Gamma)$ -algebra.

This article is a self-contained review based on [29, 30]. Our construction exhibits several new features of W -algebras:

Edge (bifundamental) mass-deformation of W -algebra

In general, quiver gauge theory consists of the vector multiplet assigned to a node of quiver i , and the hypermultiplet in bifundamental representation for an edge e with multiplicative mass parameter μ_e . (We will deal with (anti)fundamental hypermultiplets separately.) We will show that the bifundamental mass gives rise to new mass-deformation of W -algebra, which is reduced to Frenkel–Reshetikhin's construction in the limit $\mu_e \rightarrow 1$ [44, 45].

Affine quiver W -algebra

Our construction of W -algebra is applicable to generic quiver gauge theory. Applying this construction to affine quiver theory, e.g., $\mathcal{N} = 2^*$ theory (\hat{A}_0 -quiver), we obtain new W -algebra associated to affine Lie algebra. The Seiberg–Witten curve (and its quantization) is described by the affine character, which is given as an infinite series.

Quiver elliptic W -algebra

We can construct elliptically deformed W -algebras using six-dimensional quiver gauge theory on $\mathbb{R}^4 \times T^2$, which corresponds to the elliptic integrable system.

This article contains several new features compared with the original papers [29, 30]: We introduce the A -operator to discuss the i Weyl reflection in Sec. 6. We show the explicit OPE relations for A_2 quiver in Sec. 6.2. We also provide the fundamental qq -characters for D_4 quiver in Sec. 6.4. Furthermore, generalization to fractional quiver, which gives rise to non-simply-laced algebra, is found in [31].

The rest of this article is organized as follows. We start in Sec. 2 to discuss the classical Seiberg–Witten geometry and its 1st quantization. The resultant quantum spectral curve obtained in the NS limit is a difference operator, which turns out to be Baxter’s TQ-relation, and the Q -operator plays a role of the wavefunction. In Sec. 3 we discuss 2nd quantization of the spectral curve. To obtain an operator relation, we introduce the higher time variable as a Heisenberg oscillator, leading to modification of the UV prepotential. We then show in Sec. 4 how the generating current of q -deformed Virasoro algebra arises in 5d gauge theory. The construction using the qq -character yields the free field realization of the current. In Sec. 5 we show that the t -extended gauge theory partition function is given by a state, which we call the Z -state, in the Hilbert space through the operator/state correspondence. In particular, the Z -state is constructed with the screening charge of the corresponding conformal algebra. We also show that the (anti)fundamental matter contribution is realized by a vertex operator applied to the state. In Sec. 6 we show how to construct $W(\Gamma)$ -algebra associated with generic quiver Γ from the mass-deformed q -Cartan matrix characterizing the quiver gauge theory. We discuss the regularity of the qq -character which provides the generating current, and show it commutes with the screening charge of $W(\Gamma)$ -algebra. We demonstrate our construction with several examples reproducing known results in the limit. In Sec. 7 our construction is applied to affine quiver theory, involving the adjoint matter, corresponding to 4d $\mathcal{N} = 2^*$ (5d $\mathcal{N} = 1^*$) theory. This quiver doesn’t correspond to finite Lie algebra, and thus provides a new W -algebra such that the generating current is given by an infinite sum of the fundamental current. In Sec. 8 we discuss elliptic generalization of quiver W -algebra using 6d gauge theory defined on $\mathbb{R}^4 \times T^2$. The whole construction discussed above is straightforwardly lifted, and thus we obtain elliptically deformed W -algebras associated with generic quiver.

2. Seiberg–Witten spectral curve and 1st quantization

The low-energy effective dynamics of 4d $\mathcal{N} = 2$ gauge theory is completely determined by the algebraic curve called the Seiberg–Witten curve [1, 2]. For $U(n)$ SYM theory it is characterized by the algebraic relation

$$(2.1) \quad y(x) + \frac{1}{y(x)} = T_n(x)$$

where $T_n(x)$ is a degree n monic polynomial in x^{-1} variable,² and the vev of Coulomb moduli parameter is encoded into its coefficients.³ This algebraic relation can be obtained from the saddle point analysis of gauge theory partition function [9, 11, 46], and its derivation is quite analogous to the large N analysis of matrix models: y -variable is realized as a one-point function of a generating function of gauge invariant observables (the Y -operator) [14]

$$(2.3) \quad y(x) = \langle Y(x) \rangle,$$

²While we usually use the convention such that $T_n(x)$ is a polynomial in x , we apply the opposite convention in this article for latter convenience.

³In this article we basically use the 5d notation:

$$(2.2) \quad q_1 = e^{\epsilon_1}, \quad q_2 = e^{\epsilon_2}, \quad q = q_1 q_2, \quad \{\nu_{i,n}\}_{i \in \Gamma_0, \alpha \in [1, \dots, n_i]} = \{e^{\alpha_i, n}\}_{i \in \Gamma_0, \alpha \in [1, \dots, n_i]}.$$

These parameters correspond to the standard notation of q -deformed W -algebras: $(q_1, q_2) = (t^{-1}, q)$.

which plays a similar role to the resolvent function having a singularity in the complex plane. Incorporating the one-form differential defined on the curve, called the Seiberg–Witten differential,⁴

$$(2.4) \quad \lambda = \begin{cases} (4d) & x d \log y & (x, y) \in \mathbb{C} \times \mathbb{C}^\times \\ (5d) & \log x d \log y & (x, y) \in \mathbb{C}^\times \times \mathbb{C}^\times \end{cases},$$

they define the Seiberg–Witten geometry describing the SUSY vacuum structure. We remark that the curve given by (2.1) coincides with the classical spectral curve of the affine n -site Toda chain, and also the inhomogeneous limit of $SU(2)$ rational/trigonometric/elliptic spin chain for $4d/5d/6d$ gauge theory.

Although the spectral curve is obtained in the classical limit $\epsilon_1, \epsilon_2 \rightarrow 0$ ($q_1, q_2 \rightarrow 1$), a similar structure can be found even after turning on the equivariant parameters. Taking into account one of the equivariant parameters $(\epsilon_1, \epsilon_2) \rightarrow (\hbar, 0)$, which is called the Nekrasov–Shatashvili limit, we obtain a difference equation instead of the algebraic relation for $5d$ theory [13, 14]⁵

$$(2.5) \quad y(x) + \frac{1}{y(q_1^{-1}x)} = T_n(x).$$

This is again obtained by the saddle point analysis with respect to $\epsilon_2 \rightarrow 0$, and the other finite parameter $\epsilon_1 = \hbar$ plays a role of the Planck constant. Indeed the difference equation (2.5) is interpreted as a quantization of the spectral curve (2.1) with respect to a canonical pair $(\log x, \log y)$,

$$(2.6) \quad [\log y, \log x] = \hbar,$$

which corresponds to the one-form (2.4) or the associated symplectic two-form

$$(2.7) \quad \Omega = d \log x \wedge d \log y.$$

This quantization implies the y -variable behaves as a shift operator $y = \exp(\hbar \partial_{\log x})$, and thus the classical curve (2.1) becomes a difference operator

$$(2.8) \quad [e^{\hbar \partial_{\log x}} + e^{-\hbar \partial_{\log x}} - T_n(x)] Q(x) = 0,$$

which is equivalent to (2.5) under identification⁶

$$(2.9) \quad y(x) = \frac{Q(q_1 x)}{Q(x)}.$$

In this way, quantization of spectral curve gives a differential/difference equation. In particular, in this case, the quantum curve (2.8) is equivalent to Baxter’s TQ-relation for $SU(2)$ spin chain,⁷ which is a consequence of the connection between gauge theory and integrable system at the quantum level.

⁴The higher-dimensional lift is equivalent to imposing periodicity: The cylindrical coordinate for $5d$ theory on $\mathbb{R}^4 \times S^1$ is given by $\mathbb{C}^\times = \mathbb{C}/\mathbb{Z}$. The geometry for $6d$ gauge theory compactified on a torus $\mathbb{R}^4 \times T^2$ is parametrized by a coordinate $(x, y) \in E_{T^2} \times \mathbb{C}^\times$ where the elliptic curve E_{T^2} is doubly periodic, $E_{T^2} = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ with the torus modulus τ .

⁵This is also called the q -character due to its representation theoretical origin in quantum affine algebra [15].

⁶This transformation is analogous to that for the Riccati-type differential equation, widely used to derive quantum spectral curves.

⁷To obtain a precise agreement, we have to take into account the Chern–Simons factor and the (anti)fundamental hypermultiplets. See [13, 14] for more arguments.

In addition, it has been recently revealed that a similar polynomial relation holds even away from the saddle point analysis, which is called the qq -character equation [17, 20, 23, 25, 26]

$$(2.10) \quad y(x) + \frac{1}{y(q^{-1}x)} = T_n(x).$$

Precisely speaking, the second factor should be $y(q^{-1}x)^{-1} \rightarrow \langle \mathbf{Y}(q^{-1}x)^{-1} \rangle$ as discussed below. By the analogy with the matrix model and other related theories, this is quite natural for quantum curve. For example, in the case of the matrix model, the characteristic polynomial average plays a similar role to the wavefunction $Q(x)$ in (2.8), and actually satisfies the quantum curve equation at finite N under identification $\hbar = 1/N$. In general, quantization scheme is a way to go back to the finite coupling regime from the classical limit. We remark that, in the unrefined limit $q_1q_2 = q = 1$ ($\epsilon_1 + \epsilon_2 = 0$), it is not distinguishable from the classical relation (2.1), since the summation plays a role of the Planck constant in this case, $\epsilon_1 + \epsilon_2 = \hbar$. However, when we consider the higher-weight character, an extra factor has to be taken into account as discussed later.

