

Airy structures and symplectic geometry of topological recursion

Maxim Kontsevich and Yan Soibelman

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1. Introduction

1.1. Linear algebra data of the topological recursion. Conventional approach to topological recursion (TR for short, see [EO07]) produces a collection of meromorphic polydifferentials $\omega_{g,n}$ on a spectral curve Σ starting with the “initial data” $\omega_{0,1}, \omega_{0,2}$. From $\omega_{0,1}, \omega_{0,2}$ one derives a certain integral kernel $K(z, w)$, which behaves like a differential form in one variable and vector field in the other one. Forms $\omega_{g,n}$ except of $\omega_{0,1}, \omega_{0,2}$ belong to the symmetric powers $Sym^n(V)$ of a certain infinite-dimensional vector space V of meromorphic 1-forms. The form $\omega_{0,1}$ is auxiliary and is used only via so-called recursion kernel. The form $\omega_{0,2}$ is not an honest element of $Sym^2(V)$.

In this paper we suggest to change the set of initial data, by replacing the above forms by four tensors assigned to V , which we denote by A, B, C, ϵ . These tensors can be derived from $\omega_{0,1}, \omega_{0,2}$. The tensor A is essentially $\omega_{0,3}$, and the tensor ϵ is $\omega_{1,1}$. There are explicit formulas for the tensors B and C which we will discuss in the paper. The vector space V is the space of such meromorphic 1-forms on the spectral curve which have trivial residues at the ramification points.

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From this new point of view the topological recursion becomes a special case of the deformation quantization. More precisely it gives a cyclic vector (wave function) of a deformation quantization module (DQ -module for short) corresponding to a certain quadratic¹ Lagrangian subvariety in the infinite-dimensional symplectic vector space $V \oplus V^*$. The Lagrangian subvariety can be described explicitly in terms of the tensors A, B, C, ϵ . The construction is purely algebraic and it does not depend on the notion of spectral curve. Furthermore, it makes V into an algebra over a certain PROP defined by the above four tensors.

From this perspective spectral curves are not fundamental objects in the formalism of topological recursion, contrary to the conventional point of view.

In particular, the vector space V can be arbitrary (e.g. finite-dimensional or infinite-dimensional).²

1.2. Lie algebra of the topological recursion. The notion of *Airy structure* underlying our approach to TR is named after the archetypical example of the “Airy spectral curve” $x = y^2$. In fact there are two notions: *classical Airy structure* and *quantum Airy structure*.

Starting with the tensors A, B, C we define a *classical* Airy structure. It is equivalent to the above-mentioned quadratic Lagrangian subvariety. Skew-symmetrization of the tensor C endows V with a Lie coalgebra structure, hence a Lie algebra structure on $\mathfrak{g} := V^*$.

The tensor ϵ is responsible for the quantization of the classical Airy structure. It gives a trivialization (i.e. a way to represent the cocycle as a coboundary) of a certain 2-cocycle on the Lie algebra \mathfrak{g} . The cocycle itself is a restriction to the above Lie algebra of the famous “Japanese cocycle” for the Lie algebra $gl(V \oplus V^*)$. In the case when $\dim V < \infty$ the restricted cocycle is naturally a coboundary. In the infinite-dimensional case (which we are mostly interested in) a choice of trivializing 1-cochain ϵ is an additional piece of information. In the case of the Airy spectral curve one finds that $\mathfrak{g} \simeq \mathbf{C}[[x]] \partial_x \times (\mathbf{C}[[x]]/\mathbf{C})$.

1.3. Spectral curves and the corresponding Airy structures. Our notion of spectral curve is more general than the one in the conventional TR. Namely, we consider a smooth complex Poisson surface P (which can be non-compact), such that the divisor D of zeros of the Poisson bivector field is smooth. We assume that the open symplectic leaf $S := P - D$ is endowed with a Lagrangian foliation \mathcal{F} , which extends by continuity to P and the extension is tangent to D . Then an (\mathcal{F}, D) -transversal spectral curve is a smooth compact curve $\Sigma \subset P$ which intersects transversally D at finitely many points p_β with multiplicities $k_\beta \geq 1$, and which is tangent with order 1 to leaves of the foliation \mathcal{F} at finitely many points r_α .

These data determine a germ of complex analytic space \mathcal{B}_1 , which is the moduli space of deformations of Σ in the class of spectral curves. It is a subspace of a bigger moduli space \mathcal{B}_0 of spectral curves defined in the same way, but without the tangency condition.

¹Here we mean that the subvariety under consideration is the set of common zeroes of a collection of polynomials of degree at most 2.

²In [ABCO] the authors study the dynamics of Young diagrams via an “abstract” version of topological recursion which seems to give an Airy structure in our sense.

Consider the family of symplectic vector spaces

$$\mathcal{H}_\Sigma := \mathbb{H}^1(\Sigma, \Omega_\Sigma^0(-\sum_{\beta} k_{\beta} p_{\beta})) \rightarrow \Omega_\Sigma^1(\sum_{\beta} k_{\beta} p_{\beta}).$$

We prove that the family \mathcal{H}_Σ gives rise to a symplectic vector bundle on \mathcal{B}_0 , endowed with an *affine* flat connection. It follows that (locally) \mathcal{B}_0 can be embedded into any affine symplectic vector space \mathcal{H}_Σ as a Lagrangian subvariety. In other words we have a (germ of) Lagrangian submanifold in an affine symplectic space.

Then the general story of deformation quantization ensures that there is a cyclic DQ -module which quantizes the Lagrangian image of \mathcal{B}_1 . This cyclic DQ -module can be realized as a representation of the (completed) Weyl algebra of \mathcal{H}_Σ . Locally (along the Lagrangian germ) it is unique up to isomorphism. Nevertheless there is no preferable choice of a cyclic vector.

It turns out that our quantum Airy structure gives a canonical cyclic vector $\psi_{\mathcal{B}_1}$ (wave function). In fact it exists even for a bigger space \mathcal{B}_0 . More precisely, the finite-dimensional symplectic vector space \mathcal{H}_Σ can be represented as the quotient G/G^\perp , where G is an infinite-dimensional coisotropic subspace of a symplectic vector space W of meromorphic 1-forms with trivial residue on the disjoint union of formal discs about r_α (here G^\perp is the skew-orthogonal to G). Then \mathcal{B}_1 can be obtained as a symplectic (a.k.a Hamiltonian) reduction of an infinite-dimensional quadratic Lagrangian submanifold $L \subset W$, i.e. \mathcal{B}_1 is isomorphic to the image of $L \cap G$ under the natural projection $G \rightarrow G/G^\perp = \mathcal{H}_\Sigma$. The Lagrangian submanifold L gives rise to a classical Airy structure on any Lagrangian complement to a tangent space to L .

Any Lagrangian complement \mathcal{V}_Σ to the finite-dimensional Lagrangian subspace $T_\Sigma \mathcal{B}_0 \subset \mathcal{H}_\Sigma$ gives (after the pullback by epimorphism $G \twoheadrightarrow \mathcal{H}_\Sigma$) an infinite-dimensional Lagrangian complement V_Σ to the tangent space to L . This gives us a classical Airy structure on V_Σ , which can be quantized. The corresponding wave function ψ is given by the TR, and it corresponds to $\psi_{\mathcal{B}_1}$. Notice that the choice of V_Σ corresponds to a choice of the Bergman kernel in the language of conventional TR.

1.4. Topological recursions. Another interesting observation is that there are different versions of TR. In particular, we define two sets of tensors: the set $(A_{TR}, B_{TR}, C_{TR}, \epsilon_{TR})$ corresponding to the conventional (Eynard-Orantin) version of the TR, and the set (A, B, C, ϵ) mentioned above, which is based on certain residue constraints arising in the deformation theory of the spectral curve Σ . These two sets of tensors are different. Furthermore, the set $(A_{TR}, B_{TR}, C_{TR}, \epsilon_{TR})$ does not satisfy the axiomatics of Airy structures. Nevertheless the comparison of the two approaches can be made. For that we introduce a certain subspace $V_{odd} \subset V$ respected by both sets of tensors. Furthermore they define on it the same Airy structure (which is a natural Airy substructure of the one given by (A, B, C, ϵ)). This leads to a new recursion formulas, which are different from those of Eynard-Orantin, but produce the same collection of polydifferentials $\omega_{g,n}$, $n \geq 1, 2 - 2g - n < 0$.

Moreover, our modified recursion is in a sense simpler than the original Eynard-Orantin TR. Our formulas do not involve local involutions (compare Sections 3.1 and 3.4).

1.5. Contents of the paper. Classical and quantum Airy structures are introduced in Section 2. We also prove there the result that connects TR with deformation quantization. The proof is purely algebraic, so it works for Airy structures over an arbitrary field of characteristic zero. An important corollary of our formalism is the fact that the tensors arising in this “abstract” topological recursion are symmetric.

In Section 3 we consider the local version of TR. First we recall the basics on the conventional TR and derive the corresponding tensors $(A_{TR}, B_{TR}, C_{TR}, \epsilon_{TR})$. As we have already pointed out they do not satisfy the axioms of the Airy structure, contrary to the alternative approach via the residue constraints. We explain in Section 3 that two approaches agree if we restrict both tensors to the smaller subspace V_{odd} .

Finally we compare two approaches in general. The comparison reveals the role of Bergman kernel and the notion of primitive Airy structure which we introduced in Section 2.

In Section 4 we discuss the notion of spectral curve as well the deformation theory of spectral curves. It turns out that the natural framework for that is the one of a complex Poisson surface which has an open symplectic leaf endowed with an analytic foliation. This allows us to consider e.g. non-compact spectral curves, like those which are associated with irregular Hitchin systems (see e.g. [KS14]).

In Section 5 we discuss the notion of affine symplectic connection which allows us to embed the germ of the moduli space of spectral curves into an affine symplectic space as a germ of Lagrangian submanifold.

In Section 6 we discuss an infinite-dimensional version of the material of Section 5 in the case of spaces of embedded formal discs. There we use the ideas of formal differential geometry of Gelfand and Kazhdan (see [GelKaz]).

In Section 7 we explain the formal version of the embedded Lagrangian germ. It is described in terms of the Hamiltonian reduction of a certain infinite-dimensional Lagrangian submanifold of a Tate vector space with respect to a coisotropic subspace. In that construction only the coisotropic space depends on the global geometry of the spectral curve. Rest of the data are local: they are derived from the formal neighborhoods of points, where the spectral curve is tangent to the foliation. An important corollary is that TR solves the so-called Holomorphic Anomaly Equation, i.e. quantizes the Lagrangian image of \mathcal{B}_0 in a covariantly constant way.

In Section 8 we discuss a possible extension of the quantization of the moduli space of smooth spectral curves to the case when there is no foliation at all. In the case when the foliation exists, the moduli space of smooth spectral curves can be embedded locally as a Lagrangian submanifold in an affine symplectic space (e.g. in examples it can be an open dense subvariety of the base of Hitchin integrable system). In the case when there is no foliation, the moduli space still can be embedded locally as a Lagrangian submanifold, but the ambient symplectic manifold does not have a global affine structure. The ambient space has a weaker structure which we call a *semi-affine symplectic structure*.

In Section 9 we propose two speculations. One is about a possible relation of the quantization of the moduli space of smooth spectral curves (e.g. the above-mentioned open subset of the Hitchin base) with the quantization of an individual spectral curve. Another speculation is about differential-graded Airy structures derived from the moduli spaces of *compact* 3-dimensional Calabi-Yau varieties.

Few final remarks.

We warn the reader that this paper is not about applications of the topological recursion. It is about the origin of the recursion relations. For that reason we make only a basic comparison of the formalism of Airy structures with the one of Eynard-Orantin. We leave for the future work many interesting aspects of the story, which (as we hope) admit a natural explanation in the framework of Airy structures. Those include e.g. the relation of the TR to WKB quantization of an *individual* spectral curve (see [DuMu]), symplectic invariance in TR (see [EynOr2]). We do not discuss the passage from the multi-differentials $\omega_{g,n}$ to the corresponding functions $F_{g,n}$. There are many other questions which arise naturally in the framework of our formalism, but which we prefer to leave for the future work. This includes e.g. the generalization of TR to non-smooth spectral curves and also to so-called quantum curves.