3. Operator formalism and 2nd quantization

As mentioned above, the y -variable plays a similar role to the resolvent function, and thus the classical spectral curve (2.1) and its quantizations (2.5) and (2.10) yield relations between (one-point) correlation functions in gauge theory. To emphasize it, let us rewrite the qq -character with the \mathbf{Y} -operator

$$(3.1) \quad \langle \mathbf{Y}(x) \rangle + \left\langle \frac{1}{\mathbf{Y}(q^{-1}x)} \right\rangle = T_n(x).$$

Now the degree of $T_n(x)$ depends on the gauge group rank. However, the meaning of average $\langle \mathcal{O}(x) \rangle$ actually depends on the theory that we consider (gauge group, matter content, and so on). From this point of view, it is natural to define T -operator whose average with $U(n)$ SYM theory is given by

$$(3.2) \quad \langle T(x) \rangle = T_n(x).$$

Then we obtain a universal operator relation

$$(3.3) \quad \mathbf{Y}(x) + \frac{1}{\mathbf{Y}(q^{-1}x)} = T(x).$$

This is the 2nd quantization of Seiberg–Witten spectral curve in our sense, and this equation defines the T -operator in terms of the \mathbf{Y} -operator. We remark this operator relation is independent of gauge group rank, in contrast to the polynomial relation. In this sense, it can be seen as a universal relation for gauge theory.

Precisely speaking, in order to obtain a proper operator relation, we need to take into account the higher time variables, namely the potential term in gauge theory

$$(3.4) \quad Z_{\text{pot}}(t) = \exp \left(\sum_{n=1}^{\infty} t_n \mathbf{Y}^{[n]} \right),$$

so that the t -extended partition function plays a role of generating function of the gauge theory observable $\mathbf{Y}^{[n]}$, which is a trace of the n -th power adjoint scalar $\text{Tr } \Phi^n$ for 4d, or its loop operator along a compactified circle for 5d theory. Inclusion of

the potential term corresponds to deformation of the UV prepotential with the holomorphic operators [32]

$$(3.5) \quad \mathcal{F}_{\text{UV}} \longrightarrow \mathcal{F}_{\text{UV}} + \sum_{n=1}^{\infty} t_n \text{Tr} \Phi^n .$$

Then since the situation is completely parallel with the matrix model, let us explain the reason why we need the potential term (3.4) by the analogy. The resolvent operator of matrix model is a generating function of the gauge invariant observable, which is a single trace operator,

$$(3.6) \quad \text{Tr} \frac{1}{x - X} = \sum_{n=0}^{\infty} x^{-n-1} \text{Tr} X^n .$$

This observable is equivalent to derivative with respect to the coupling constant, playing a role of the time variable, in the potential term

$$(3.7) \quad \frac{\partial}{\partial t_n} e^{\text{Tr} V(X)} = \text{Tr} X^n e^{\text{Tr} V(X)}$$

where

$$(3.8) \quad V(x) = \sum_{n=1}^{\infty} t_n x^n .$$

In this sense the single trace observable is replaced with the derivative with t_n under the presence of the potential term.

4. From double quantization to Virasoro/W-algebra

In terms of the gauge theory observable, the Y-operator (in particular, its average) is obtained [9, 14]

$$(4.1) \quad \langle Y(x) \rangle = \exp \left(- \sum_{n=1}^{\infty} \frac{x^{-n}}{n} \mathbf{Y}^{[n]} \right) .$$

Taking into account the higher time variables, the Y-operator shows t -dependence, which is written in terms of (q_1, q_2) -modified free field operators [29]

$$(4.2) \quad Y(x) = q_1^{\frac{1}{2}} : \exp \left(y_0 + \sum_{n \neq 0} y_n x^{-n} \right) :$$

where the free fields are defined

$$(4.3) \quad (n > 0) \quad y_{-n} = \frac{(1 - q_1^n)(1 - q_2^n)}{1 + q^n} t_n, \quad y_n = -\frac{1}{n} \frac{\partial}{\partial t_n}, \quad y_0 = -\frac{1}{2} t_0 \log q_2 .$$

A constant and zero mode factors are added for convenience. According to the operator relation (3.3), it defines the T -operator in the free field realization, and it turns out to coincide with the generating current of q -deformed Virasoro algebra [44, 47]. Indeed the generator appearing in the mode expansion

$$(4.4) \quad T(x) = \sum_{n \in \mathbb{Z}} T_n x^{-n}$$

satisfies the algebraic relation

$$(4.5) \quad [T_n, T_m] = - \sum_{k=1}^{\infty} f_k (T_{n-k} T_{m+k} - T_{m-k} T_{n+k}) - \frac{(1-q_1)(1-q_2)}{1-q} (q^n - q^{-n}) \delta_{n+m,0}$$

where

$$(4.6) \quad \sum_{k=0}^{\infty} f_k x^k = \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{(1-q_1^n)(1-q_2^n)}{1+q^n} x^n \right) =: f(x).$$

In practice, it is convenient to see the q -analog of OPE from the two-point function of the generating current, which is equivalent to the algebraic relation (4.5),

$$(4.7) \quad f \left(\frac{y}{x} \right) T(x)T(y) - f \left(\frac{x}{y} \right) T(y)T(x) = - \frac{(1-q_1)(1-q_2)}{1-q} \left(\delta \left(\frac{qy}{x} \right) - \delta \left(\frac{qx}{y} \right) \right)$$

where the delta function is defined

$$(4.8) \quad \delta(x) = \sum_{n \in \mathbb{Z}} x^n.$$

This shows that double quantization of Seiberg–Witten spectral curve provides the generating current of q -Virasoro algebra in the free field realization. We remark that such an idea on the realization of generating current through the quantization of Seiberg–Witten curve was already discussed in the context of the (non q -deformed) AGT relation [33].

What we should remark is that the conformal algebra obtained here using the gauge theory does not depend on the gauge group rank: we obtain (q -deformed) Virasoro algebra for $U(n)$ gauge theory for arbitrary $n \in \mathbb{Z}_{\geq 1}$. This situation is actually different from the AGT relation where the underlying conformal algebra is associated to the gauge group.

5. Z -state

Let us see how the gauge theory partition function is interpreted in our construction. In particular, the partition function extended with the time variables (3.4) explicitly depends on them. Since the time variable behaves as an operator, the t -extended partition function $Z(t)$ has to be identified as an operator. This interpretation directly leads to the notion of Z -state through the operator/state correspondence of CFT.

We show that the t -extended full partition function (including both one-loop and instanton contributions) for $U(n)$ SYM theory is a state given by [29]

$$(5.1) \quad |Z\rangle = \sum_{\mathcal{X} \in \mathfrak{M}^T} \prod_{x \in \mathcal{X}} S(x) |1\rangle$$

where the product is a radial-ordered product, and \mathfrak{M}^T is the torus action fixed point in the instanton moduli space, characterized by a set of partitions [11] with

$$(5.2) \quad \mathcal{X} = \{x_{\alpha,k}\}_{\alpha \in [1 \dots n], k \in [1 \dots \infty]}, \quad x_{\alpha,k} = \nu_{\alpha} q_1^{k-1} q_2^{\lambda_{\alpha,k}}.$$

The vacuum state is denoted by $|1\rangle$, which is a constant with respect to the time variables, $\partial_{t_n} |1\rangle = 0$, namely a primary state, and $S(x)$ is a screening current operator

$$(5.3) \quad S(x) = : \exp \left(s_0 \log x + \tilde{s}_0 + \frac{\kappa}{2} (\log_{q_2}^2 x - \log_{q_2} x) + \sum_{n \neq 0} s_n x^{-n} \right) : .$$

As well as the Y -operator, they are (q_1, q_2) -deformed free field operators

$$(5.4) \quad (n > 0) \quad s_{-n} = (1 - q_1^n) t_n, \quad s_n = -\frac{1 + q^{-n}}{n(1 - q_2^{-n})} \frac{\partial}{\partial t_n}, \quad s_0 = t_0$$

obeying the commutation relation

$$(5.5) \quad [s_n, s_m] = -\frac{1}{n} \frac{1 - q_1^n}{1 - q_2^{-n}} (1 + q^{-n}) \delta_{n+m, 0} .$$

The zero mode t_0 corresponds to the gauge coupling $t_0 = \log_{q_2} q$, but we need some redefinition to obtain precise agreement with the gauge theory result. The additional zero mode satisfies

$$(5.6) \quad [\tilde{s}_0, s_n] = -2\beta \delta_{n, 0}$$

where $\beta = -\epsilon_1/\epsilon_2$, and the κ -term is added to reproduce the Chern–Simons term in gauge theory. See [29] for details.

The Z -state can be also written using the screening charge, which commutes with the generating current $T(x)$. First of all, the summation over the fixed points \mathfrak{M}^\top in the partition function can be replaced with $\mathbb{Z}^{\mathcal{X}_0}$, which is a set of arbitrary integer sequences terminating by zeros. \mathcal{X}_0 is the ground configuration $\lambda_{\alpha, k} = 0$ for $k \in \mathbb{Z}_{>0}$, and $\hat{x}_{\alpha, k} = \nu_\alpha q_1^{k-1}$. If the partition $\lambda_{\alpha, k}$ is not a non-increasing sequence, there appears a zero factor in the expression

$$(5.7) \quad \prod_{x \in \mathcal{X}} S(x) |1\rangle = 0$$

for $\mathcal{X} \in \mathbb{Z}^{\mathcal{X}_0}$ but $\mathcal{X} \notin \mathfrak{M}^\top$. Therefore the Z -state is given by

$$(5.8) \quad |Z\rangle = \prod_{\hat{x} \in \mathcal{X}_0} S(\hat{x}) |1\rangle$$

where we define the screening charge⁸

$$(5.10) \quad S(\hat{x}) = \sum_{k \in \mathbb{Z}} S(q_2^k \hat{x}) .$$

For q -deformed Virasoro/ W -algebra, the screening charge is a discrete sum of the screening current. Precisely speaking, this sum is given by the Jackson integral to obtain a proper convergence property.

⁸This infinite sum can be written as the Jackson integral with respect to q_2 ,

$$(5.9) \quad S(x) = \int dx_{q_2} S(x) ,$$

which is more convenient to discuss its convergence property. In the limit $q_2 \rightarrow 1$, the discrete sum is replaced with the usual integral.

Let us mention an important property of the screening charge. The OPE between the Y -operator and the screening current is given by

$$(5.11) \quad Y(x)S(x') = \frac{1 - x'/x}{1 - q_1 x'/x} : Y(x)S(x') : , \quad S(x')Y(x) = q_1^{-1} \frac{1 - x/x'}{1 - q_1^{-1} x/x'} : Y(x)S(x') :$$

and thus the commutation relation shows

$$(5.12) \quad [Y(x), S(x')] = (1 - q_1^{-1}) \delta\left(q_1 \frac{x'}{x}\right) : Y(x)S(x') :$$

where the delta function defined (4.8). Similarly we obtain

$$(5.13) \quad [Y(q^{-1}x)^{-1}, S(x')] = -(1 - q_1^{-1}) \delta\left(q \frac{x'}{x}\right) : Y(x)S(x') :$$

where we used the formula

$$(5.14) \quad q_1^{-1} : Y(x)Y(q^{-1}x) : = : S(q^{-1}x)S(q_1^{-1}x)^{-1} :$$

obtained by comparing the y and s fields, (4.3) and (5.4). We have almost the same OPEs for Y and Y^{-1} except for the q_2 -shift. This discrepancy is canceled in the screening charge because it is defined as a summation over the q_2 -shifted screening currents. Therefore the T -operator turns out to commute with the screening charge

$$(5.15) \quad [T(x), S(x')] = 0 .$$

This assures the regularity of the T -operator, and the mode expansion (4.4). This commutativity also corresponds to the pole cancellation mechanism in the qq -character.

In order to obtain the non-extended plain partition function, we apply a dual vacuum state $\langle 1|$ because it plays a role of the projector into the $t = 0$ sector, $\langle 1| t_n = 0$. Thus the plain partition function is given by a correlator

$$(5.16) \quad Z(t = 0) = \langle 1|Z\rangle = \langle 1| \prod_{\hat{x} \in \mathcal{X}_0} S(\hat{x}) |1\rangle .$$

See also [41–43] for a similar construction using little string theory.⁹ This expression resembles the AGT relation [33], which is the coincidence between the gauge theory partition function and the Liouville/Toda CFT correlator. In the case of the AGT relation, we need to subtract the $U(1)$ factor to obtain agreement between the instanton partition function and the conformal block. In this case, on the other hand, we don't need to care about such an extra factor. We can directly see the agreement. In addition, as mentioned before, the underlying conformal algebra does not depend on the gauge group. We will come back to this issue later.

So far we have focused on the vector multiplet contribution. We can also incorporate the fundamental hypermultiplet in the operator formalism. Since the

⁹In these papers, the authors discuss a correlator which consists of finite screening currents by considering the defect operators.

fixed point contribution to the gauge theory observable is given by¹⁰

$$(5.18) \quad \mathbf{Y}^{[n]} = (1 - q_1^n) \sum_{x \in \mathcal{X}} x^n,$$

the potential term (3.4) yields the fundamental matter contribution by the shift of time variables

$$(5.19) \quad t_n \longrightarrow t_n + \frac{1}{n} \frac{1}{(1 - q_1^n)(1 - q_2^n)} \mu^{-n}$$

where $\mu \in \mathbb{C}^\times$ is the multiplicative mass parameter. We introduce an operator which induces the time shift

$$(5.20) \quad \mathbf{V}(x) = : \exp \left(\sum_{n \neq 0} v_n x^{-n} \right) :$$

where

$$(5.21) \quad (n > 0) \quad v_{-n} = -\frac{1}{1 + q^n} t_n, \quad v_n = \frac{1}{n} \frac{1}{(1 - q_1^n)(1 - q_2^n)} \frac{\partial}{\partial t_n}.$$

We compute the OPE of \mathbf{V} and S operators

$$(5.22) \quad \mathbf{V}(x)S(x') = \left(\frac{x'}{x}; q_2 \right)_{\infty}^{-1} : \mathbf{V}(x)S(x') : , \quad S(x')\mathbf{V}(x) = \left(\frac{q_2 x}{x'}; q_2 \right)_{\infty} : \mathbf{V}(x)S(x') : .$$

They correspond to the fundamental and antifundamental hypermultiplet contributions, while the OPE of \mathbf{V} and \mathbf{V} does not yield dynamical contribution. Thus the extended partition function in the presence of (anti)fundamental matters is obtained by inserting the \mathbf{V} -operators

$$(5.23) \quad |Z\rangle = \left(\prod_{x \in \mathcal{X}_f} \mathbf{V}(x) \right) \left(\prod_{\hat{x} \in \mathcal{X}_0} S(\hat{x}) \right) \left(\prod_{x \in \tilde{\mathcal{X}}_f} \mathbf{V}(x) \right) |1\rangle$$

where $\mathcal{X}_f = \{\mu_f\}_{f \in [1 \dots n_f]}$ and $\tilde{\mathcal{X}}_f = \{\tilde{\mu}_f\}_{f \in [1 \dots \tilde{n}_f]}$ are sets of fundamental and anti-fundamental mass parameters. This \mathbf{V} -operator puts a singularity on the spectral curve at $x = \mu_f$. Then the plain partition function is given by a correlator with additional vertex operators

$$(5.24) \quad Z(t=0) = \langle 1 | \left(\prod_{x \in \mathcal{X}_f} \mathbf{V}(x) \right) \left(\prod_{\hat{x} \in \mathcal{X}_0} S(\hat{x}) \right) \left(\prod_{x \in \tilde{\mathcal{X}}_f} \mathbf{V}(x) \right) |1\rangle .$$

This correlator is given by an infinite discrete sum, which can be interpreted as a q -analog of Dotsenko–Fateev’s integral formula of the conformal block [48, 49].

¹⁰This expression can be interpreted as a q -deformation of n -th Casimir element

$$(5.17) \quad C_n = \frac{1}{n} \sum_{k=1}^{\infty} \left(\left(\lambda_k - k + \frac{1}{2} \right)^n - \left(-k + \frac{1}{2} \right)^n \right) .$$

6. Quiver W-algebra

The construction shown above is applicable to generic quiver gauge theory, and we correspondingly obtain the W-algebra associated to the quiver structure. Let us first fix the notations. Let Γ be a quiver with the set of nodes Γ_0 and the set of edges Γ_1 . A quiver Γ defines a $|\Gamma_0| \times |\Gamma_0|$ matrix

$$(6.1) \quad c_{ij}^{[n]} = (1 + q^{-n})\delta_{ij} - \sum_{e:i \rightarrow j} \mu_e^{-n} - \sum_{e:j \rightarrow i} \mu_e^n q^{-n}$$

where $\mu_e \in \mathbb{C}^\times$ is the multiplicative bifundamental mass parameter associated to each edge $e \in \Gamma_1$. This is (q, μ_e) -deformation of Cartan matrix, reduced to the ordinary Cartan matrix in the limit $n \rightarrow 0$. In the analogy with matrix model, the quiver theory with gauge group $\times_{i \in \Gamma_0} U(n_i)$ corresponds to the ADE multi-matrix model [50, 51], so that we have $|\Gamma_0|$ sets of *eigenvalues*

$$(6.2) \quad \mathcal{X}_i = \{x_{i,\alpha,k}\}_{\alpha \in [1 \dots n_i], k \in [1 \dots \infty]}, \quad \mathcal{X} = \bigsqcup_{i \in \Gamma_0} \mathcal{X}_i.$$

Let $i: \mathcal{X} \rightarrow \Gamma_0$ be the node label so that $i(x) = i$ for $x \in \mathcal{X}_i$.

Then the construction is quite parallel with the previous situation: The t -extended partition function is given by a state

$$(6.3) \quad |Z\rangle = \prod_{\hat{x} \in \mathcal{X}_0} S_{i(\hat{x})}(\hat{x}) |1\rangle.$$

The screening charge is defined as a summation over the screening current defined

$$(6.4) \quad S_i(x) = : \exp \left(s_{i,0} \log x + \tilde{s}_{i,0} + \frac{\kappa_i}{2} (\log_{q_2}^2 x - \log_{q_2} x) + \sum_{n \neq 0} s_{i,n} x^{-n} \right) : .$$

where the (q_1, q_2) modified free fields are written using the deformed Cartan matrix

$$(6.5) \quad (n > 0) \quad s_{i,-n} = (1 - q_1^n)t_{i,n}, \quad s_{i,n} = -\frac{1}{n(1 - q_2^{-n})} c_{ji}^{[n]} \frac{\partial}{\partial t_{j,n}}, \quad s_{i,0} = t_{i,0}$$

which obey the commutation relation

$$(6.6) \quad [s_{i,n}, s_{j,m}] = -\frac{1}{n} \frac{1 - q_1^n}{1 - q_2^{-n}} c_{ji}^{[n]} \delta_{n+m,0}.$$

Again the zero mode is related to the gauge coupling $t_{i,0} = \log_{q_2} \mathbf{q}_i$, under some redefinition. The additional zero mode similarly satisfies

$$(6.7) \quad [\tilde{s}_{i,0}, s_{j,n}] = -\beta \delta_{n,0} c_{ji}^{[0]}.$$

We can show that the Z -state (6.3) coincides with the t -extended gauge theory partition function [29].

Let us then discuss how the generating current is constructed using the Y -operator. For quiver gauge theory, the Y -operator is defined for each node $i \in \Gamma_0$, corresponding to the fundamental weight,

$$(6.8) \quad Y_i(x) = q_1^{\tilde{\rho}_i} : \exp \left(y_{i,0} + \sum_{n \neq 0} y_{i,n} x^{-n} \right) :$$

where $\tilde{\rho}_i = \sum_{j \in \Gamma_0} \tilde{c}_{ji}^{[0]}$ is the Weyl vector for non-affine quiver, and $\tilde{\rho}_i = 0$ for affine quiver. The modified free fields are given by¹¹

$$(6.9) \quad (n > 0) \quad y_{i,-n} = (1 - q_1^n)(1 - q_2^n) \tilde{c}_{ji}^{[-n]} t_{j,n},$$

$$y_{i,n} = -\frac{1}{n} \frac{\partial}{\partial t_{i,n}}, \quad y_{i,0} = -\tilde{c}_{ji}^{[0]} t_{j,0} \log q_2$$

with the commutation relation

$$(6.10) \quad [y_{i,n}, y_{j,m}] = -\frac{1}{n} (1 - q_1^n)(1 - q_2^n) \tilde{c}_{ij}^{[-n]} \delta_{n+m,0}$$

$$= -\frac{1}{n} (1 - q_1^{-n})(1 - q_2^{-n}) \tilde{c}_{ji}^{[n]} \delta_{n+m,0}.$$

In this case, we can define the T -operator which starts with each operator $Y_i(x)$ for $i \in \Gamma_0$. It is generated by the local reflection, which is the mass deformed iWeyl reflection [17]¹²

$$(6.11) \quad Y_i(x) \longrightarrow Y_i(x) \times \left(\frac{1}{Y_i(x)Y_i(q^{-1}x)} \prod_{e:i \rightarrow j} Y_j(\mu_e^{-1}x) \prod_{e:j \rightarrow i} Y_j(\mu_e q^{-1}x) \right)$$

where we assume that there is no self-connecting edge in the quiver. See Sec. 7 for \widehat{A}_0 quiver. This reflection formula is obtained by comparing configurations at the fixed points which differ by one instanton factor. This means that we shall see the behavior under the partition shift $\lambda_{i,\alpha,k} \rightarrow \lambda_{i,\alpha,k} + 1$, which is equivalent to $x \rightarrow q_2x$ for $x \in \mathcal{X}_i$. Although the Y -operator itself has a singularity, it is cancelled by the reflection. Then we obtain

$$(6.12) \quad T_i(x) = Y_i(x) + : Y_i(q^{-1}x)^{-1} \prod_{e:i \rightarrow j} Y_j(\mu_e^{-1}x) \prod_{e:j \rightarrow i} Y_j(\mu_e q^{-1}x) : + \dots$$

If the reflection gives rise to another singularity, we apply another reflection to cancel it. For the finite-type quiver, this procedure closes within a finite time.