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2. Airy structures

2.1. Classical Airy structures. Let \mathbf{k} be a field of characteristic zero, V an n -dimensional \mathbf{k} -vector space. Let us choose an ordered basis on V and denote by y_1, \dots, y_n the corresponding coordinates. Denote by x^1, \dots, x^n the dual coordinates on $V^* = Hom_{\mathbf{k}}(V, \mathbf{k})$.

On the vector space $W := T^*(V^*) = V \oplus V^*$ we have the standard symplectic structure. The corresponding Poisson bracket in the coordinates are given by the formulas $\{y_i, x^j\} = \delta_{ij}$, $\{x^i, x^j\} = \{y_i, y_j\} = 0$.

We denote by $Sym_{\leq 2}(W)$ the Lie algebra $Sym^0(W) \oplus Sym^1(W) \oplus Sym^2(W)$ of polynomial functions on W^* of degree less or equal than 2, endowed with the natural Lie algebra structure induced by the above Poisson bracket.

DEFINITION 2.1.1. Classical Airy structure on V is given by a collections of at most quadratic polynomials (i.e. elements of $Sym_{\leq 2}(W)$) of the form

$$H_i := -y_i + \sum_{jk} a_{ijk} x^j x^k + 2 \sum_{jk} b_{ij}^k x^j y_k + \sum_{jk} c_i^{jk} y_j y_k, 1 \leq i \leq n,$$

such that the vector space $\oplus_{1 \leq i \leq n} \mathbf{k} \cdot H_i$ is closed under the Poisson bracket.

To save the language we will often skip the word “at most” and call such polynomials *quadratic*. The same terminology will apply to the variety defined by equations $H_i = 0, 1 \leq i \leq n$.

In a coordinate-free form a classical Airy structure on V is defined by three tensors:

- a) tensor $A = (a_{ijk}) \in V^{\otimes 3}$,
- b) tensor $B = (b_{ij}^k) : V \rightarrow V \otimes V$,
- c) tensor $C = (c_i^{jk}) : V \otimes V \rightarrow V$,

such that A and C are symmetric with respect to the permutation of the indices j, k .

Let us analyze the constraints on A, B, C .

By definition, any classical Airy structure gives rise to a Lie subalgebra

$$\mathfrak{g} := V^* \simeq \oplus_{1 \leq i \leq n} \mathbf{k} \cdot H_i \subset Sym_{\leq 2}(W).$$

Let us compute its structure constants. We have

$$\begin{aligned} \{H_{i_1}, H_{i_2}\} &= \{-y_{i_1} + 2 \sum_{j,k} b_{i_1 j}^k x^j y_k + \dots, -y_{i_2} + 2 \sum_{j,k} b_{i_2 j}^k x^j y_k + \dots\} \\ &= 2 \sum_k (b_{i_1 i_2}^k - b_{i_2 i_1}^k) y_k + \dots \end{aligned}$$

Since in the RHS we must have a linear combination of H_1, \dots, H_n we conclude that

$$\{H_{i_1}, H_{i_2}\} = \sum_k g_{i_1 i_2}^k H_k,$$

where $g_{i_1 i_2}^k = 2(b_{i_2 i_1}^k - b_{i_1 i_2}^k)$ are the structure constants of \mathfrak{g} .

Since there are no linear terms in $x^i, 1 \leq i \leq n$ in the expression for the generator H_k , we can see (by directly computing the coefficient in front of x^i in the LHS of the above formula) that the tensor a_{ijk} is symmetric with respect to arbitrary permutations of the indices i, j, k .

In a similar way comparing in the above formula the coefficients in front of the quadratic monomials in x^i and y_j we can reformulate the notion of a classical Airy structure in the following way: it is a collection of tensors

$$\begin{aligned} A &\in (V^{\otimes 3})^{S_3} \subset V^{\otimes 3}, \\ B &\in \text{Hom}(V, V \otimes V), \\ C &\in \text{Hom}((V \otimes V)_{S_2}, V) = \text{Hom}(V \otimes V, V)^{S_2} \subset \text{Hom}(V \otimes V, V), \end{aligned}$$

where the notation like E^{S_m} (resp E_{S_m}) refers to invariants (resp. coinvariants) of the action of the symmetric group S_m on an S_m -vector space E . The above three tensors are required to satisfy the following three relations: for any $1 \leq i_1, i_2, i_3, i_4 \leq n$

$$(1) \quad \begin{aligned} &\text{Coeff}_{x^{i_3} x^{i_4}} \Delta_{i_1, i_2} = 0, \text{Coeff}_{x^{i_3} y_{i_4}} \Delta_{i_1, i_2} = 0, \text{Coeff}_{y_{i_3} y_{i_4}} \Delta_{i_1, i_2} = 0 \\ &\text{where } \Delta_{i_1, i_2} := \{H_{i_1}, H_{i_2}\} - 2 \sum_k (b_{i_2 i_1}^k - b_{i_1 i_2}^k) H_k. \end{aligned}$$

These relations can be considered in a coordinate-free way as certain cumbersome purely quadratic in A, B, C expressions with values in respectively

$$\wedge^2 V \otimes \text{Sym}^2 V, \text{Hom}(V, \wedge^2 V \otimes V), \text{Hom}(\text{Sym}^2 V, \wedge^2 V).$$

Furthermore, we can reformulate the notion of classical Airy structure geometrically. Let M be a \mathbf{k} -affine symplectic space of dimension $2n$ and \mathfrak{g} is a Lie algebra over \mathbf{k} of dimension n which acts on M by Hamiltonian transformations. Assume that the corresponding moment map $\mu : M \rightarrow \mathfrak{g}^*$ is at most quadratic. This means that the moment map is given explicitly by $\mu(p) = (H_1(p), \dots, H_n(p))$, where each H_i is a polynomial of degree less or equal than two on $M \simeq \mathbf{k}^{2n}$.

Let $p \in \mu^{-1}(0)$ be a smooth point. The germ L at p of the algebraic variety $\mu^{-1}(0)$ is a smooth coisotropic subvariety of M of dimension n , hence it is Lagrangian. We will call such Lagrangian submanifolds (or germs of such submanifolds) *quadratic Lagrangian submanifolds*.

The tangent space $T_p L \subset T_p M$ is a Lagrangian subspace. Let us choose a Lagrangian vector subspace $V \subset T_p M$ which is transversal to $T_p L$ (we call it *Lagrangian complement*).

PROPOSITION 2.1.2. *The vector space V carries a natural Airy structure. Conversely any classical Airy structure is isomorphic in the natural sense to the one given by this geometric construction.*

PROOF. Indeed the above choices identify symplectically M with $V \oplus V^* = T^*(V^*)$, as long as we identified V^* with T_pL . The quadratic Hamiltonians H_i has the form

$$H_i := -y_i + \sum_{jk} a_{ijk} x^j x^k + 2 \sum_{jk} b_{ij}^k x^j y_k + \sum_{jk} c_i^{jk} y_j y_k, 1 \leq i \leq n.$$

Hence we have a classical Airy structure on V .

Conversely, if we have a classical Airy structure on V , we observe that $\mathfrak{g} = V^*$ acts in the Hamiltonian way on $M = V \oplus V^*$, the moment map $\mu : V \oplus V^* \rightarrow \mathfrak{g}^* \simeq V$ is given by the formula $\mu(p) = (H_1(p), \dots, H_n(p))$ with the Hamiltonians as above, and that $p = 0$ is a smooth point of $\mu^{-1}(0)$. This completes the proof. \square

REMARK 2.1.3. The space $\mathcal{O}(M)_{\leq 2}$ of polynomial functions on M of degree less or equal than 2 is a Lie subalgebra of the algebra $\mathcal{O}(M)$ of polynomial functions on M . Then the above considerations mean that a classical Airy structure is given by a choice of an n -dimensional Lie algebra \mathfrak{g} over \mathbf{k} together with a monomorphism of Lie algebras $\phi : \mathfrak{g} \rightarrow \mathcal{O}(M)_{\leq 2}$, a choice of a point $p \in \mu^{-1}(0)$ where $\mu^{-1}(0)$ is a non-empty smooth variety of dimension n and a choice of a transversal Lagrangian complement to the tangent space to $\mu^{-1}(0)$ at p .

Suppose that the field $\mathbf{k} = \mathbf{C}$ or \mathbf{R} . Then we have the Lie group $G = \exp(\mathfrak{g})$ (say, connected and simply-connected). This group acts on $\mu^{-1}(0)$ by affine symplectic isomorphisms. On a small neighborhood of the smooth point p the action is locally transitive with the discrete stabilizer. This means that for any two points p_1, p_2 which are sufficiently close to p there is a unique element $g = g_{p_1, p_2} \in G$ close to id_G which maps p_1 to p_2 and hence inducing an identifications of affine spaces of Lagrangian complements to $T_{p_i} \mu^{-1}(0) \subset T_{p_i} M$, and an equivalence of the corresponding Airy structures. It is not clear a priori (although very likely) that the obtained equivalence depends *algebraically* (and not only analytically) on the Lie subalgebra $\mathfrak{g} \subset \mathcal{O}(M)_{\leq 2}$, and a pair of smooth points $p_1, p_2 \in \mu^{-1}(0)$.

Finally, let us write the formula for the change of tensors A, B, C with respect to the change of the transversal Lagrangian subspace at a given smooth point p . Any two Lagrangian complements $V_p^{(1)}, V_p^{(2)}$ to $T_pL \subset T_pM$ are both canonically identified with $T_p^*L = T_pM/T_pL$, hence identified with each other. One of them can be seen as the graph of linear map $V \rightarrow V^*$ which comes from a symmetric bilinear pairing $Sym^2(V) \rightarrow \mathbf{k}$. In local coordinates the calculation reduces to the application of a linear symplectic change of variables

$$y_i \rightarrow y_i, \quad x^i \rightarrow x^i + \sum_j s^{ij} y_j,$$

where $s = (s^{ij})_{1 \leq i, j \leq n}$ is a symmetric matrix. A direct calculation gives us the following **gauge transformation**:

$$\begin{aligned}
 a_{ijk} &\rightarrow a_{ijk}, \\
 b_{ij}^k &\rightarrow b_{ij}^k + \sum_l a_{ijl} s^{lk}, \\
 c_i^{jk} &\rightarrow c_i^{jk} + \sum_{l_1, l_2} a_{il_1 l_2} s^{l_1 j} s^{l_2 k} + \sum_l b_{il}^k s^{lj} + \sum_l b_{il}^j s^{lk}.
 \end{aligned}
 \tag{2}$$

2.2. Quantum Airy structures. We keep the notation of the previous subsection. The vector space of at most quadratic functions $Sym_{\leq 2}(W), W = V \oplus V^*$ is spanned by

$$\{1, x^i, y_i, x^i x^j, x^i y_j, y_i y_j\}, \quad 1 \leq i, j \leq n.$$

We define an associative \mathbf{k} -algebra $Sym^{quant}(W)$ generated by $1, \hbar, x^i, \hbar \partial_i$, with 1 and \hbar being central elements, subject to the relations $(\hbar \partial_i) x^j - x^j (\hbar \partial_i) = \hbar \delta_{ij}$ (i.e. $\partial_i = \partial / \partial x^i$ if we allow the division by \hbar). We define $Sym_{\leq 2}^{quant}(W)$ as the vector subspace of $Sym^{quant}(W)$ spanned by

$$\{1, \hbar, x^i, \hbar \partial_i, x^i x^j, \hbar(x^i \partial_j + \partial_j x^i), \hbar^2 \partial_i \partial_j\}, \quad 1 \leq i, j \leq n.$$

It is a Lie algebra with the bracket equal to $(ab - ba) / \hbar$. Clearly modulo \hbar it is isomorphic to the Lie algebra $Sym_{\leq 2}(W)$.

DEFINITION 2.2.1. A **quantization** of a classical Airy structure is given by a monomorphism $\mathfrak{g} \simeq V^* \rightarrow Sym_{\leq 2}^{quant}(W)$ such that modulo \hbar it coincides with the monomorphism $\mathfrak{g} \rightarrow Sym_{\leq 2}(W)$. We will call the former monomorphism quantum Airy structure.

We have quantized Hamiltonians

$$\widehat{H}_i = -\hbar \partial_i + \sum_{j,k} a_{ijk} x^j x^k + 2\hbar \sum_{j,k} b_{ij}^k x^j \partial_k + \hbar^2 \sum_{j,k} c_i^{jk} \partial_j \partial_k + \hbar \varepsilon_i \cdot 1,$$

where $\varepsilon = (\varepsilon_i \in \mathbf{k})_{1 \leq i \leq n}$ are some constants.

In the coordinate-free notation ε is an element of V . The constraint on tensor ε follows from the relation saying that

$$\forall i_1, i_2 \quad \text{Coeff}_{\hbar^{-1}} \left(\frac{1}{\hbar} \left(\widehat{H}_{i_1} \widehat{H}_{i_2} - \widehat{H}_{i_2} \widehat{H}_{i_1} \right) - 2 \sum_k (b_{i_2 i_1}^k - b_{i_1 i_2}^k) \widehat{H}_k \right) = 0.
 \tag{3}$$

In the coordinate-free form, this constraint means vanishing of certain purely quadratic expression in A, B, C, ε with values in $\wedge^2 V$. Together with classical constraints (1) we obtain an explicit algebraic description of quantum Airy structures.

The gauge transformation formula (2) extends to the quantum case as

$$\varepsilon_i \rightarrow \varepsilon_i + \sum_{j,k} a_{ijk} s^{jk}.
 \tag{4}$$

(Apply transformation $\hbar \partial_i \rightarrow \hbar \partial_i, x^i \rightarrow x^i + \sum_j s^{ij} \hbar \partial_j$ where $s = (s^{ij})_{1 \leq i, j \leq n}$ is a symmetric matrix).

For any classical Airy structure on a *finite-dimensional* space V there exists a *canonical* choice of its quantization³. Namely, we set

$$\widehat{H}_i^{can} := -\hbar\partial_i + \sum_{j,k} a_{ijk}x^jx^k + \hbar \sum_{j,k} b_{ij}^kx^j\partial_k + \hbar \sum_{j,k} b_{ij}^k\partial_kx^j + \hbar^2 \sum_{j,k} c_i^{jk}\partial_j\partial_k.$$

In other words, we set $\varepsilon_i^{can} := \sum_j b_{ij}^j$. A general solution to (3) is

$$(5) \quad \varepsilon = \varepsilon^{can} + \chi, \text{ where } \chi \in V = \mathfrak{g}^* \text{ is any character of } \mathfrak{g} \Leftrightarrow \chi \in H^1(\mathfrak{g}, \mathbf{k}).$$

2.3. Airy structures in the infinite-dimensional case. Four tensors A, B, C, ε satisfy four relations, see (1) and (3).

Notice that since all four relations include acyclic tensors, our definitions of Airy structures (both classical and quantum) make sense in the case of any \mathbf{k} -linear tensor category. In particular, they make sense in the case when $\dim V = \infty$. Notice that in the infinite-dimensional case Lie algebra \mathfrak{g} is naturally endowed with linear topology being dual to an ordinary vector space V (endowed with the discrete topology). In other words, V is a Lie coalgebra.

Let us analyze constraint (3). Rewrite it as

$$(6) \quad \forall i_1, i_2 \quad 2 \sum_{j,k} (c_{i_1}^{jk} a_{i_2jk} - c_{i_2}^{jk} a_{i_1jk}) = \sum_k g_{i_1 i_2}^k \varepsilon_k.$$

where $g_{i_1 i_2}^k := 2(b_{i_2 i_1}^k - b_{i_1 i_2}^k)$ are structure constants of \mathfrak{g} . One sees that the RHS is the (i_1, i_2) -component of the coboundary of the continuous cochain (ε_k) on the Lie algebra $\mathfrak{g} = V^*$.

Let us look more closely on the expression in the LHS in (6). As the indices i_1, i_2 are fixed, we see that $a_{i_1jk}, a_{i_2jk} \in \text{Sym}^2(V)$, and $c_{i_1}^{jk}, c_{i_2}^{jk} \in \widehat{\text{Sym}}^2(V^*)$, where the hat notation over a vector space indicates the natural topological completion.⁴

Notice that the Lie algebra $gl(W)$ carries a natural 2-cocycle ψ (“Japanese cocycle”). Namely, splitting $W = V \oplus V^*$ allows us to represent any linear map S as a block 2×2 matrix $(s_{ij})_{1 \leq i, j \leq 2}$. Then the cocycle ψ can be written as $\psi(S, T) = \text{Tr}(t_{21}s_{12} - s_{21}t_{12})$. Comparing it with the above expression for the LHS of (6), we see that the latter is twice of the pull-back of the Japanese 2-cocycle to the Lie algebra \mathfrak{g} under the composition of natural morphisms

$$\mathfrak{g} \rightarrow \widehat{\text{Sym}}_{\leq 2} W \rightarrow \widehat{\text{Sym}}^2(W) \simeq sp(W) \hookrightarrow gl(W).$$

Therefore existence of a quantization of a classical Airy structure in these terms corresponds to triviality of the pullback of the cohomology class ψ .

Ansatz (5) shows that in the finite-dimensional case one can recover the 1-cochain ε from the other data by taking $\varepsilon_i = \sum_j b_{ij}^j + \chi(H_j)$, where $\chi : \mathfrak{g} \rightarrow \mathbf{k}$ is any character. In the infinite-dimensional case this formula does not make sense since it involves taking the trace of the tensor B . In this case the *continuous* 1-cochain ε is a new datum for the quantum Airy structure.

³In the next section we will talk about Airy structures in the infinite-dimensional case where the analogous assertion is false in general.

⁴We hope that the reader will not mix it with the hat indicating quantization, like in the previous section.

In general, if a quantization of a classical Airy structure given by a Lie algebra \mathfrak{g} exists, all such quantizations form a torsor over the space of characters $Hom(\mathfrak{g}, \mathbf{k}) = H^1_{cont}(\mathfrak{g}, \mathbf{k})$. In particular, if $H^1_{cont}(\mathfrak{g}, \mathbf{k}) = H^2_{cont}(\mathfrak{g}, \mathbf{k}) = 0$ then there exists a unique quantization of the classical Airy structure.

Finally, we remark that the tensors A, B, C, ε satisfying the above four quadratic relations make V into an algebra over a certain PROP. Obviously, (quantum or classical) Airy structures form a category (there is a straightforward notion of a morphism). Also, for any family of Airy structures on spaces $(V_i)_{i \in I}$ the direct sum space $\oplus_{i \in I} V_i$ carries a natural direct sum Airy structure.

2.4. Cyclic DQ-modules. We start with considerations which hold in a bigger generality than the framework of Airy structures. The corollary of these considerations will be the (non-trivial) fact that the polydifferential forms $\omega_{g,n}(z_1, \dots, z_n)$ defined via the topological recursion are symmetric with respect to the variables z_1, \dots, z_n .

Let us first assume that V is a finite-dimensional vector space over a field \mathbf{k} of characteristic zero. For convenience, we introduce the coordinates $x^i, 1 \leq i \leq \dim V$ on V^* as we discussed previously⁵. In order to keep track of the invariant (= coordinate-free) meaning, we use the natural identification of the algebra of formal power series $\mathbf{k}[[x^\bullet]]$ with $\prod_{n \geq 0} Sym^n(V)$, by associating with every series f the collection of tensors (Taylor coefficients)

$$Taylor_n(f) \in (V^{\otimes n})^{Sym^n} \subset V^{\otimes n}, \quad Taylor_n(f)_{i_1, \dots, i_n} := \partial_{i_1} \dots \partial_{i_n}(f)(0).$$

Let \mathbf{D} be the completed algebra of differential operators on V^* . More precisely, $\mathbf{D} = \mathbf{k}[[\hbar]] [[x^\bullet]] [[\hbar \partial_\bullet]]$ endowed with the standard product. Let us introduce a grading on \mathbf{D} such that $deg \hbar = 2, deg x^i = deg \hbar \partial_i = 1$. Denote by $\mathbf{I} \subset \mathbf{D}$ the closed 2-sided ideal spanned by elements of degree ≥ 2 .

Let now $\widehat{H}_i, 1 \leq i \leq \dim V$ be elements of \mathbf{D} which satisfy the following properties:

- 1) $\widehat{H}_i = -\hbar \partial_i + P_i$, where $P_i \in \mathbf{I}$;
- 2) for any $i \neq j$ the commutator $[\widehat{H}_i, \widehat{H}_j]$ belongs to the left ideal generated by $\hbar \widehat{H}_\bullet$, i.e. there exists (non-unique) “structure constants” $g^k_{i,j} \in \mathbf{D}$ such that

$$\frac{1}{\hbar} [\widehat{H}_i, \widehat{H}_j] = \sum_k g^k_{i,j} \widehat{H}_k.$$

Notice that the case when $\widehat{H}_i, 1 \leq i \leq \dim V$ form a basis of a Lie algebra over \mathbf{k} , i.e. $g^k_{i,j} \in \mathbf{k} \subset \mathbf{D}$ (and, more generally, over $\mathbf{k}[[\hbar]]: g^k_{i,j} \in \mathbf{k}[[\hbar]] \subset \mathbf{D}$) is a particular case of the above condition.

Having a formal expression

$$\psi = \exp(S) = \exp \left(S_0/\hbar + \sum_{g \geq 1} \hbar^{g-1} S_g \right),$$

with

$$S_g = \sum_{n \geq 0} S_{g,n} \in \prod_{n \geq 0} Sym^n(V) = \mathbf{k}[[x^\bullet]], \forall g \geq 0$$

⁵In this section we *do not* denote the dimension of V by n , the letter n is reserved for the homogeneity degree of polynomials in x^\bullet .

we define a continuous automorphism $T_\psi : \mathbf{D} \rightarrow \mathbf{D}$ which corresponds informally to the conjugation $D \mapsto \psi^{-1}D\psi$. Explicitly, the automorphism T_ψ is defined on the generators by the formulas $T_\psi(x^i) = x^i, T_\psi(\hbar\partial_i) = \hbar\partial_i + \partial_i(S_0 + \sum_{g \geq 1} \hbar^g S_g)$. Of course this is motivated by the well-known formula $e^{-f}\partial_i e^f = \partial_i + \partial_i(f)$. By definition we have $T_{\psi\phi} = T_\psi T_\phi$, where the product $\psi\phi$ is defined naturally, as the exponent of the sum of two expressions. The formula $\widehat{H}(e^S f) = e^S(T_{e^S}(\widehat{H})(f))$ which is valid for the differential operators \widehat{H} and functions S, f motivates the following definition.

DEFINITION 2.4.1. We say that the above formal expression ψ satisfies the system of equations $\widehat{H}_i(\psi) = 0$ if $T_\psi(\widehat{H}_i)(1) = 0, 1 \leq i \leq \dim V$.

In the language of quantum mechanics, ψ is a wave function in WKB form.

THEOREM 2.4.2. Suppose that the Hamiltonians $\widehat{H}_i, 1 \leq i \leq \dim V$ satisfy the above conditions 1) and 2). Then there exists a unique solution to the system of equations $\widehat{H}_i(\psi) = 0, 1 \leq i \leq \dim V$ of the form $\psi = \exp(S_0/\hbar + \sum_{g \geq 1} \hbar^{g-1} S_g)$, where $S_0 = \sum_{n \geq 3} S_{0,n}$ and for $g \geq 1$ $S_g = \sum_{n \geq 1} S_{g,n}$ with $S_{g,n} \in \text{Sym}^n(V)$ for any $g, n \geq 0$ (in particular $S_{0,\leq 2} = S_{\geq 0,0} = 0$).