Since the Z -state (6.3) is constructed by the screening charges, we consider the shift of screening current,

$$(6.13) \quad : \frac{S_i(x)}{S_i(q_2x)} : = q_1^{-1} A_i(x)$$

where the A -operator is defined

$$(6.14) \quad A_i(x) = q_1 : \exp \left(a_{i,0} + \sum_{n \neq 0} a_{i,n} x^{-n} \right) :$$

with the free field operators

$$(6.15) \quad a_{i,n} = (1 - q_2^{-n}) s_{i,n}, \quad a_{i,0} = -t_{i,0} \log q_2.$$

¹¹For the quiver whose Cartan matrix is not invertible $\det c^{[0]} = 0$, we cannot incorporate the coupling constant into the zero mode $t_{i,0} = \log_{q_2} q_i$. One needs to put the q_i factor for every iWeyl reflection by hand.

¹²In the classical limit $q_1, q_2, \mu_e \rightarrow 1$, this reflection is reduced to the ordinary Weyl reflection, and thus the classical Seiberg–Witten geometry is expressed using the character of group G_Γ , associated to Dynkin-quiver diagram Γ [9].

These A and Y operators correspond to “root” and “fundamental weight” respectively [45]. Thus the oscillators are related as follows:

$$(6.16) \quad a_{i,n} = y_{j,n} c_{j_i}^{[n]},$$

or explicitly,

$$(6.17) \quad A_i(x) = : \frac{Y_i(x)Y_i(qx)}{\prod_{e:i \rightarrow j} Y_j(\mu_e^{-1}qx) \prod_{e:j \rightarrow i} Y_j(\mu_e x)} : .$$

This leads to the pole cancellation in the i Weyl reflection

$$(6.18) \quad \text{Res}_{x' \rightarrow q^{-1}x} [Y_i(x)S_i(q_2x')] + \text{Res}_{x' \rightarrow q^{-1}x} [: Y_i(x)A_i(q^{-1}x)^{-1} : S_i(x')] = 0.$$

Similarly we can show that the T -operator commutes with the screening charge defined as a summation over the screening operator (5.10),

$$(6.19) \quad [T_i(x), S_j(x')] = 0.$$

This shows that the T -operator yields the generating current of $W(\Gamma)$ -algebra associated to Γ -quiver in the free field realization.

We remark that when there appears a product of the Y -operators from the same node $i \in \Gamma_0$ in the reflection (6.12), one needs an extra structure

$$(6.20) \quad : Y_i(x)Y_i(x') : + S\left(\frac{x'}{x}\right) : \frac{Y_i(x)Y_i(x')}{A_i(q^{-1}x)} : + S\left(\frac{x}{x'}\right) : \frac{Y_i(x)Y_i(x')}{A_i(q^{-1}x')} : \\ + : \frac{Y_i(x)Y_i(x')}{A_i(q^{-1}x)A_i(q^{-1}x')} :$$

where the factor corresponding to the OPE between Y and A operators defined

$$(6.21) \quad S(x) = \frac{(1 - q_1x)(1 - q_2x)}{(1 - qx)(1 - x)}.$$

Furthermore, in the collision limit $x' \rightarrow x$, it involves a derivative term

$$(6.22) \quad : Y_i(x)^2 : + : \left(\mathfrak{c}(q_1, q_2) - \frac{(1 - q_1)(1 - q_2)}{1 - q} \partial_{\log x} \log A_i(q^{-1}x) \right) \frac{Y_i(x)^2}{A_i(q^{-1}x)} : \\ + : \frac{Y_i(x)^2}{A_i(q^{-1}x)^2} :$$

where the constant defined

$$(6.23) \quad \mathfrak{c}(q_1, q_2) = \lim_{x \rightarrow 1} (S(x) + S(x^{-1})) = \frac{1 - 6q + q^2 + (q_1 + q_2)(1 + q)}{(1 - q)^2}.$$

In the NS limit $q_2 \rightarrow 1$, the derivative term vanishes, and the factor becomes $\mathfrak{c}(q_1, 1) = 2$ with $S(x) \rightarrow 1$. Thus the reflection structure becomes simple. In general, one can consider the product of more than two Y -operators. In that case, we obtain correspondingly higher derivative terms.

We notice that the free fields (6.9) explicitly depend on the bifundamental mass parameter, which may affect the algebraic relation of W-algebra. Since our construction coincides with the definition of W-algebra given by [45] in the massless limit $\mu_e \rightarrow 1$ for $e \in \Gamma_1$, the algebra obtained from quiver gauge theory yields a new mass deformation of W-algebra. Furthermore, the bifundamental mass can be absorbed by redefinition of the Coulomb moduli parameter when there is no loop edge [9, 14].

6.1. A_1 quiver. We show more explicit examples. The simplest example is the A_1 quiver. The T -operator is given by

$$(6.24) \quad T_1(x) = Y_1(x) + \frac{1}{Y_1(q^{-1}x)},$$

which is a double quantum deformation of the fundamental representation character for $G_\Gamma = SU(2)$

$$(6.25) \quad \chi_2^{SU(2)}(y) = y + y^{-1}.$$

In addition, we compute the degree n current, corresponding to $(n/2)$ -spin representation of $SU(2)$, with the weight (w_1, \dots, w_n)

$$(6.26) \quad \begin{aligned} T_1^{[w]}(x) &= : Y_1(w_1x)Y_1(w_2x) \cdots Y_1(w_nx) : + \cdots \\ &= \sum_{I \cup J = \{1 \dots n\}} \prod_{i \in I, j \in J} \mathcal{S}\left(\frac{w_i}{w_j}\right) : \prod_{i \in I} Y_1(w_ix) \prod_{j \in J} Y_1(w_jq^{-1}x)^{-1} : . \end{aligned}$$

The degree two current is especially used to define the algebraic relation for the q -Virasoro algebra (4.7).

6.2. A_2 quiver. The next example is the A_2 quiver, consisting of two nodes and an edge connecting them. We have two T -operators

$$(6.27) \quad T_1(x) = Y_1(x) + : \frac{Y_2(\mu^{-1}x)}{Y_1(q^{-1}x)} : + \frac{1}{Y_2(\mu^{-1}q^{-1}x)},$$

$$(6.28) \quad T_2(x) = Y_2(x) + : \frac{Y_1(\mu q^{-1}x)}{Y_2(q^{-1}x)} : + \frac{1}{Y_1(\mu q^{-2}x)},$$

where $\mu = \mu_{1 \rightarrow 2} = \mu_{2 \rightarrow 1}^{-1}q$. They correspond to the fundamental and antifundamental representation characters of $G_\Gamma = SU(3)$

$$(6.29) \quad \chi_{\mathbf{3}}^{SU(3)}(y_1, y_2) = y_1 + y_1^{-1}y_2 + y_2^{-1}, \quad \chi_{\bar{\mathbf{3}}}^{SU(3)}(y_1, y_2) = y_2 + y_1y_2^{-1} + y_1^{-1}.$$

The OPEs for these T -operators are given by

$$\begin{aligned}
 f_{11} \left(\frac{y}{x} \right) T_1(x)T_1(y) - f_{11} \left(\frac{x}{y} \right) T_1(y)T_1(x) \\
 (6.30) \qquad \qquad \qquad = -\frac{(1-q_1)(1-q_2)}{1-q} \left(\delta \left(q \frac{y}{x} \right) T_2(\mu^{-1}x) - \delta \left(q^{-1} \frac{y}{x} \right) T_2(\mu^{-1}qx) \right)
 \end{aligned}$$

$$\begin{aligned}
 f_{12} \left(\frac{y}{x} \right) T_1(x)T_2(y) - f_{21} \left(\frac{x}{y} \right) T_2(y)T_1(x) \\
 (6.31) \qquad \qquad \qquad = -\frac{(1-q_1)(1-q_2)}{1-q} \left(\delta \left(\mu q \frac{y}{x} \right) - \delta \left(\mu q^{-2} \frac{y}{x} \right) \right)
 \end{aligned}$$

$$\begin{aligned}
 f_{22} \left(\frac{y}{x} \right) T_2(x)T_2(y) - f_{22} \left(\frac{x}{y} \right) T_2(y)T_2(x) \\
 (6.32) \qquad \qquad \qquad = -\frac{(1-q_1)(1-q_2)}{1-q} \left(\delta \left(q \frac{y}{x} \right) T_1(\mu q^{-1}x) - \delta \left(q^{-1} \frac{y}{x} \right) T_1(\mu x) \right)
 \end{aligned}$$

where

$$(6.33) \qquad f_{ij}(x) = \exp \left(\sum_{n=1}^{\infty} \frac{(1-q_1^n)(1-q_2^n)}{n} \tilde{c}_{ij}^{[-n]} x^n \right).$$

Now the symmetry under exchange $Y_1 \leftrightarrow Y_2$ is obvious. We can read off the algebraic relation for the (q, μ) -deformed $W(A_2)$ algebra from these OPEs, which explicitly depends on the mass parameter μ . We remark that the diagonal element $f_{ii}(x)$ does not depend on the mass parameter $(\mu_e)_{e \in \Gamma_1}$ in general. It appears only in the off-diagonal element $f_{ij}(x)$ for $i \neq j$.