PROOF. We will proceed by induction in g , and for each g we will proceed by induction in n . We start with $g = 0$ case. The equation for $g = 0$ says that for any i

$$H_i(x^\bullet, \partial_\bullet S_0) = 0 \in \mathbf{k}[[x^\bullet]].$$

Here the classical limit $H_i \in \mathbf{k}[[x^\bullet, y_\bullet]]$ is defined as the image of \widehat{H}_i in $\mathbf{D}/\hbar\mathbf{D} \simeq \mathbf{k}[[x^\bullet]]/[[\hbar\partial_\bullet]]$, when we identify variables y_i with the images of $\hbar\partial_i, 1 \leq i \leq \dim V$ in the quotient algebra $\mathbf{D}/\hbar\mathbf{D}$. The condition 1) implies that $H_i = -y_i$ modulo quadratic and higher order terms in x^\bullet, y_\bullet . The classical limit of the commutation relation 2) implies that H_\bullet generate a Poisson ideal. Therefore, the formal germ at 0 of a submanifold L in $\mathbf{k}^{2 \dim V} = V \oplus V^*$ given by equation $H_i(x^\bullet, y_\bullet) = 0$ is coisotropic. It is of dimension $\dim V$, hence it is Lagrangian. This implies that it is given by the graph of dS_0 for some formal power series $S_0 \in \mathbf{k}[[x_\bullet]]$. We can assume that the constant term is zero. The tangent space to L at 0 is given by equations $y_i = 0, 1 \leq i \leq \dim V$, hence S_0 starts with a cubic term.

It is useful to formulate the proof purely in terms of linear algebra using the induction on $n \geq 3$ (the order of the term in the Taylor expansion). In what follows we will use a shorthand notation where the upper index describes the constraints on the homogeneity degree of monomials in variables x^\bullet or y_\bullet . First, let us prove that there exists a unique cubic polynomial $S_{0,3} = x^{\geq 3} \in \mathbf{k}[x^\bullet]$ such that $H_i(x^\bullet, \partial_\bullet S_{0,3}) = x^{\geq 3} \forall i$. Indeed, we have

$$H_i = -y_i + H_{i,2} + x^{\geq 3} + x^{\geq 1}y^{\geq 1} + x^{\geq 0}y^{\geq 2},$$

where $H_{i,2} = x^{\geq 2}$ is a homogeneous polynomial in x of degree 2. The equation on $S_{0,3}$ says that $-\partial_i S_{0,3} + H_{i,2} = 0 \forall i$. The Poisson bracket

$$\{H_{i_1}, H_{i_2}\} = \partial_{i_1} H_{i_2,2} - \partial_{i_2} H_{i_1,2} + x^{\geq 2} + x^{\geq 0}y^{\geq 1}$$

belongs to the ideal generated by H_\bullet . Hence $-\partial_{i_1} H_{i_2,2} + \partial_{i_2} H_{i_1,2} = 0 \forall i_1, i_2$, i.e. 1-form $\sum_i H_{i,2} dx^i$ is closed, therefore exact and given by $dS_{0,3}$ for a homogeneous cubic polynomial $S_{0,3}$.

Now, let us assume that we have already a solution of $H_i(x^\bullet, \partial_\bullet S_0) = 0$ up to order $n \geq 3$, and want to extend it to a solution up to order $n + 1$. Applying the conjugation by $\exp(S_0/\hbar)$ we may assume that the solution up to order n vanishes, i.e.

$$S_0 = x^{\geq(n+1)}, \quad H_i(x^\bullet, \partial_\bullet S_0) = x^{\geq(n+1)} \quad \forall i.$$

This implies that H_i has the form

$$H_i = -y_i + H_{i,n} + x^{\geq(n+1)} + x^{\geq 1}y^1 + x^{\geq 0}y^{\geq 2}.$$

Similarly to the previous arguments, the Poisson bracket has the form

$$\{H_{i_1}, H_{i_2}\} = -\partial_{i_1}H_{i_2,n} + \partial_{i_2}H_{i_1,n} + x^{\geq n} + x^{\geq 0}y^{\geq 1}.$$

Hence homogeneous 1-form $\sum_i H_{i,n}dx^i$ is closed and can be written as $dS_{0,n+1}$ where $S_{0,n+1} = x^{=(n+1)}$. This proves the n -th step of induction for the case $g = 0$.

Finally, we proceed with the induction on g . Assume that for $g \geq 1$ and $n \geq 0$ we have a solution S up to $\hbar^{g-1}x^{\geq(n+1)} + \hbar^{\geq g}x^{\geq 1}$. At beginning of our induction, when $n = 0$, we may assume that $S_{g,0} = 0$. Indeed adding a multiple of \hbar^{g-1} to S does not change the constraint $\widehat{H}_i(\exp(S)) = 0$. Applying the conjugation, we may assume that

$$S = \hbar^{g-1}x^{\geq(n+1)} + \hbar^{\geq g}x^{\geq 1}, \quad \widehat{H}_i(\exp(S)) = 0 \text{ modulo } \hbar^g x^{\geq n} + \hbar^{\geq(g+1)}x^{\geq 0},$$

where $\exp(S) := 1 + S + S^2/2! + \dots$ is a well-defined invertible element of $\mathbf{k}[[\hbar]][[x^\bullet]] \subset \mathbf{D}$ (unlike the formal expression $\psi = \exp(S_0/\hbar + \dots)$ when $S_0 \neq 0$). The latter condition on \widehat{H}_i, S implies that

$$\widehat{H}_i = -\hbar\partial_i + \hbar^g x^{\geq n} + \hbar^{\geq(g+1)}x^{\geq 0} + \hbar^{\geq 0}x^{\geq 1}(\hbar\partial)^{\geq 1} + \hbar^{\geq 1}x^{\geq 0}(\hbar\partial)^{\geq 1}.$$

Let us denote by $H_{g;i,n} \in \mathbf{k}[x^\bullet]$ the homogeneous term of the type $\hbar^g x^n$ in \widehat{H}_i . The commutator has the form

$$[\widehat{H}_{i_1}, \widehat{H}_{i_2}] = -\hbar\partial_{i_1}H_{g;i_2,n} + \hbar\partial_{i_2}H_{g;i_1,n} + \text{higher order terms},$$

where the higher order terms are of the form

$$\hbar^{g+1}x^{\geq n} + \hbar^{\geq(g+2)}x^{\geq 0} + \hbar^{\geq 0}x^{\geq 1}(\hbar\partial)^{\geq 1} + \hbar^{\geq 1}x^{\geq 0}(\hbar\partial)^{\geq 1}.$$

The constraint 2)

$$[\widehat{H}_{i_1}, \widehat{H}_{i_2}] \in \hbar \sum_j \mathbf{D} \cdot \widehat{H}_j$$

implies that $\hbar\partial_{i_1}H_{g;i_2,n} = \hbar\partial_{i_2}H_{g;i_1,n}$, hence there exists a unique homegenous $S_{g,n} \in \mathbf{k}[x^\bullet]$ of degree $n+1$ such that $H_{g;i,n} = \hbar^g\partial_i S_{g,n} \quad \forall i$. Then $S = \hbar^{g-1}S_{g,n} + \dots$ is a solution up to higher order terms $\hbar^{g-1}x^{\geq(n+2)} + \hbar^{\geq g}x^{\geq 1}$. We proved the induction step $(g, n) \rightarrow (g, n + 1)$, hence proved the theorem. \square

Let us discuss algebraic meaning of the above theorem. First of all, the algebra \mathbf{D} is a topologically free module over the central subalgebra $\mathbf{k}[[\hbar]]$. Furthermore $[\mathbf{D}, \mathbf{D}] \subset \hbar\mathbf{D}$. We claim that the condition 2) on the collection of elements \widehat{H}_i is equivalent to the constraint *only* on the left ideal $J := \sum_i \mathbf{D} \cdot \widehat{H}_i \subset \mathbf{D}$:

$$J = \mathbf{D}J, \quad [J, J] \subset \hbar J.$$

The implication in one direction is obvious. Conversely, assuming condition 2) we have for $\forall a_1, a_2 \in \mathbf{D}, \forall i_1, i_2$

$$[a_1 \widehat{H}_{i_1}, a_2 \widehat{H}_{i_2}] = [a_1 \widehat{H}_{i_1}, a_2] \widehat{H}_{i_2} + a_2 a_1 [\widehat{H}_{i_1}, \widehat{H}_{i_2}] + a_2 [a_1, \widehat{H}_{i_2}] \widehat{H}_{i_1} \in \hbar J.$$

Theorem means that under conditions 1), 2) the cyclic module $\mathcal{M} := \mathbf{D}/J$ is isomorphic to the canonical holonomic \mathbf{D} -module associated with the germ of Lagrangian submanifold $graph(dS_0)$. In particular it is a topologically free $\mathbf{k}[[\hbar]]$ -module.

In the case (relevant to Airy structures) when $\mathfrak{g} := \oplus_i \mathbf{k} \cdot \widehat{H}_i$ is closed under Lie bracket, the fact that cyclic module $\mathcal{E} := \mathbf{D}/\sum_i \mathbf{D}\widehat{H}_i$ is topologically free over $\mathbf{k}[[\hbar]]$, has an easy alternative proof. Indeed, let us consider the \hbar -enveloping algebra $U_\hbar(\mathfrak{g})$ of \mathfrak{g} (i.e. the commutation relations are $ab - ba = \hbar[a, b], a, b, \in \mathfrak{g}$). It is a deformation quantization of $Sym^*(\mathfrak{g})$ endowed with the Lie-Kirillov-Kostant Poisson bracket. The natural completion $\widehat{U}_\hbar(\mathfrak{g})$ is a topologically free $\mathbf{k}[[\hbar]]$ -algebra, which is naturally embedded to the topologically free $\mathbf{k}[[\hbar]]$ -algebra $\mathbf{k}[[x_i, \hbar\partial_i, \hbar]]$ of \hbar -differential operators (it is an embedding modulo \hbar which implies an embedding over $\mathbf{k}[[\hbar]]$). Moreover, \mathbf{D} is topologically free as right $U_\hbar(\mathfrak{g})$ -module. Finally,

$$\mathcal{E} = \mathbf{D}/\sum_i \mathbf{D}\widehat{H}_i = \mathbf{D} \otimes_{U_\hbar(\mathfrak{g})} \mathbf{k}[[\hbar]],$$

where we treat $\mathbf{k}[[\hbar]]$ as the trivial rank one representation of $\widehat{U}_\hbar(\mathfrak{g})$. Hence \mathcal{E} is topologically free over $\mathbf{k}[[\hbar]]$. This argument generalizes immediately to the case of Lie algebra over $\mathbf{k}[[\hbar]]$.

2.5. Application to quantum Airy structures: Abstract topological recursion. Assume that our Hamiltonians $\widehat{H}_i, 1 \leq i \leq n$ come from a quantum Airy structure. The general result from the previous subsection implies that there exists a unique WKB solution $\exp(S_0/\hbar + \dots)$. We obtain a collection of symmetric tensors

$$S_{g,n} \in Sym^n(V), \quad g = 0, n \geq 3 \text{ or } g \geq 1, n \geq 1,$$

$$S_{g,n;i_1,\dots,i_n} := \partial_{i_1} \dots \partial_{i_n} S_g(0).$$

The equation $\widehat{H}_i(\exp(S)) = 0$ for $S = \sum_g \hbar^{g-1} S_g, S_g = \sum_n S_{g,n}$ reads as

$$\begin{aligned} \hbar\partial_i S &= \sum_{jk} a_{ijk} x^j x^k + 2 \sum_{jk} b_{ij}^k x^j \hbar\partial_k S + \\ &+ \sum_{jk} c_i^{jk} \hbar\partial_j(S) \hbar\partial_k(S) + \sum_{jk} c_i^{jk} \hbar^2 \partial_j \partial_k(S) + \hbar\varepsilon_i. \end{aligned}$$

Expanding the RHS of the above equation into series in \hbar, x^\bullet we obtain recursive formulas for $\partial_i S_{g,n}$ in terms of tensors A, B, C, ε and previously known $S_{g',n'}$ with either $g' < g$ or $g' = g, n' < n$. It is *not obvious* a priori that the RHS which should be equal by induction to the 1-form $dS_{g,n} = \sum_i \partial_i S_{g,n} dx^i$ is in fact a closed 1-form (which would imply exactness). Theorem guarantees that this is the case.

The recursion starts with the cases $g = 0, n = 3$ and $g = 1, n = 1$. Then

at the level of tensors $S_{0,3} = 2A \iff$ as a polynomial $S_{0,3} = \frac{1}{3} \sum_{ijk} a_{ijk} x^i x^j x^k;$

at the level of tensors $S_{1,1} = \varepsilon \iff$ as a polynomial $S_{1,1} = \sum_i \varepsilon_i x^i.$

In general, for all other possible values of g, n (i.e. for $g = 0, n \geq 4$, or $g = 1, n \geq 2$ or $g \geq 2, n \geq 1$) we have

$$\begin{aligned}
 S_{g,n;i,i_1,\dots,i_{n-1}} &= 2 \sum_{\alpha=1}^{n-1} \sum_k b_{ii_\alpha}^k S_{g,n-1;k,i_{\{1,\dots,n-1\}\setminus\{\alpha\}}} \\
 (7) \quad &+ \sum_{g_1+g_2=g; J_1 \sqcup J_2 = \{1,\dots,n-1\}} \sum_{jk} c_i^{jk} S_{g_1, \#J_1+1; j, i_{J_1}} S_{g_2, \#J_2+1; k, i_{J_2}} \\
 &+ \sum_{jk} c_i^{jk} S_{g-1, n+2; j, k, i_1, \dots, i_{n-1}},
 \end{aligned}$$

where at the RHS we use only terms $S_{g',n'}$ with either $g' = 0, n' \geq 3$ or $g' \geq 1, n' \geq 1$. Also, here for any subset $J \subset \{1, \dots, n - 1\}$ such that $J = \{\alpha_1, \dots, \alpha_{\#J}\}$ we denote by i_J the sequence $i_{\alpha_1}, \dots, i_{\alpha_{\#J}}$. This sequence depends on the order in which we enumerate elements of J . Nevertheless this dependence is irrelevant for the recursion formula because tensors $S_{g,n}$ are symmetric.

We call the above formula **abstract topological recursion**.

EXAMPLE 2.5.1. Suppose we are given a homogeneous cubic polynomial P_3 as well as the non-degenerate quadratic form P_2 , both in variables y_\bullet .

Consider the $1/\hbar$ -Fourier transform

$$I(\hbar, x) = \int_Z e^{-\frac{1}{\hbar}xy} e^{\frac{1}{\hbar}(P_2(y)+P_3(y))} dy.$$

Here we understand $I(\hbar, x)$ in terms of formal Feynman diagram expansion as $\hbar \rightarrow 0$. The “integration cycle” Z is the germ at the origin of a real closed n -dimensional ball in \mathbf{C}^n such that $Re(P_2 + P_3)|_{Z-\{0\}} < 0$.

Then one has

$$\log(I(\hbar, x)) \sim const + \sum_{g \geq 0, n \geq 0, 2-2g-n < 0} \hbar^{g-1} x^n S_{g,n}$$

with the coefficients computed by Feynman rules. This means that

$$S_{g,n} = \sum_{G(g,n)} R_{G(g,n)} / |Aut(G(g, n))|,$$

where the sum is taken over all connected 3-valent graphs of genus g with n numbered tails, each tail is assigned the corresponding variable x^\bullet . The degree n homogeneous polynomial $R_{G(g,n)}(x)$ is obtained by formal application of the Feynman rules (i.e. every edge is assigned the inverse to the quadratic form P_2 , while each 3-valent vertex is assigned the cubic form P_3). Then our theorem claims that $S_{g,n}$ satisfy the abstract topological recursion formulas. The corresponding Lie algebra \mathfrak{g} is abelian, and the quantum Airy structure is obtained by the gauge transformation (see (2) and (4)) from the following Ansatz:

$$\widehat{H}_i := -\hbar \partial_i + \sum_{j,k} a_{ijk} x^j x^k, \quad \widehat{H}_i (\exp(\hbar^{-1} S_{0,3}(x))) = 0.$$

Here $S_{0,3} = \frac{1}{3} \sum_{ijk} a_{ijk} x^i x^j x^k$ is obtained by a change of variables in P_3 from y_\bullet to x^\bullet via the inverse of the quadratic form P_2 .

2.6. Primitive Airy structures. The terminology below will be used both in classical and quantum cases. For that reason sometimes we will skip the words “classical” or “quantum”, and will speak about Airy structures, etc. The mathematical content of this section is almost empty, we introduce it only to handle the fact that the topological recursion as originally defined by Eynard and Orantin, does *not* satisfy the axiomatics of Airy structures. Nevertheless, the data of the conventional TR contain an Airy substructure. Hence by our general theory of Airy structures the conventional TR produces a collection of symmetric tensors (usually denoted by $\omega_{g,n}$).

DEFINITION 2.6.1. A **classical pre-Airy structure** on vector space V given by a tuple (A, B, C) such that

$$A \in V^{\otimes 3}, B(V) \subset V \otimes V, C(V \otimes V) \subset V.$$

A **quantum pre-Airy structure** on V is given by a tuple (A, B, C, ε) such that (A, B, C) is a classical pre-Airy structure and $\varepsilon \in V$ is a vector. A **substructure** of a (quantum or classical) pre-Airy structure is given by a vector subspace $U \subset V$ such that

$$A \in U^{\otimes 3}, B(U) \subset U \otimes U, C(U \otimes U) \subset U.$$

and $\varepsilon \in U$ in the quantum case.

Obviously, any quantum pre-Airy structure on V induces a classical pre-Airy structure.

DEFINITION 2.6.2. A (quantum or classical) pre-Airy structure on space V is called **primitive** if it does not contain a nontrivial substructure (i.e. the one with $U \neq V$).

Obviously, if U is an Airy substructure, then it carries the induced classical Airy structure. Equivalently, an Airy substructure is the same as a morphism of Airy structures which is an inclusion at the level of underlying vector spaces. It is easy to see that any (quantum or classical) pre-Airy structure contains a unique primitive substructure.

In both quantum or classical cases, an Airy structure is a particular case of a pre-Airy structure. A substructure (in the sense of pre-structures) of an Airy structure is again an Airy structure.

If a classical Airy structure has an Airy substructure then the generating series S_0 from the Theorem 2.4.2 is constant along fibers of the linear surjection $V^* \twoheadrightarrow U^*$. Equivalently, for a basis in V such that U is spanned by a part of the basis, the series S_0 depends only on the corresponding subset of the coordinates in V^* . Classical Hamiltonians corresponding to elements of $(V/U)^* \subset V^*$ are linear (i.e. they do not contain non-trivial quadratic forms).

Geometrically, the existence of a non-trivial classical Airy substructure implies that the germ L of the corresponding Lagrangian submanifold in an affine symplectic space M lies in a non-trivial affine coisotropic submanifold $N \subset M$. Hence L is the pullback of a Lagrangian germ $L' \subset M'$, where M' is a symplectic affine space which is the quotient of N by the constant foliation given by the skew-orthogonal $T_N^\perp \subset T_N$ (equivalently, by the kernel of the symplectic form restricted to N).

Similarly to the above, one can speak about substructures and primitivity of *quantum* Airy structures, by adding the condition $\varepsilon \in U$. Quantum Hamiltonians corresponding to $(V/U)^* \subset V^*$ are just derivations along corresponding directions,

and the equation on wave function means that $\exp(S)$ does not depend on a part of coordinates.

It follows from the above considerations that if one uses a quantum Airy structure only as a technical tool for producing the series $S_g, g \geq 0$ then it suffices to work with any quantum Airy substructure, in particular with the primitive one.

REMARK 2.6.3. The modification formulas (2) and (4) make sense for pre-Airy structures as well, where the symmetric matrix $s = (s^{ij})$ controlling the gauge change is understood in coordinate-free manner as a symmetric bilinear form on V . Obviously, if $U \subset V$ is a substructure, then the restriction of bilinear form s to U gives a gauge change of the induced structure on U .

3. Comparison with topological recursion

3.1. Reminder: Topological recursion. Here we recall the classical Eynard-Orantin formalism for topological recursion. Let Σ be a smooth complex curve (not necessarily connected, e.g. a disjoint union of discs). We assume that we are given a finite collection of distinct points $(r_\alpha \in \Sigma)_{\alpha \in Ram}$ where Ram is a finite set, a collection of germs of involutions $\sigma = \sigma_\alpha$ defined in the vicinity of r_α such that

$$\sigma(r_\alpha) = r_\alpha, \sigma|_{T_{r_\alpha}\Sigma} = -id.$$

In what follows we will consider “tensor fields” on the product of copies of Σ . They are sections of tensor products of pull-backs the line bundles K_Σ and K_Σ^{-1} , where $K_\Sigma = T_\Sigma^*$ is the canonical bundle of Σ . For example, the expression $dz_1 \dots dz_n$ on \mathbf{C}^n where $\Sigma = \mathbf{C}$ and $(z_i)_{i=1, \dots, n}$ are coordinates on Σ^n is not the top degree differential form, it is a polydifferential. Almost all our tensor fields will be polydifferentials, i.e. the pull-backs of K_Σ^{-1} will be absent (exception is the recursion kernel below). We denote by $(\pi_i)_{i=1, \dots, n}$ the projections $\Sigma^n \rightarrow \Sigma$ to factors.

The input data for the topological recursion consist of:

- (1) The Bergman kernel $\omega_{0,2} \in \Gamma(\Sigma \times \Sigma \setminus \{diag\}, \pi_1^*K_\Sigma \otimes \pi_2^*K_\Sigma)$ such that $\omega_{0,2} = \omega_{0,2}(z_1, z_2)$ is symmetric under permutation $z_1 \leftrightarrow z_2$ and has second order pole at the diagonal with the leading coefficient of the polar part equal to 1: in local coordinates near every point

$$\omega_{0,2}(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2} + \text{regular terms for } z_1 \text{ close to } z_2.$$

- (2) A germ of 1-form $\omega_{0,1} = \omega_{0,1}(z_1)$ defined in a vicinity of each point r_α , satisfying the condition that

$$\omega_{0,1}(z_1) - \omega_{0,1}(\sigma(z_1)) \text{ vanishes with the order 2 at } r_\alpha.$$

In particular the form $\omega_{0,1}$ vanishes at r_α .

In a typical example, Σ is an affine curve in the plane $\mathbf{C} \times \mathbf{C}$ endowed with coordinates (x, y) , such that the projection to x -coordinate \mathbf{C} has only double ramification points r_α (hence the notation Ram for the set of ramification points). The local involution σ_α interchanges two branches of the projection $x : \Sigma \rightarrow \mathbf{C}$ near r_α , and (the germ of) the 1-form $\omega_{0,1}$ is $y dx$.

DEFINITION 3.1.1. The **recursion kernel** $K = K(z_1, z_2 = z)$ is a collection labeled by $\alpha \in Ram$ of sections of $\pi_1^* K_\Sigma \otimes \pi_2^* K_\Sigma^{-1}$ defined for arbitrary $z_1 \in \Sigma$ such that $z_1 \neq r_\alpha$ for all $\alpha \in Ram$, and for arbitrary z lying in the vicinity of r_α for some α , $z \neq z_1, r_\alpha$, given by the formula

$$K(z_1, z) := \frac{-1}{2} \cdot \frac{\int_{z'=\sigma(z)}^{z'=z} \omega_{0,2}(z_1, z')}{\omega_{0,1}(z) - \omega_{0,1}(\sigma(z))},$$

where in the numerator we use arbitrary integration path from $z' = \sigma(z)$ to $z' = z$ in the vicinity of r_α .

EXAMPLE 3.1.2. The most important for us example is the *Airy curve* $\Sigma = \{(x, y) \in \mathbf{C}^2 | x = y^2\}$ parametrized by the coordinate z so that $x = z^2, y = z$. In this case we have only one ramification point $z = 0$. The involution is given by $\sigma(z) = -z$. The most natural choice of the Bergman kernel is $\omega_{0,2}(z_1, z_2) := \frac{dz_1 dz_2}{(z_2 - z_1)^2}$. Also, the 1-form $\omega_{0,1}$ is $\omega_{0,1}(z) := y(z)dx(z) = 2z^2 dz$.