6.3. A_r quiver. Let us demonstrate our construction with the linear quiver theory $\Gamma = A_r$. The T -operators are computed using the iWeyl reflection

$$(6.34) \qquad T_i(\mu_i^{-1}x) = \sum_{1 \leq j_1 \leq \dots \leq j_i \leq r+1} : \prod_{k=1}^i \Lambda_{j_k}(q^{-i+k}x) :$$

where

$$(6.35) \qquad \Lambda_i(x) = Y_i(\mu_i^{-1}x)Y_{i-1}(\mu_{i-1}^{-1}q^{-1}x)^{-1}$$

with $Y_0(x) = Y_{r+1}(x) = 1$, and the mass product defined

$$(6.36) \qquad \mu_i := \mu_{1 \rightarrow 2} \mu_{2 \rightarrow 3} \cdots \mu_{i-1 \rightarrow i}.$$

We remark that these T -operators are given by (q, μ) -deformation of fundamental representation characters for $G_\Gamma = SU(r+1)$. This construction coincides with the quantum Miura transformation for the q -deformed $W(A_r)$ -algebra in the massless limit $\mu_e \rightarrow 1$ [44, 52]

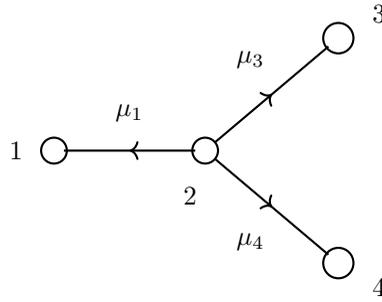


FIGURE 2. D_4 quiver decorated with bifundamental mass $\mu_i = \mu_{2 \rightarrow i}$.

6.4. D_r quiver. We consider D -type quiver gauge theory. The simplest case is $\Gamma = D_4$, shown in Fig. 2. This quiver shows a symmetry under exchange $1 \leftrightarrow 3 \leftrightarrow 4$, which is known as the $SO(8)$ triality. Putting the bifundamental mass parameters $\mu_i := \mu_{2 \rightarrow i}$ for $i = 1, 3, 4$, we compute the T -operator

$$\begin{aligned}
 (6.37) \quad T_1(x) = & Y_1(x) + : \frac{Y_2(\mu_1 q^{-1}x)}{Y_1(q^{-1}x)} : + : \frac{Y_3(\mu_1 \mu_3^{-1} q^{-1}x) Y_4(\mu_1 \mu_4^{-1} q^{-1}x)}{Y_2(\mu_1 q^{-2}x)} : \\
 & + : \frac{Y_4(\mu_1 \mu_4^{-1} q^{-1}x)}{Y_3(\mu_1 \mu_3^{-1} q^{-2}x)} : \\
 & + : \frac{Y_3(\mu_1 \mu_3^{-1} q^{-1}x)}{Y_4(\mu_1 \mu_4^{-1} q^{-2}x)} : + : \frac{Y_2(\mu_1 q^{-2}x)}{Y_3(\mu_1 \mu_3^{-1} q^{-2}x) Y_4(\mu_1 \mu_4^{-1} q^{-2}x)} : \\
 & + : \frac{Y_1(q^{-2}x)}{Y_2(\mu_1 q^{-3}x)} : + \frac{1}{Y_1(q^{-3}x)}.
 \end{aligned}$$

The operators $T_3(x)$ and $T_4(x)$ are similarly obtained by permutation. These three T -operators correspond to three $\mathbf{8}$ -representations of $SO(8)$. The remaining $T_2(x)$ operator, corresponding to $\mathbf{28}$ -representation, involves collision and derivative terms, (6.20) and (6.22),

$$\begin{aligned}
 (6.38) \quad T_2(x) = & T_2^+(x) + T_2^-(x) + \mathcal{S}(q) \left(: \frac{Y_1(\mu_1^{-1}x)}{Y_1(\mu_1^{-1} q^{-1}x)} : + : \frac{Y_3(\mu_3^{-1}x)}{Y_3(\mu_3^{-1} q^{-1}x)} : + : \frac{Y_4(\mu_4^{-1}x)}{Y_4(\mu_4^{-1} q^{-1}x)} : \right) \\
 & + \left(\mathfrak{c}(q_1, q_2) - \frac{(1-q_1)(1-q_2)}{1-q} \partial_{\log x} \log \left(\frac{Y_2(q^{-1}x) Y_2(q^{-2}x)}{Y_1(\mu_1^{-1} q^{-1}x) Y_3(\mu_3^{-1} q^{-1}x) Y_4(\mu_4^{-1} q^{-1}x)} \right) \right) : \\
 & \frac{Y_2(q^{-1}x)}{Y_2(q^{-2}x)} :
 \end{aligned}$$

where

(6.39)

$$\begin{aligned}
 T_2^+(x) = & Y_2(x) + : \frac{Y_1(\mu_1^{-1}x)Y_3(\mu_3^{-1}x)Y_4(\mu_4^{-1}x)}{Y_2(q^{-1}x)} : + : \frac{Y_3(\mu_3^{-1}x)Y_4(\mu_4^{-1}x)}{Y_1(\mu_1^{-1}q^{-1}x)} : \\
 & + : \frac{Y_1(\mu_1^{-1}x)Y_4(\mu_4^{-1}x)}{Y_3(\mu_3^{-1}q^{-1}x)} : + : \frac{Y_1(\mu_1^{-1}x)Y_3(\mu_3^{-1}x)}{Y_4(\mu_4^{-1}q^{-1}x)} : \\
 & + : \frac{Y_1(\mu_1^{-1}x)Y_2(q^{-1}x)}{Y_3(\mu_3^{-1}q^{-1}x)Y_4(\mu_4^{-1}q^{-1}x)} : + : \frac{Y_3(\mu_3^{-1}x)Y_2(q^{-1}x)}{Y_1(\mu_1^{-1}q^{-1}x)Y_4(\mu_4^{-1}q^{-1}x)} : \\
 & + : \frac{Y_4(\mu_4^{-1}x)Y_2(q^{-1}x)}{Y_1(\mu_1^{-1}q^{-1}x)Y_3(\mu_3^{-1}q^{-1}x)} : + : \frac{Y_2(q^{-1}x)^2}{Y_1(\mu_1^{-1}q^{-1}x)Y_3(\mu_3^{-1}q^{-1}x)Y_4(\mu_4^{-1}q^{-1}x)} : \\
 & + : \frac{Y_1(\mu_1^{-1}x)Y_1(\mu_1^{-1}q^{-1}x)}{Y_2(q^{-2}x)} : + : \frac{Y_3(\mu_3^{-1}x)Y_3(\mu_3^{-1}q^{-1}x)}{Y_2(q^{-2}x)} : \\
 & + : \frac{Y_4(\mu_4^{-1}x)Y_4(\mu_4^{-1}q^{-1}x)}{Y_2(q^{-2}x)} :,
 \end{aligned}$$

(6.40)

$$\begin{aligned}
 T_2^-(x) = & \frac{Y_1(\mu_1^{-1}q^{-1}x)Y_3(\mu_3^{-1}q^{-1}x)Y_4(\mu_4^{-1}q^{-1}x)}{Y_2(q^{-2}x)^2} + : \frac{Y_2(q^{-1}x)}{Y_1(\mu_1^{-1}q^{-1}x)Y_1(\mu_1^{-1}q^{-2}x)} : \\
 & + : \frac{Y_2(q^{-1}x)}{Y_3(\mu_3^{-1}q^{-1}x)Y_3(\mu_3^{-1}q^{-2}x)} : + : \frac{Y_2(q^{-1}x)}{Y_4(\mu_4^{-1}q^{-1}x)Y_4(\mu_4^{-1}q^{-2}x)} : \\
 & + : \frac{Y_3(\mu_3^{-1}q^{-1}x)Y_4(\mu_4^{-1}q^{-1}x)}{Y_1(\mu_1^{-1}q^{-2}x)Y_2(q^{-2}x)} : + : \frac{Y_1(\mu_1^{-1}q^{-1}x)Y_4(\mu_4^{-1}q^{-1}x)}{Y_3(\mu_3^{-1}q^{-2}x)Y_2(q^{-2}x)} : \\
 & + : \frac{Y_1(\mu_1^{-1}q^{-1}x)Y_3(\mu_3^{-1}q^{-1}x)}{Y_4(\mu_4^{-1}q^{-2}x)Y_2(q^{-2}x)} : + : \frac{Y_1(\mu_1^{-1}q^{-1}x)}{Y_3(\mu_3^{-1}q^{-2}x)Y_4(\mu_4^{-1}q^{-2}x)} : \\
 & + : \frac{Y_3(\mu_3^{-1}q^{-1}x)}{Y_1(\mu_1^{-1}q^{-2}x)Y_4(\mu_4^{-1}q^{-2}x)} : + : \frac{Y_4(\mu_4^{-1}q^{-1}x)}{Y_1(\mu_1^{-1}q^{-2}x)Y_3(\mu_3^{-1}q^{-2}x)} : \\
 & + : \frac{Y_2(q^{-2}x)}{Y_1(\mu_1^{-1}q^{-2}x)Y_3(\mu_3^{-1}q^{-2}x)Y_4(\mu_4^{-1}q^{-2}x)} : + Y_2(q^{-3}x)^{-1}.
 \end{aligned}$$

In the classical limit, this T_2 -operator is reduced to the character of 28 dimensional representation of $SO(8)$. The \mathfrak{S} -factor appears at the zero weight terms.

7. Affine quiver W-algebra

Applying our construction to affine quiver gauge theory, we can define a new W-algebra, associated to affine Lie algebra. Let us consider the simplest affine quiver $\Gamma = \widehat{A}_0$, corresponding to 4d $\mathcal{N} = 2^*$ (5d $\mathcal{N} = 1^*$) theory, whose deformation of Cartan matrix is given by

$$(7.1) \quad c^{[n]} = 1 + q^{-n} - \mu^{-n} - \mu^n q^{-n} \xrightarrow{n \rightarrow 0} 0.$$

This Cartan matrix can be nontrivial only in the presence of the multiplicative adjoint mass parameter $\mu (\neq 1)$. This means that the mass deformation of the Cartan matrix is essential for the affine quiver, and this deformation goes beyond

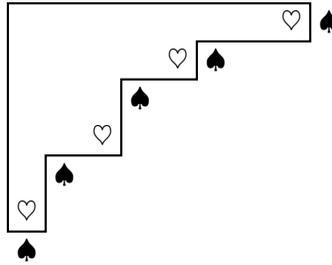


FIGURE 3. The outer and inner boundaries of the partition, $\partial_+ \lambda$ and $\partial_- \lambda$, denote the to-be-added and to-be-removed boxes, ♠ and ♡, respectively. The number of ♠ is always bigger than ♡ by one.