The integral in the definition of the recursion kernel is

$$\begin{aligned} \int_{z'=-z}^{z'=z} \frac{dz' dz_1}{(z' - z_1)^2} &= dz_1 \cdot \int_{z'=-z}^{z'=z} d \frac{1}{z_1 - z'} = dz_1 \cdot \left(\frac{1}{z_1 - z} - \frac{1}{z_1 + z} \right) = \frac{2z dz_1}{(z_1^2 - z^2)} \\ \implies K(z_1, z) &:= \frac{1}{4(z^2 - z_1^2)z} \frac{dz_1}{dz}. \end{aligned}$$

Finally, given the input data $\omega_{0,1}, \omega_{0,2}$ one defines a collection of new tensor fields $\omega_{g,n}$ for $g = 0, n \geq 3$ or $g \geq 1, n \geq 1$ which are sections of $\pi_1^* K_\Sigma \otimes \dots \otimes \pi_n^* K_\Sigma$ on $(\Sigma - \{r_\alpha | \alpha \in Ram\})^n$ inductively by the formula

$$\begin{aligned} &\text{for } g = 0, n \geq 3 \text{ or } g \geq 1, n \geq 1 \quad \omega_{g,n}(z_1, z_2, \dots, z_n) \\ (8) \quad &= \sum_\alpha Res_{z=r_\alpha} K(z_1, z) \cdot \omega_{g-1, n+1}(z, \sigma(z), z_2, \dots, z_n) \\ &+ \sum_\alpha \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2, \dots, n\}}}^* Res_{z=r_\alpha} K(z_1, z) \cdot \omega_{g_1, 1+\#I_1}(z, z_{I_1}) \cdot \omega_{g_2, 1+\#I_2}(\sigma(z), z_{I_2}). \end{aligned}$$

The first term in the RHS appears only if $g \geq 1$. The symbol \sum^* in the second term indicates that we *do not* allow cases $g_1 = 0, \#I_1 = 0$ or $g_2 = 0, \#I_2 = 0$, i.e. only $\omega_{0,2}, \omega_{0,\geq 3}$ and $\omega_{\geq 1, \geq 1}$ can appear in the RHS. Notice that $\omega_{0,1}$ does not appear explicitly in the RHS, it participates in the whole formalism only via the denominator $\omega_{0,1}(z) - \omega_{0,1}(\sigma(z))$ for the recursion kernel.

An amazing property of the above formalism is that the tensor fields $\omega_{g,n}$ defined recursively are invariant under the action of the symmetric group S_n . The recursion formulas for cases $g = 0, n = 3$ and $g = 1, n = 1$ are somewhat different from the rest:

$$(9) \quad \mathbf{A-type} : \omega_{0,3}(z_1, z_2, z_3) = 2 \sum_\alpha Res_{z=r_\alpha} K(z_1, z) \omega_{0,2}(z, z_2) \omega_{0,2}(\sigma(z), z_3),$$

$$(10) \quad \mathbf{\varepsilon-type} : \omega_{1,1}(z_1) = \sum_\alpha Res_{z=r_\alpha} K(z_1, z) \omega_{0,2}(z, \sigma(z)).$$

In further cases $g = 0, n \geq 4$, or $g = 1, n \geq 2$ or $g = 2, n \geq 1$ in the RHS of the recursion formula for $\omega_{g,n}$ one has *two* types of summands:

- (1) **B-type:** the terms including decomposition $g = g_1 + g_2, \{2, \dots, n\} = I_1 \sqcup I_2$ with *either* $g_1 = 0, \#I_1 = 1$ or $g_2 = 0, \#I_2 = 1$. These terms are given by the action of a linear operator whose argument $f(z) dz$ is a collection of germs of meromorphic 1-forms at points r_α ,

$$(11) \quad f(z)dz \mapsto \sum_{\alpha} \text{Res}_{z=r_\alpha} K(z_1, z) \omega_{0,2}(\sigma(z), z_2) f(z)dz.$$

- (2) **C-type:** all other terms, corresponding to a bilinear operator on pairs $(f(z_1) dz_1, g(z_2) dz_2)$ of collections of germs of meromorphic forms as above,

$$(12) \quad f(z_1)dz_1 \otimes g(z_2)dz_2 \mapsto \sum_{\alpha} \text{Res}_{z=r_\alpha} K(z_1, z) f(z)dz g(\sigma(z)) d(\sigma(z)).$$

We can recast the topological recursion by saying that $\omega_{0,3}, \omega_{1,1}$ are the *initial data* (of correspondingly A-type and ε -type), and the recursive steps use only operations of B-type or C-type. Comparing formula (8) with the formula (7) from Section 2.5, we see that the perfect match:

$$\omega_{g,n} \longleftrightarrow S_{g,n}, \quad g = 0, n \geq 3 \text{ or } g \geq 1, n \geq 0.$$

Nevertheless, in order to identify the topological recursion with the formalism of Airy structures, one has to choose an appropriate space V of 1-forms on $\Sigma - \{r_\alpha \mid \alpha \in \text{Ram}\}$.

3.2. Topological recursion, the case of Airy curve. Recall from Example 3.1.2 Bergman and recursion kernels for the Airy curve:

$$\omega_{0,2}(z_1, z_2) := \frac{dz_1 dz_2}{(z_2 - z_1)^2}, \quad K(z_1, z) := \frac{1}{4(z^2 - z_1^2)z} \frac{dz_1}{dz}.$$

Let $V := z^{-1}\mathbf{C}[z^{-1}] \frac{dz}{z}$. We endow it with the basis

$$e_n := z^{-n} \frac{dz}{z}, \quad n = 1, 2, \dots$$

Define 4 tensors of topological recursion

$$A_{TR} \in \text{Sym}^3(V), B_{TR} \in \text{Hom}(V, V \otimes V), C_{TR} \in \text{Hom}(V \otimes V, V)^{S_2}, \varepsilon_{TR} \in V$$

via formulas (see (9),(10),(11), (12)):

$$\begin{aligned} A_{TR} &:= && \text{Res}_z K(z_1, z) \omega_{0,2}(z, z_2) \omega_{0,2}(-z, z_3), \\ B_{TR} : f(z)dz &\mapsto && \text{Res}_z K(z_1, z) \omega_{0,2}(z, z_2) f(-z)d(-z), \\ C_{TR} : f(z)dz \otimes g(z)dz &\mapsto && \text{Res}_z K(z_1, z) f(z)dz g(-z)d(-z), \\ \varepsilon_{TR} &:= && \text{Res}_z K(z_1, z) \omega_{0,2}(z, -z), \end{aligned}$$

where $Res_z := Res_{z=0}$. Let us calculate the structure constants of these tensors in the basis $(e_n)_{n \geq 1}$. We have:

$$\begin{aligned}
 A_{TR} &= -\frac{1}{4} Res_z \frac{1}{(z_1^2 - z^2)z} \frac{dz_1}{dz} \frac{dz dz_2}{(z_2 - z)^2} \frac{-dz dz_3}{(z_3 + z)^2} \\
 &= \frac{dz_1 dz_2 dz_3}{4} Res_z \frac{dz}{z} \left(\frac{1}{z_1^2} + \frac{z^2}{z_1^4} + \dots \right) \left(\frac{1}{z_2^2} + \frac{2z}{z_2^3} + \dots \right) \left(\frac{1}{z_3^2} - \frac{2z}{z_3^3} + \dots \right) \\
 &= \frac{1}{4} \frac{dz_1}{z_1^2} \frac{dz_2}{z_2^2} \frac{dz_3}{z_3^2} = \frac{1}{4} e_1 \otimes e_1 \otimes e_1, \\
 B_{TR} : e_n &= z^{-n} \frac{dz}{z} \mapsto -\frac{1}{4} Res_z \frac{1}{(z_1^2 - z^2)z} \frac{dz_1}{dz} \frac{dz dz_2}{(z_2 - z)^2} (-z)^{-n} \frac{dz}{z} \\
 &= -\frac{1}{4} dz_1 dz_2 Res_z \frac{dz}{z} (-1)^n z^{-n-1} \left(\frac{1}{z_1^2} + \frac{z^2}{z_1^4} + \dots \right) \left(\frac{1}{z_2^2} + \frac{2z}{z_2^3} + \frac{3z^2}{z_2^4} + \dots \right) \\
 &= -\frac{1}{4} dz_1 dz_2 (-1)^n \sum_{n_1, n_2 \geq 0; 2n_1 + n_2 = n+1} (n_2 + 1) z_1^{-2n_1-2} z_2^{-n_2-2} \\
 &= -\frac{1}{4} \sum_{n_1, n_2 \geq 0; 2n_1 + n_2 = n+1} (-1)^{n_2+1} (n_2 + 1) e_{2n_1+1} \otimes e_{n_2+1}, \\
 C_{TR} : e_{n_1} \otimes e_{n_2} &= \frac{dz}{z^{n_1+1}} \otimes \frac{dz}{z^{n_2+1}} \mapsto -\frac{1}{4} Res_z \frac{1}{(z_1^2 - z^2)z} \frac{dz_1}{dz} \frac{dz d(-z)}{z^{n_1+1} (-z)^{n_2+1}} \\
 &= -\frac{1}{4} (-1)^{n_2} dz_1 Res_z \frac{dz}{z} z^{-(n_1+n_2+2)} \left(\frac{1}{z_1^2} + \frac{z^2}{z_1^4} + \dots \right) \\
 &= \begin{cases} 0 & \text{if } n_1 + n_2 \in 1 + 2\mathbf{Z} \\ -\frac{1}{4} (-1)^{n_2} e_{2m+1} & \text{if } n_1 + n_2 + 2 = 2m \geq 2, \end{cases} \\
 \varepsilon_{TR} &= -\frac{1}{4} Res_z \frac{1}{(z_1^2 - z^2)z} \frac{dz_1}{dz} \frac{dz d(-z)}{4z^2} = -\frac{1}{4} dz_1 Res_z \frac{dz}{z} \frac{-1}{4z^2} \left(\frac{1}{z_1^2} + \frac{z^2}{z_1^4} + \dots \right) \\
 &= \frac{dz_1}{16 z_1^4} = \frac{1}{16} e_3.
 \end{aligned}$$

For the future reference we combine above formulas:

(13) $A_{TR} = \frac{1}{4} e_1 \otimes e_1 \otimes e_1,$

(14) $B_{TR} : e_n \mapsto \frac{1}{4} \sum_{n_1, n_2 \geq 1; n_1 + n_2 = n+3; n_1 \text{ odd}} (-1)^{n_2+1} n_2 e_{n_1} \otimes e_{n_2},$

(15) $C_{TR} : e_{n_1} \otimes e_{n_2} \mapsto \begin{cases} 0 & \text{if } n_1 + n_2 \text{ odd} \\ \frac{1}{4} (-1)^{n_2+1} e_{n_1+n_2+3} & \text{if } n_1 + n_2 \text{ even,} \end{cases}$

(16) $\varepsilon_{TR} = \frac{1}{16} e_3.$

We will see below that the data $(V, A_{TR}, B_{TR}, C_{TR}, \varepsilon_{TR})$ define a quantum pre-Airy structure on V , which is *not* a quantum Airy structure. In order to fix the problem let us consider $V_{odd} := z^{-1} \mathbf{C}[z^{-2}] \frac{dz}{z} \subset V$, which is the vector subspace spanned by e_1, e_3, e_5, \dots . We will see that the restrictions of the tensors $A_{TR}, B_{TR}, C_{TR}, \varepsilon_{TR}$ to V_{odd} give rise to a quantum Airy structure.

3.3. Airy structure from residue constraints. Let us consider the infinite-dimensional symplectic vector space

$$W_{Airy} := \left\{ \eta \in \mathbf{C}((z)) dz \mid Res_z(\eta) = 0 \right\} = \left\{ J \cdot \frac{dz}{z} \mid J = \sum_{n \in \mathbf{Z}; n \neq 0; n \ll +\infty} J_n z^{-n} \right\}$$

with the symplectic pairing given by

$$(17) \quad \langle df_1, df_2 \rangle := Res_z(f_1 df_2) \iff \left\langle z^{-n} \frac{dz}{z}, z^n \frac{dz}{z} \right\rangle = \frac{1}{n} \iff \{J_n, J_{-n}\} = n.$$

Let us introduce a family of residue constraints on the elements η which are polynomials of degree ≤ 2 in the coordinates⁶ x, y defined below⁷:

$$\forall n \geq 1 \ Res_z(y x^n dx) = 0, \quad \forall m \geq 0 \ Res_z(y^2 x^m dx) = 0,$$

where

$$x := z^2 \in \mathbf{C}((z)), \quad y = y(\eta) := z - \frac{J}{2x} = z - \frac{J}{2z^2}, \quad J = \frac{\eta}{z^{-1} dz}.$$

Equivalently, $y(\eta) dx = zd(z^2) - \eta$.