Frenkel–Reshetikhin’s construction of W-algebra [44, 45]. The free field operators for Y and S fields obey the commutation relation

$$(7.2) \quad [y_{1,n}, y_{1,m}] = -\frac{1}{n} \frac{(1 - q_1^n)(1 - q_2^n)}{(1 - \mu^n)(1 - \mu^{-n}q^n)} \delta_{n+m,0},$$

and

$$(7.3) \quad [s_{1,n}, s_{1,m}] = -\frac{1}{n} \frac{1 - q_1^n}{1 - q_2^{-n}} (1 - \mu^{-n})(1 - \mu^n q^{-n}) \delta_{n+m,0}.$$

The s -oscillator becomes trivial in the limit $\mu \rightarrow 1$ (and also $\mu \rightarrow q$), which means that the Nekrasov factor becomes trivial in $\mathcal{N} = 4$ theory, while the y -oscillator becomes singular in this limit.

The T -operator in this case is given by the iWeyl reflection as well, but it does not terminate within a finite sum: the affine character needs an infinite sum parametrized by the partition

$$(7.4) \quad \begin{aligned} T_1(x) &= Y_1(x) + q S(\mu^{-1}) : Y_1(q^{-1}x)^{-1} Y_1(\mu^{-1}x) Y_1(\mu q^{-1}x) : + \dots \\ &= \sum_{\lambda} q^{|\lambda|} Z_{\lambda}^{\widehat{A}_0}(\tilde{q}_1, \tilde{q}_2, \tilde{\mu}) : \prod_{s \in \partial_+ \lambda} Y_1(qx/\tilde{x}(s)) \prod_{s \in \partial_- \lambda} Y_1(x/\tilde{x}(s))^{-1} : \end{aligned}$$

where the coupling constant q is now explicitly appearing because the Cartan matrix $c = (0)$ is not invertible for affine quiver $\Gamma = \widehat{A}_0$. $\partial_+ \lambda$ and $\partial_- \lambda$ are the outer and inner boundary of the partition λ , as shown in Fig. 3, and we define

$$(7.5) \quad \tilde{x}(s) = (\mu^{-1}q)^{s_1-1} \mu^{s_2-1} q,$$

which reads the dual variables

$$(7.6) \quad \tilde{q}_1 = \mu^{-1}q, \quad \tilde{q}_2 = \mu, \quad \tilde{\mu} = q_2, \quad \tilde{\nu} = q.$$

We remark that $\tilde{q} := \tilde{q}_1 \tilde{q}_2 = q$. The Nekrasov instanton function $Z_{\lambda}^{\widehat{A}_0}(\tilde{q}_1, \tilde{q}_2, \tilde{\mu})$ is again for \widehat{A}_0 quiver, but $U(1)$ gauge theory

$$(7.7) \quad Z_{\lambda}^{\widehat{A}_0}(q_1, q_2, \mu) = \prod_{(s_1, s_2) \in \lambda} \frac{(1 - \mu q_1^{l(s)+1} q_2^{-a(s)})(1 - \mu^{-1} q_1^{l(s)} q_2^{-a(s)-1})}{(1 - q_1^{l(s)+1} q_2^{-a(s)})(1 - q_1^{l(s)} q_2^{-a(s)-1})}$$

where the arm and leg lengths are defined: $a(s) = \lambda_{s_1} - s_2$, $l(s) = \lambda_{s_2} - s_1$. The higher degree current is given as a summation over multiple partition. This dual expansion is naturally explained using the 8-dimensional setup [17].

8. Quiver elliptic W-algebra

We have constructed the q -deformed W-algebra $W_{q_1, q_2}(\Gamma)$ associated with Γ -quiver gauge theory in 5d, which corresponds to trigonometric (relativistic) integrable system. According to such a connection with integrable systems, it is natural to consider its elliptic deformation using 6d gauge theory compactified on the elliptic curve $\mathbb{R}^4 \times T^2$ [30]. See also [21, 24, 25, 53–58] for recent results along this direction.

In this case the spectral parameter is periodic

$$(8.1) \quad x \simeq px$$

where τ is the modulus of T^2 that defines $p = e^{2\pi i\tau} \in \mathbb{C}^\times$. One needs to impose this periodicity on the index functor. Recalling that the 5d gauge theory partition function on $\mathbb{R}^4 \times S^1$ is given by the index along S^1 defined

$$(8.2) \quad \mathbb{I} \left[\sum_{\alpha} x_{\alpha} \right] = \prod_{\alpha} (1 - x_{\alpha}^{-1}) ,$$

the elliptic index (equivariant elliptic genus on T^2) yields the 6d partition function for $\mathbb{R}^4 \times T^2$

$$(8.3) \quad \mathbb{I}_p \left[\sum_{\alpha} x_{\alpha} \right] = \prod_{\alpha} \theta(x_{\alpha}^{-1}; p)$$

where the Jacobi theta function is given by¹³

$$(8.7) \quad \begin{aligned} \theta(x; p) &= (x; p)_{\infty} (px^{-1}; p)_{\infty} \\ &= \exp \left(- \sum_{n \neq 0} \frac{1}{n} \frac{x^n}{1 - p^n} \right) . \end{aligned}$$

We assume $|p| < 1$ and this elliptic system is reduced to 5d theory in the limit $p \rightarrow 0$.

¹³In this article we apply the Dolbeault convention, which obeys

$$(8.4) \quad \mathbb{I} [x^{-1}] = -x^{-1} \mathbb{I} [x] , \quad \mathbb{I}_p [x^{-1}] = -x^{-1} \mathbb{I}_p [x] .$$

This convention can be converted into the Dirac index,

$$(8.5) \quad \mathbb{I} \left[\sum_{\alpha} x_{\alpha} \right] = \prod_{\alpha} (x_{\alpha}^{1/2} - x_{\alpha}^{-1/2}) , \quad \mathbb{I}_p \left[\sum_{\alpha} x_{\alpha} \right] = \prod_{\alpha} \theta_1(x_{\alpha}; p)$$

with

$$(8.6) \quad \mathbb{I} [x^{-1}] = -\mathbb{I} [x] , \quad \mathbb{I}_p [x^{-1}] = -\mathbb{I}_p [x] .$$

Under the conformal condition (8.14), these two conventions are consistent.

Since the elliptic index (8.3) involves both positive and negative powers due to the expansion (8.7), we need to introduce “positive” and “negative” oscillators [30, 59, 60]

$$(8.8) \quad \left[y_{i,n}^{(\pm)}, y_{j,m}^{(\pm)} \right] = \mp \frac{1}{n} \frac{(1 - q_1^{\pm n})(1 - q_2^{\pm n})}{1 - p^{\pm n}} \tilde{c}_{ij}^{[\mp n]} \delta_{n+m,0},$$

$$(8.9) \quad \left[s_{i,n}^{(\pm)}, s_{j,m}^{(\pm)} \right] = \mp \frac{1}{n} \frac{1 - q_1^{\pm n}}{(1 - p^{\pm n})(1 - q_2^{\mp n})} c_{ij}^{[\pm n]} \delta_{n+m,0},$$

and thus

$$(8.10) \quad Y_i(x) = q_1^{\tilde{p}_i} : \exp \left(y_{i,0} + \sum_{n \neq 0} y_{i,n}^{(+)} x^{-n} + \sum_{n \neq 0} y_{i,n}^{(-)} x^{+n} \right) :,$$

$$(8.11) \quad S_i(x) = : \exp \left(s_{i,0} \log x + \tilde{s}_{i,0} + \sum_{n \neq 0} s_{i,n}^{(+)} x^{-n} + \sum_{n \neq 0} s_{i,n}^{(-)} x^{+n} \right) : .$$

Then the remaining construction is completely parallel with 5d theory: The T -operator is generated by the iWeyl reflection with the Y -operator, and the Z -state is constructed using the screening charge, built with the elliptic oscillators (8.8) and (8.9). In this procedure, the whole rational functions are replaced with the elliptic function. For example, the scalar factor (6.21) due to the Y and A OPE becomes

$$(8.12) \quad \mathcal{S}(x) = \frac{\theta(q_1 x; p) \theta(q_2 x; p)}{\theta(qx; p) \theta(x; p)},$$

and the commutation relation between the Y and S operators is given by

$$(8.13) \quad \left[Y_i(x), S_j(x') \right] = \delta_{i,j} \frac{\theta(q_1^{-1}; p)}{(p; p)_{\infty}^2} \delta \left(q_1 \frac{x'}{x} \right) : Y_i(x) S_j(x') :$$

which becomes zero in the limit $q_1 \rightarrow 1$.

We can obtain the non-extended partition function from the Z -state as a correlator of the elliptic algebra as well as (5.16). In this case, the elliptic correlator can be also written in terms of 5d operators¹⁴

$$(8.15) \quad Z(t=0) = \langle 1 | \prod_{\hat{x} \in \mathcal{X}_0} S_{i(\hat{x})}^{6d}(\hat{x}) | 1 \rangle = \text{Tr} \left[p^{L_0} \prod_{\hat{x} \in \mathcal{X}_0} S_{i(\hat{x})}^{5d}(\hat{x}) \right]$$

where the energy operator is defined

$$(8.16) \quad L_0 = \sum_{i \in \Gamma_0} \sum_{m=1}^{\infty} m t_{i,m} \frac{\partial}{t_{i,m}} .$$

The meaning of this trace formula is as follows. One possibility to impose the periodicity (8.1) is doubling the free fields. Another possible way is to change the definition of the correlator to the trace, as shown in the bottom of Fig. 4, as well as the usual CFT correlator on a torus. They are actually two equivalent methods

¹⁴To assure the modularity of (non-extended) gauge theory partition function, we need to assign the anomaly-free condition for 6d theory

$$(8.14) \quad \sum_{j \in \Gamma_0} c_{ij} n_j = n_i^{\text{flavor}} \quad \text{for } \forall i \in \Gamma_0 .$$

This is equivalent to the conformal condition in 4d quiver gauge theory.

The figure consists of two parts. The top part shows a diagrammatic equation: a sphere with a dashed line on its left side multiplied by a cylinder, which is then multiplied by another sphere with a dashed line on its right side. This is equal to a sphere with a dashed line on its left side multiplied by a longer cylinder. Below this is the equation: $\langle 1 | \times \prod_{\hat{x} \in \mathcal{X}_0} S_{i(\hat{x}), \hat{x}}^{5d} \times | 1 \rangle = \langle 1 | \times | Z^{5d} \rangle$.

The bottom part shows a diagrammatic equation: a sphere with a dashed line on its left side multiplied by a sequence of cylinders labeled $p^{-1}x$, x , and px , with ellipses on either side. This is equal to the trace of p^{L_0} multiplied by a sphere with a dashed line on its right side. Below this is the equation: $\langle 1 | \times \prod_{\hat{x} \in \mathcal{X}_0} S_{i(\hat{x}), \hat{x}}^{6d} \times | 1 \rangle = \text{Tr} \left[p^{L_0} \times \right]$ (8.15).

FIGURE 4. Conformal blocks as the partition function of five-dimensional (top) and six-dimensional theory (bottom). The six-dimensional block has two equivalent expressions.

to compute the torus correlation function [30, 59]. From this point of view, the Z -state itself is not necessarily modular invariant, while the torus correlator should be.

Let us show the generating current of elliptic W -algebra with A_r linear quiver examples. The generating current itself is constructed by the Y -operator in the same way as before. The elliptic $W(A_1)$ generating current is given by

$$(8.17) \quad T_1(x) = Y_1(x) + \frac{1}{Y_1(q^{-1}x)},$$

as discussed in Sec. 6.1, and $W(A_2)$ currents are given by

$$(8.18) \quad T_1(x) = Y_1(x) + : \frac{Y_2(\mu^{-1}x)}{Y_1(q^{-1}x)} : + \frac{1}{Y_2(\mu^{-1}q^{-1}x)},$$

$$(8.19) \quad T_2(x) = Y_2(x) + : \frac{Y_1(\mu q^{-1}x)}{Y_2(q^{-1}x)} : + \frac{1}{Y_1(\mu q^{-2}x)},$$

where $\mu = \mu_{1 \rightarrow 2} = \mu_{2 \rightarrow 1}^{-1}q$, as shown in Sec. 6.2. The construction is also parallel for the linear quiver A_r shown in Sec. 6.3.

These generating currents exhibit a similar OPE relation. For A_1 quiver we have [56]

$$(8.20) \quad \begin{aligned} f_{11} \left(\frac{y}{x} \right) T_1(x)T_1(y) - f_{11} \left(\frac{x}{y} \right) T_1(y)T_1(x) \\ = - \frac{\theta(q_1; p)\theta(q_2; p)}{(p; p)_{\infty}^2 \theta(q; p)} \left(\delta \left(\frac{qy}{x} \right) - \delta \left(\frac{qx}{y} \right) \right) \end{aligned}$$

and for A_2 quiver we have

$$\begin{aligned} f_{11} \left(\frac{y}{x} \right) T_1(x)T_1(y) - f_{11} \left(\frac{x}{y} \right) T_1(y)T_1(x) \\ (8.21) \end{aligned} \\ = -\frac{\theta(q_1; p)\theta(q_2; p)}{(p; p)_\infty^2 \theta(q; p)} \left(\delta \left(q \frac{y}{x} \right) T_2(\mu^{-1}x) - \delta \left(q^{-1} \frac{y}{x} \right) T_2(\mu^{-1}qx) \right)$$

$$\begin{aligned} f_{12} \left(\frac{y}{x} \right) T_1(x)T_2(y) - f_{21} \left(\frac{x}{y} \right) T_2(y)T_1(x) \\ (8.22) \end{aligned} \\ = -\frac{\theta(q_1; p)\theta(q_2; p)}{(p; p)_\infty^2 \theta(q; p)} \left(\delta \left(\mu q \frac{y}{x} \right) - \delta \left(\mu q^{-2} \frac{y}{x} \right) \right)$$

$$\begin{aligned} f_{22} \left(\frac{y}{x} \right) T_2(x)T_2(y) - f_{22} \left(\frac{x}{y} \right) T_2(y)T_2(x) \\ (8.23) \end{aligned} \\ = -\frac{\theta(q_1; p)\theta(q_2; p)}{(p; p)_\infty^2 \theta(q; p)} \left(\delta \left(q \frac{y}{x} \right) T_1(\mu q^{-1}x) - \delta \left(q^{-1} \frac{y}{x} \right) T_1(\mu x) \right)$$

where

$$(8.24) \quad f_{ij}(x) = \exp \left(\sum_{n \neq 0} \frac{(1 - q_1^n)(1 - q_2^n)}{n(1 - p^n)} \tilde{c}_{ij}^{[-n]} x^n \right).$$

These OPE relations characterize the algebraic relation for the elliptic W-algebra generators.

Acknowledgments

The author would like to thank V. Pestun for fruitful collaboration which materializes this article and useful comments on the draft. This article is based on a talk presented at 2016 AMS von Neumann Symposium: Topological Recursion and its Influence in Analysis, Geometry, and Topology, July 4–8, 2016, Hilton Charlotte University Place, Charlotte, NC. The author is grateful to the symposium organizers for giving the opportunity to present our work in a stimulating atmosphere.

References

- [1] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory*, Nuclear Phys. B **426** (1994), no. 1, 19–52, DOI 10.1016/0550-3213(94)90124-4. MR1293681
- [2] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, Nuclear Phys. B **431** (1994), no. 3, 484–550, DOI 10.1016/0550-3213(94)90214-3. MR1306869
- [3] A. Gorsky, I. M. Krichever, A. Marshakov, A. Mironov, and A. Morozov, *Integrability and Seiberg-Witten exact solution*, Phys. Lett. B **355** (1995), no. 3-4, 466–474, DOI 10.1016/0370-2693(95)00723-X. MR1345598
- [4] E. J. Martinec and N. P. Warner, *Integrable systems and supersymmetric gauge theory*, Nuclear Phys. B **459** (1996), no. 1-2, 97–112, DOI 10.1016/0550-3213(95)00588-9. MR1372221
- [5] R. Donagi and E. Witten, *Supersymmetric Yang-Mills theory and integrable systems*, Nuclear Phys. B **460** (1996), no. 2, 299–334, DOI 10.1016/0550-3213(95)00609-5. MR1377167

- [6] N. Seiberg and E. Witten, *Gauge dynamics and compactification to three dimensions*, The mathematical beauty of physics (Saclay, 1996), Adv. Ser. Math. Phys., vol. 24, World Sci. Publ., River Edge, NJ, 1997, pp. 333–366. MR1490862
- [7] A. Gorsky, A. Marshakov, A. Mironov, and A. Morozov, *$N = 2$ supersymmetric QCD and integrable spin chains: rational case $N_f < 2N_c$* , Phys. Lett. B **380** (1996), no. 1-2, 75–80, DOI 10.1016/0370-2693(96)00480-7. MR1396968
- [8] N. Nekrasov, *Five-dimensional gauge theories and relativistic integrable systems*, Nuclear Phys. B **531** (1998), no. 1-3, 323–344, DOI 10.1016/S0550-3213(98)00436-2. MR1657857
- [9] N. Nekrasov and V. Pestun, *Seiberg–Witten geometry of four dimensional $N = 2$ quiver gauge theories* (2012), arXiv:1211.2240 [hep-th].
- [10] G. Moore, N. Nekrasov, and S. Shatashvili, *Integrating over Higgs branches*, Comm. Math. Phys. **209** (2000), no. 1, 97–121, DOI 10.1007/s002200050016. MR1736944
- [11] N. A. Nekrasov, *Seiberg–Witten prepotential from instanton counting*, Adv. Theor. Math. Phys. **7** (2003), no. 5, 831–864. MR2045303
- [12] N. A. Nekrasov and S. L. Shatashvili, *Supersymmetric vacua and Bethe ansatz*, Nuclear Phys. B Proc. Suppl. **192/193** (2009), 91–112, DOI 10.1016/j.nuclphysbps.2009.07.047. MR2570974
- [13] N. A. Nekrasov and S. L. Shatashvili, “Quantization of Integrable Systems and Four Dimensional Gauge Theories,” in *XVth International Congress on Mathematical Physics* (2009), pp. 265–289. arXiv:0908.4052 [hep-th].
- [14] N. Nekrasov, V. Pestun, and S. Shatashvili, *Quantum geometry and quiver gauge theories*, Comm. Math. Phys. **357** (2018), no. 2, 519–567, DOI 10.1007/s00220-017-3071-y. MR3767702
- [15] E. Frenkel and N. Reshetikhin, *The q -characters of representations of quantum affine algebras and deformations of \mathcal{W} -algebras*, Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998), Contemp. Math., vol. 248, Amer. Math. Soc., Providence, RI, 1999, pp. 163–205, DOI 10.1090/conm/248/03823. MR1745260
- [16] T. Hollowood, A. Iqbal, and C. Vafa, *Matrix models, geometric engineering and elliptic genera*, J. High Energy Phys. **3** (2008), 069, 81, DOI 10.1088/1126-6708/2008/03/069. MR2391051
- [17] N. Nekrasov, *BPS/CFT correspondence: non-perturbative Dyson–Schwinger equations and qq -characters*, J. High Energy Phys. **1603** (2016) 181. arXiv:1512.05388 [hep-th].
- [18] N. Nekrasov, *BPS/CFT correspondence II: instantons at crossroads, moduli and compactness theorem*, Adv. Theor. Math. Phys. **21** (2017), no. 2, 503–583, DOI 10.4310/ATMP.2017.v21.n2.a4. MR3678005
- [19] N. Nekrasov, *BPS/CFT correspondence III: Gauge origami partition function and qq -characters*, Comm. Math. Phys. **358** (2018), no. 3, 863–894, DOI 10.1007/s00220-017-3057-9. MR3778345
- [20] J.-E. Bourgin, Y. Mastuo, and H. Zhang, *Holomorphic field realization of SH^c and quantum geometry of quiver gauge theories*, J. High Energy Phys. **1604** (2016) 167.
- [21] A. Mironov, A. Morozov, and Y. Zenkevich, *On elementary proof of AGT relations from six dimensions*, Phys. Lett. **B756** (2016) 208–211.
- [22] H.-C. Kim, *Line defects and 5d instanton partition functions*, J. High Energy Phys. **1603** (2016) 199.
- [23] J.-E. Bourgin, M. Fukuda, Y. Matsuo, H. Zhang, and R.-D. Zhu, *Coherent states in quantum $\mathcal{W}_{1+\infty}$ algebra and qq -character for 5d super Yang–Mills*, PTEP. Prog. Theor. Exp. Phys. **12** (2016), 123B05, 41. MR3628699
- [24] A. Mironov, A. Morozov, and Y. Zenkevich, *Spectral duality in elliptic systems, six-dimensional gauge theories and topological strings*, J. High Energy Phys. **5** (2016), 121, front matter+43. MR3521840
- [25] A. Mironov, A. Morozov, and Y. Zenkevich, *Ding–Iohara–Miki symmetry of network matrix models*, Phys. Lett. B **762** (2016), 196–208, DOI 10.1016/j.physletb.2016.09.033. MR3569486
- [26] H. Awata, H. Kanno, T. Matsumoto, A. Mironov, A. Morozov, A. Morozov, Y. Ohkubo, and Y. Zenkevich, *Explicit examples of DIM constraints for network matrix models*, J. High Energy Phys. **7** (2016), 103, front matter+66. MR3550198
- [27] H. Awata, H. Kanno, A. Mironov, A. Morozov, A. Morozov, Y. Ohkubo, and Y. Zenkevich, *Toric Calabi–Yau threefolds as quantum integrable systems. \mathcal{R} -matrix and \mathcal{RTT} relations*, J. High Energy Phys. **10** (2016), 047, front matter+48. MR3577516