The first constraint $Res_z(y x^n dx) = 0$ gives

$$Res_z\left(\frac{J}{z^2} z^{2n} z dz\right) = Res_z\left(\frac{dz}{z} J z^{2n}\right) = 0 \iff J_{2n} = 0.$$

The second constraint $Res_z(y^2 x^m dx) = 0$ for $m \geq 0$ gives

$$Res_z\left(z - \frac{J}{2z^2}\right)^2 z^{2m} z dz = Res_z \frac{dz}{z} \left(-J z^{2m+1} + \frac{1}{4} J^2 z^{2m-2}\right) = 0 \iff$$

$$-J_{2m+1} + \frac{1}{4} \sum_{k+l=2m-2; k, l \neq 0} J_k J_l = 0.$$

Therefore, the first and the second residue constraints can be written together as a system of equations $(H_n = 0)_{n \geq 1}$ where:

$$\begin{aligned} H_1 &:= -J_1 + \frac{1}{4} J_{-1}^2 + \frac{1}{2} (J_{-3} J_1 + J_{-4} J_2 + \dots), \\ H_2 &:= -J_2, \\ H_3 &:= -J_3 + \frac{1}{2} (J_{-1} J_1 + J_{-2} J_2 + \dots), \\ H_4 &:= -J_4, \\ H_5 &:= -J_5 + \frac{1}{4} J_1^2 + \frac{1}{2} (J_{-1} J_3 + J_{-2} J_4 + \dots), \\ H_6 &:= -J_6, \\ H_7 &:= -J_7 + \frac{1}{4} J_2^2 + \frac{1}{2} J_1 J_3 + \frac{1}{2} (J_{-1} J_5 + J_{-2} J_6 + \dots), \\ &\dots \end{aligned}$$

REMARK 3.3.1. The infinite-dimensional topological vector or affine spaces considered in this paper are Tate spaces, i. e. they are direct sum of vector or affine spaces endowed with discrete topology and dual to such spaces. One can also say that Tate spaces are vector spaces endowed with the topology of the vector space of Laurent series. Formal germ at $0 \in W$ of the Tate space W (e.g. $W = W_{Airy}$)

⁶In formulas below x^n means the power of the variable x . The reader should not confuse it with the notation for the coordinates (x^1, \dots, x^n) in Section 2.

⁷The geometric meaning of the coordinates and the constraints will be explained below in Section 6.3. At the moment the reader should consider all that as a “black box” producing an Airy structure.

is understood as a functor on the category of finite-dimensional commutative local Artin algebras:

$$R \mapsto m_R^* \otimes W,$$

where $m_R \subset R$ is the maximal ideal. In this way one can also define the notion of subvariety, intersection of subvarieties, etc.

LEMMA 3.3.2. *The complete topological vector subspace of the space $Sym_{\leq 2}(W)$ spanned by $(H_n)_{n \geq 1}$ is closed under the Poisson bracket given in coordinates $(J_k)_{k \neq 0}$ by $\{J_k, J_{-k}\} = k$. Moreover, there exists a unique lift of $\prod_{n \geq 1} \mathbf{C} \cdot H_n$ to a Lie algebra of \hbar -differential operators.*

PROOF. It is convenient to add a central variable J_0 and denote for $\forall n \in \mathbf{Z}$

$$L_n := \frac{1}{4} \sum_{k+l=2n} J_k J_l.$$

In the end we will specialize results to the hypersurface $J_0 = 0$. Functions L_\bullet form a Lie algebra with respect to the Poisson bracket:

$$\begin{aligned} \{L_{n_1}, L_{n_2}\} &= \frac{1}{16} \sum_{k_1+l_1=2n_1, k_2+l_2=2n_2} \{J_{k_1} J_{l_1}, J_{k_2} J_{l_2}\} \\ &= \frac{1}{4} \sum_{k_1+l_1=2n_1, -k_1+l_2=2n_2} k_1 J_{l_1} J_{l_2} \\ &= \frac{1}{4} \sum_k k J_{2n_1-k} J_{2n_2+k} = \frac{1}{8} \sum_{k_1+k_2=2(n_1+n_2)} ((2n_1 - k_1) - (2n_2 - k_2)) J_{k_1} J_{k_2} \\ &= \frac{1}{8} \sum_{k_1+k_2=2(n_1+n_2)} 2(n_1 - n_2) J_{k_1} J_{k_2} = (n_1 - n_2) L_{n_1+n_2}. \end{aligned}$$

The reader can recognize here the structure constants of the Lie algebra $\mathbf{C}[[x]]\partial_x$ of formal vector fields on the line. Also, let us calculate the bracket

$$\{L_n, J_m\} = \frac{1}{4} \sum_{k+l=2n} \{J_k J_l, J_m\} = \frac{1}{2} (-m) J_{2n+m}.$$

We have

$$H_{2n+3} = -J_{2n+3} + L_n \quad \forall n \geq -1, \quad H_{2m} = -J_{2m} \quad \forall m \geq 1.$$

This formulas implies

$$(18) \quad \begin{aligned} \{H_{2n_1+3}, H_{2n_2+3}\} &= (n_1 - n_2) H_{2n_1+2n_2+3}, \\ \{H_{2n+3}, H_{2m}\} &= (-m) H_{2n+2m}, \\ \{H_{2m_1}, H_{2m_2}\} &= 0. \end{aligned}$$

The above calculation shows that the Lie algebra $\mathfrak{g} = \prod_{n \geq 1} \mathbf{C} \cdot H_n$ is isomorphic to the quotient of the algebra of differential operators of order ≤ 1 on $\mathbf{C}[[x]]$ by constants $\mathbf{C} \cdot 1$:

$$H_{2n+3} \leftrightarrow -x^{n+1} \partial_x \quad \forall n \geq -1, \quad H_{2m} \leftrightarrow x^m \quad \forall m \geq 1.$$

Let us introduce coordinates x_1, x_2, \dots on an infinite-dimensional space, and realize quantum operators $(\hat{J}_n)_{n \in \mathbf{Z}, n \neq 0}$ as

$$\hat{J}_n = \hbar \frac{\partial}{\partial x_n}, \quad \hat{J}_{-n} := n x_n \quad \forall n \geq 1.$$

The direct calculation shows that the same commutation relations as in Equation (18) hold for

$$\begin{aligned} \widehat{H}_{2n+3} &:= -\widehat{J}_{2n+3} + \frac{1}{4} \sum_{k+l=2n; k, l \neq 0} \widehat{J}_k \widehat{J}_l, \quad n = -1, 1, 2, 3, \dots, \\ \widehat{H}_3 &:= -\widehat{J}_3 + \frac{1}{2} (\widehat{J}_{-1} \widehat{J}_1 + \widehat{J}_{-2} \widehat{J}_2 + \dots) + \frac{\hbar}{16}, \\ \widehat{H}_{2m} &:= -\widehat{J}_{2m}, \quad m \geq 1. \end{aligned}$$

More precisely, instead of the Poisson bracket we use the Lie bracket $(ab - ba)/\hbar$.

The uniqueness of the quantum lift follows from the vanishing $H^1_{cont}(\mathfrak{g}, \mathbf{C})$. The existence was proven above via explicit formulas.⁸

Let us split $W = W_{Airy}$ into the direct sum of two complementary Lagrangian subspaces:

$$W = V^* \oplus V, \quad V^* := \mathbf{C}[[z]]dz, \quad V := z^{-2}\mathbf{C}[z^{-1}]dz.$$

This splitting gives us a quantum Airy structure on V . More precisely, the germ of the quadratic Lagrangian submanifold $L = L_{Airy}$ defined by the system of equations $H_n = 0, n \geq 1$ has the tangent space at 0 given by the system of linear equations $J_1 = J_2 = \dots = 0$, which is isomorphic to V^* .

From the above splitting we derive the tensors A, B, C, ε . E.g. the tensor B is encoded in the bracketed sum in the formulas for $(H_n)_{n \geq 1}$.

In the basis $(e_n)_{n \geq 1}$ we have

$$\begin{aligned} A &= \frac{1}{4} e_1 \otimes e_1 \otimes e_1, \\ B : e_n &\mapsto \frac{1}{4} \sum_{n_1, n_2 \geq 1; n_1 + n_2 = n + 3; n_1 \text{ odd}} n_2 e_{n_1} \otimes e_{n_2}, \\ (19) \quad C : e_{n_1} \otimes e_{n_2} &\mapsto \begin{cases} 0 & \text{if } n_1 + n_2 \text{ odd} \\ \frac{1}{4} e_{n_1 + n_2 + 3} & \text{if } n_1 + n_2 \text{ even,} \end{cases} \\ \varepsilon &= \frac{1}{16} e_3. \end{aligned}$$

One can write formulas for tensors B, C using residues:

$$\begin{aligned} (20) \quad B : f(z)dz &\mapsto -Res_z K(z_1, z) \omega_{0,2}(z, z_2) f(z)dz, \\ C : f(z)dz \otimes g(z)dz &\mapsto -Res_z K(z_1, z) f(z)dz g(z)dz. \end{aligned}$$

3.4. Common Airy substructure and modified topological recursion.

We see (formulas (13), (14), (15), (16) versus (19)) that in the example of Airy curve we have *two* collections of tensors $(A_{TR}, B_{TR}, C_{TR}, \varepsilon_{TR})$ and (A, B, C, ε) which can be used for the recursive definition of $\omega_{g,n}$ with $g = 0, n \geq 3$ or $g \geq 1, n \geq 1$.

It turns out that they produce the *same* tensors $\omega_{g,n}$. It will follow from the fact that both collections of tensors (i.e. the corresponding pre-Airy structures) define quantum Airy structures on the vector subspace

$$V_{odd} := \oplus_{m \geq 0} \mathbf{C} \cdot e_{2m+1} \subset V,$$

and moreover these quantum structures coincide, since the restrictions of the above tensors on V_{odd} coincide.

Furthermore V_{odd} endowed with this quantum Airy structure is a primitive Airy structure (and the corresponding classical Airy structure is also primitive).

⁸Notice that $H^2_{cont}(\mathfrak{g}, \mathbf{C}) \neq 0$ as \mathfrak{g} has a non-trivial central extension which is the Atiyah Lie algebra $\mathbf{C}[[x]]\partial_x \oplus \mathbf{C}[[x]]$, hence the existence of a lift is not automatic.

For this primitive *classical* structure the existence and uniqueness of the quantum lift is obvious, as $H_{cont}^1(\mathfrak{g}_0, \mathbf{C}) = H_{cont}^2(\mathfrak{g}_0, \mathbf{C}) = 0$ for the Lie algebra $\mathfrak{g}_0 \simeq \mathbf{C}[[x]]\partial_x$ of formal vector fields on the line (sometimes it is called Witt Lie algebra).

REMARK 3.4.1. One can characterize V_{odd} as a subspace of V intrinsically. Namely, it is the space of such meromorphic 1-forms from V that their direct image to the x -line under the projection $z \mapsto x = z^2$ has no poles.

Subspaces $V, V_{odd} \subset \mathbf{C}((z))dz$ are naturally associated with kernels $\omega_{0,2}(z_1, z)$ and $K(z_1, z)$ in the following way. Let us expand these kernels in the iterated Laurent series in the domain $0 \ll |z| \ll |z_1| \ll 1$:

$$\omega_{0,2}(z_1, z)_{|z| \ll |z_1|} = \left(\frac{1}{z_1^2} + \frac{2z}{z_1^3} + \frac{3z^2}{z_1^4} + \dots \right) dz_1 dz \in \mathbf{C}((z_1))((z)) dz_1 dz,$$

$$K(z_1, z)_{|z| \ll |z_1|} = \frac{-1}{4} \cdot \left(\frac{z^{-1}}{z_1^2} + \frac{z}{z_1^4} + \frac{z^3}{z_1^6} + \dots \right) \frac{dz_1}{dz} \in \mathbf{C}((z_1))((z)) \frac{dz_1}{dz}.$$

We see that $V^{z \rightarrow z_1}, V_{odd}^{z \rightarrow z_1}$ are the *minimal* subspaces of $\mathbf{C}((z_1))dz_1$ such that

$$\omega_{0,2}(z_1, z)_{|z| \ll |z_1|} \in V((z))dz, \quad K(z_1, z)_{|z| \ll |z_1|} \in V_{odd}((z))(dz)^{-1}.$$

Here the notation $z \rightarrow z_1$ means that we replace variable z by z_1 and realize V, V_{odd} as subspaces of $\mathbf{C}((z_1))dz_1$ instead of $\mathbf{C}((z))dz$.