- [28] T. Kimura, H. Mori, and Y. Sugimoto, *Refined geometric transition and qq-characters*, J. High Energy Phys. **1** (2018), 025, front matter+40, DOI 10.1007/jhep01(2018)025. MR3764250
- [29] T. Kimura and V. Pestun, *Quiver W-algebras*, Lett. Math. Phys. **108** (2018), no. 6, 1351–1381, DOI 10.1007/s11005-018-1072-1. MR3797750
- [30] T. Kimura and V. Pestun, *Quiver elliptic W-algebras*, Lett. Math. Phys. **108** (2018), no. 6, 1383–1405, DOI 10.1007/s11005-018-1073-0. MR3797751
- [31] T. Kimura and V. Pestun, *Fractional quiver W-algebras*, Lett. Math. Phys. (2018), [arXiv:1705.04410](https://arxiv.org/abs/1705.04410) [[hep-th](#)].
- [32] A. Marshakov and N. A. Nekrasov, *Extended Seiberg-Witten theory and integrable hierarchy*, J. High Energy Phys. **1** (2007), 104, 39, DOI 10.1088/1126-6708/2007/01/104. MR2285911
- [33] L. F. Alday, D. Gaiotto, and Y. Tachikawa, *Liouville correlation functions from four-dimensional gauge theories*, Lett. Math. Phys. **91** (2010), no. 2, 167–197, DOI 10.1007/s11005-010-0369-5. MR2586871
- [34] N. Wyllard, *A_{N-1} conformal Toda field theory correlation functions from conformal $\mathcal{N} = 2$ $SU(N)$ quiver gauge theories*, J. High Energy Phys. **11** (2009), 002, 22, DOI 10.1088/1126-6708/2009/11/002. MR2628905
- [35] H. Awata and Y. Yamada, *Five-dimensional AGT conjecture and the deformed Virasoro algebra*, J. High Energy Phys. **1** (2010), 125, 11, DOI 10.1007/JHEP01(2010)125. MR2660773
- [36] J. Teschner, *Quantization of the Hitchin moduli spaces, Liouville theory and the geometric Langlands correspondence I*, Adv. Theor. Math. Phys. **15** (2011), no. 2, 471–564. MR2924236
- [37] M. Manabe and P. Sulkowski, *Quantum curves and conformal field theory*, Phys. Rev. **D95** (2017) 126003. [arXiv:1512.05785](https://arxiv.org/abs/1512.05785) [[hep-th](#)].
- [38] S. Katz, P. Mayr, and C. Vafa, *Mirror symmetry and exact solution of 4D $N = 2$ gauge theories. I*, Adv. Theor. Math. Phys. **1** (1997), no. 1, 53–114, DOI 10.4310/ATMP.1997.v1.n1.a2. MR1489296
- [39] O. Aharony, A. Hanany, and B. Kol, *Webs of (p, q) five-branes, five-dimensional field theories and grid diagrams*, J. High Energy Phys. **01** (1998) 002. [arXiv:hep-th/9710116](https://arxiv.org/abs/hep-th/9710116) [[hep-th](#)].
- [40] L. Bao, E. Pomoni, M. Taki, and F. Yagi, *M5-branes, toric diagrams and gauge theory duality*, J. High Energy Phys. **4** (2012), 105, front matter+58, DOI 10.1007/JHEP04(2012)105. MR2967598
- [41] M. Aganagic, N. Haouzi, C. Kozçaz, and S. Shakirov, *Gauge/Liouville Triality* (2013), [arXiv:1309.1687](https://arxiv.org/abs/1309.1687) [[hep-th](#)].
- [42] M. Aganagic, N. Haouzi, and S. Shakirov, *A_n -Triality* (2014), [arXiv:1403.3657](https://arxiv.org/abs/1403.3657) [[hep-th](#)].
- [43] M. Aganagic and N. Haouzi, *ADE Little String Theory on a Riemann Surface (and Triality)* (2015), [arXiv:1506.04183](https://arxiv.org/abs/1506.04183) [[hep-th](#)].
- [44] E. Frenkel and N. Reshetikhin, *Quantum affine algebras and deformations of the Virasoro and W-algebras*, Comm. Math. Phys. **178** (1996), no. 1, 237–264. MR1387950
- [45] E. Frenkel and N. Reshetikhin, *Deformations of W-algebras associated to simple Lie algebras*, Comm. Math. Phys. **197** (1998), no. 1, 1–32. MR1646483
- [46] N. A. Nekrasov and A. Okounkov, *Seiberg-Witten theory and random partitions*, The unity of mathematics, Progr. Math., vol. 244, Birkhäuser Boston, Boston, MA, 2006, pp. 525–596, DOI 10.1007/0-8176-4467-9_15. MR2181816
- [47] J. Shiraishi, H. Kubo, H. Awata, and S. Odake, *A quantum deformation of the Virasoro algebra and the Macdonald symmetric functions*, Lett. Math. Phys. **38** (1996), no. 1, 33–51, DOI 10.1007/BF00398297. MR1401054
- [48] V. S. Dotsenko and V. A. Fateev, *Conformal algebra and multipoint correlation functions in 2D statistical models*, Nuclear Phys. B **240** (1984), no. 3, 312–348, DOI 10.1016/0550-3213(84)90269-4. MR762194
- [49] V. S. Dotsenko and V. A. Fateev, *Four-point correlation functions and the operator algebra in 2D conformal invariant theories with central charge $C \leq 1$* , Nuclear Phys. B **251** (1985), no. 5-6, 691–734, DOI 10.1016/S0550-3213(85)80004-3. MR789026
- [50] S. Kharchev, A. Marshakov, A. Mironov, A. Morozov, and S. Pakuliak, *Conformal matrix models as an alternative to conventional multi-matrix models*, Nuclear Phys. B **404** (1993), no. 3, 717–750, DOI 10.1016/0550-3213(93)90595-G. MR1234891
- [51] I. K. Kostov, *Gauge invariant matrix model for the \hat{A} - \hat{D} - \hat{E} closed strings*, Phys. Lett. B **297** (1992), no. 1-2, 74–81, DOI 10.1016/0370-2693(92)91072-H. MR1199768

- [52] H. Awata, H. Kubo, S. Odake, and J. Shiraishi, *Quantum \mathcal{W}_N algebras and Macdonald polynomials*, *Comm. Math. Phys.* **179** (1996), no. 2, 401–416. MR1400746
- [53] M.-C. Tan, *An M-Theoretic Derivation of a 5d and 6d AGT Correspondence, and Relativistic and Elliptized Integrable Systems*, *J. High Energy Phys.* **1312** (2013) 031. [arXiv:1309.4775 \[hep-th\]](#).
- [54] P. Koroteev and A. Sciarappa, *Quantum hydrodynamics from large- n supersymmetric gauge theories*, *Lett. Math. Phys.* **108** (2018), no. 1, 45–95, DOI 10.1007/s11005-017-0996-1. MR3740429
- [55] A. Iqbal, C. Kozçaz, and S.-T. Yau, *Elliptic Virasoro Conformal Blocks* (2015), [arXiv:1511.00458 \[hep-th\]](#).
- [56] F. Nieri, *An elliptic Virasoro symmetry in 6d*, *Lett. Math. Phys.* **107** (2017), no. 11, 2147–2187, DOI 10.1007/s11005-017-0986-3. MR3714778
- [57] P. Koroteev and A. Sciarappa, *On elliptic algebras and large- n supersymmetric gauge theories*, *J. Math. Phys.* **57** (2016), no. 11, 112302, 32, DOI 10.1063/1.4966641. MR3570998
- [58] M.-C. Tan, *Higher AGT Correspondences, W -algebras, and Higher Quantum Geometric Langlands Duality from M-Theory* (2016), [arXiv:1607.08330 \[hep-th\]](#).
- [59] L. Clavelli and J. A. Shapiro, *Pomeron factorization in general dual models*, *Nucl. Phys.* **B57** (1973) 490–535.
- [60] Y. Saito, *Elliptic Ding-Iohara algebra and the free field realization of the elliptic Macdonald operator*, *Publ. Res. Inst. Math. Sci.* **50** (2014), no. 3, 411–455, DOI 10.4171/PRIMS/139. MR3262444

KEIO UNIVERSITY, HIYOSHI 4-1-1, YOKOHAMA, 223-8521 JAPAN
Email address: taro.kimura@keio.jp