Let us define the recursion based on tensors A, B, C, ε and using the residue formulas from equation (20). We call it the **modified topological recursion**. Explicitly, it is given by:

$$\text{initial data: } \omega_{0,3} = 2A = \frac{1}{2} \frac{dz_1 dz_2 dz_3}{z_1^2 z_2^2 z_3^2}, \quad \omega_{1,1} = \varepsilon = \frac{1}{16} \frac{dz_1}{z_1^4},$$

$$\text{for } g = 0, n \geq 4 \text{ or } g = 1, n \geq 2 \text{ or } g \geq 2, n \geq 1$$

$$\omega_{g,n}(z_1, z_2, \dots, z_n) =$$

$$- \sum_{\alpha} \text{Res}_{z=r_{\alpha}} K(z_1, z) \cdot \omega_{g-1, n+1}(z, z, z_2, \dots, z_n)$$

$$- \sum_{\alpha} \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2, \dots, n\}}}^* \text{Res}_{z=r_{\alpha}} K(z_1, z) \cdot \omega_{g_1, 1+\#I_1}(z, z_{I_1}) \cdot \omega_{g_2, 1+\#I_2}(z, z_{I_2}).$$

The first sum in the RHS appears only if $g \geq 1$. Also, similarly to the Eynard-Orantin topological recursion, the notation \sum^* in the second sum means that we do not allow cases $g_1 = 0, \#I_1 = 0$ or $g_2 = 0, \#I_2 = 0$. In fact, the above formula holds also in the a priori forbidden case $g = 0, n = 3$ (but not for $g = 1, n = 1$ as $\omega_{0,2}(z, z)$ is ill-defined).

Let us summarize differences between the modified topological recursion and the usual one:

- a) the overall minus sign;
- b) the absence of the local involution in the RHS of the formula for the modified topological recursion;
- c) lesser range of validity of the modified topological recursion (i.e. $\omega_{0,3}$ and $\omega_{1,1}$ are treated in the modified recursion as initial data).

REMARK 3.4.2. An interesting (and a priori unexpected) corollary of the above discussion is the fact that the tensors $\omega_{g,n}$ derived from the usual (i.e. Eynard-Orantin) topological recursion satisfy also the modified topological recursion. This observation seems to be new.

3.5. Comparison: TR versus Airy structures in general. In general, in the formalism of topological recursion, one can replace the curve Σ by a disjoint union of small discs containing ramification points $\{r_\alpha\}$. Moreover, one can pick a local coordinate z on each disc (uniquely up to $\mathbf{Z}/3\mathbf{Z}$ -action $z \mapsto \xi z, \xi^3 = 1$) in such a way that

$$\sigma(z) = -z, \quad \omega_{1,0}(z) - \omega_{1,0}(-z) = 4z^2 dz.$$

The remaining input is the Bergman kernel $\omega_{0,2}$ which can be written as the sum of canonical kernels $dz_1 dz_2 / (z_1 - z_2)^2$ on each disc, plus a regular (= without poles) S_2 -symmetric section of $\pi_1^* K_\Sigma \otimes \pi_2^* K_\Sigma$. As a matrix-valued 2-differential

$$\omega_{0,2}(z_1, z) = \left(\frac{\delta_{\alpha_1, \alpha_2} dz_1 dz}{(z_1 - z)^2} + \sum_{n_1, n_2 \geq 1} P_{n_1, n_2}^{\alpha_1, \alpha_2} z_1^{n_1} z^{n_2} \frac{dz_1 dz}{z_1 z} \right)_{\alpha_1, \alpha_2 \in Ram},$$

where $(P_{n_1, n_2}^{\alpha_1, \alpha_2} \in \mathbf{C})_{\alpha_1, \alpha_2 \in Ram; n_1, n_2 \geq 1}$ obey the symmetry $P_{n_1, n_2}^{\alpha_1, \alpha_2} = P_{n_2, n_1}^{\alpha_2, \alpha_1}$.

Define the Tate topological vector space

$$W := W_{Airy}^{Ram} = \mathbf{C}^{Ram} \otimes \{\gamma \in \mathbf{C}((z)) dz \mid Res_z(\gamma) = 0\} \subset \mathbf{C}^{Ram}((z)) dz.$$

It has topological basis $([\alpha] \otimes z^n dz/z)_{\alpha \in Ram, n \neq 0}$ where $([\alpha])_{\alpha \in Ram}$ is the coordinate basis of \mathbf{C}^{Ram} . We endow W with the symplectic pairing

$$\begin{aligned} \langle [\alpha_1] \otimes df_1, [\alpha_2] \otimes df_2 \rangle &:= \delta_{\alpha_1, \alpha_2} Res_z(f_1 df_2) \\ \iff \langle [\alpha_1] \otimes z^{n_1} \frac{dz}{z}, [\alpha_2] \otimes z^{n_2} \frac{dz}{z} \rangle &:= \frac{1}{n_2} \delta_{\alpha_1, \alpha_2} \delta_{n_1, -n_2}. \end{aligned}$$

Define a Lagrangian subspace $W_+ := \mathbf{C}^{Ram}[[z]] dz \subset W$, which is topologically spanned by the base vectors $[\alpha] \otimes z^n \frac{dz}{z}$ with $n \geq 1$. It is complementary to the Lagrangian subspace $W_- \simeq W_+^*$ spanned by $[\alpha] \otimes z^{-n} \frac{dz}{z}$ with $n \geq 1$.

LEMMA 3.5.1. *Under the above assumptions formal Bergman kernels (formal in the sense that we do not put convergence condition on the regular part of $\omega_{0,2}$) are in one-to-one correspondence with Lagrangian subspaces $V \subset W$ complementary to W_+ . The correspondence is defined as follows: expand $\omega_{0,2}(z_1, z)$ into the iterated Laurent series:*

$$\omega_{0,2}(z_1, z)_{|z| \ll |z_1|} \in \mathbf{C}^{Ram} \widehat{\otimes} \mathbf{C}^{Ram}((z_1))((z)) dz_1 dz.$$

Then one has

$$(21) \quad \omega_{0,2}(z_1, z)_{|z| \ll |z_1|} \in V^{z \rightarrow z_1} \widehat{\otimes} \mathbf{C}^{Ram}[[z]] dz.$$

Proof. In the above notation one has

$$\omega_{0,2}(z_1, z) = \sum_{\alpha} \frac{[\alpha] \otimes [\alpha] dz_1 dz}{(z_1 - z)^2} + \omega_{0,2}^{reg},$$

where $\omega_{0,2}^{reg} \in \widehat{Sym^2} W_+$ (recall that the hat means topological completion). The polar part is equal to

$$\sum_{n \geq 1, \alpha} \left([\alpha] \otimes n z^{-n} \frac{dz}{z} \right) \otimes \left([\alpha] \otimes z^n \frac{dz}{z} \right) = (\text{canonical pairing})^{-1} \in W_- \widehat{\otimes} W_+.$$

The regular part is given by

$$\omega_{0,2}^{reg} = \sum_{\alpha_1, \alpha_2} \sum_{n_1, n_2 \geq 1} P_{n_1 n_2}^{\alpha_1 \alpha_2} \left([\alpha_1] \otimes z_1^{n_1} \frac{dz_1}{z_1} \right) \otimes \left([\alpha_2] \otimes z_2^{n_2} \frac{dz_2}{z_2} \right).$$

Any Lagrangian subspace $V \subset W$ complementary to W_+ is the graph in $W \simeq T^*W_-$ of the differential of a quadratic form on W_- . Such a form corresponds to an element of $\widehat{Sym^2 W_+}$. □

From now on we denote by V the Lagrangian subspace of W associated to the given Bergman kernel $\omega_{0,2}$ whose regular part could be even formal (non-convergent) matrix-valued power series in two variables.⁹ Space V has canonical basis $(e_n^\alpha)_{\alpha \in Ram; n \geq 1}$:

$$(22) \quad e_n^\alpha := [\alpha] \otimes z^{-n} \frac{dz}{z} + \sum_{\alpha'; n' \geq 1} \frac{1}{n} P_{n', n}^{\alpha', \alpha} [\alpha'] \otimes z^{n'} \frac{dz}{z}.$$

Vector e_n^α is the unique vector in V such that

$$e_n^\alpha = [\alpha] \otimes z^{-n} \frac{dz}{z} + (\text{regular part} \in \mathbf{C}^{Ram}[[z]] dz).$$

It will be convenient to write Bergman kernel as

$$(23) \quad \omega_{0,2}(z_1, z) = \sum_{\alpha; n > 0} (e_n^\alpha)^{z \rightarrow z_1} \otimes ([\alpha] \otimes nz^{n-1} dz)$$

where notation $(e_n^\alpha)^{z \rightarrow z_1}$ means the element of $\mathbf{C}^{Ram}((z_1)) dz_1$ obtained by the replacement of variables $z \rightarrow z_1$ from $e_n^\alpha \in V \subset \mathbf{C}^{Ram}((z)) dz$.

The recursion kernel $K(z_1, z)$ exists in general only as a $(Ram \times Ram)$ -matrix with coefficients in $\mathbf{C}((z_1))((z)) \frac{dz_1}{dz}$. Let us expand the recursion kernel in double series:

$$\begin{aligned} K(z_1, z)_{|z| \ll |z_1|} &= \sum_{\alpha} [\alpha] \otimes [\alpha] \cdot \frac{-1}{4} \cdot \left(\frac{z^{-1}}{z_1^2} + \frac{z}{z_1^4} + \dots \right) \frac{dz_1}{dz} \\ &+ \sum_{\alpha_1, \alpha_2} [\alpha_1] \otimes [\alpha_2] \sum_{n_1, n_2 \geq 1; n_2 \text{ odd}} \frac{-1}{4n_2} P_{n_1, n_2}^{\alpha_1, \alpha_2} z_1^{n_1} z^{n_2-2} \frac{dz_1}{dz}. \end{aligned}$$

This formula implies

$$(24) \quad K(z_1, z) = -\frac{1}{4} \sum_{\alpha; n \geq 1 \text{ odd}} (e_n^\alpha)^{z \rightarrow z_1} \otimes \left([\alpha] \otimes \frac{z^{n-2}}{dz} \right)$$

Denote by $W_{odd} \subset W$ the coisotropic subspace of meromorphic 1-forms on Σ whose pushforward to the quotient Σ/σ has no poles at the images of ramification points $(r_\alpha)_{\alpha \in Ram}$. Subspace W_{odd} is the topological span of base vectors $[\alpha] \otimes z^n \frac{dz}{z}$ with $n \geq 1$ arbitrary and $n \leq -1$ odd. In particular, W_+ is a subspace of W_{odd} . Let

$$V_{odd} := V \cap W_{odd}$$

be the subspace of V spanned by basis vectors $(e_n^\alpha)_{\alpha \in Ram, n \geq 1 \text{ odd}}$.

⁹In fact the whole formalism of the topological recursion can be generalized straightforwardly to the formal case. We are going to use this observation without further comments.

Combining together the above formulas we conclude that the residue formulas from the conventional topological recursion (see formulas (9),(10),(11), (12) from Section 3.1) give tensors (no completions for the tensor products):

$$\begin{aligned} A_{TR}^{univ} &\in V_{odd} \otimes V \otimes V, \\ B_{TR}^{univ} &: W \rightarrow V_{odd} \otimes V, \\ C_{TR}^{univ} &: W \otimes W \rightarrow V_{odd}, \\ \varepsilon_{TR}^{univ} &\in V_{odd}. \end{aligned}$$

Restricting B_{TR}^{univ} and C_{TR}^{univ} to V we see that $(A_{TR}^{univ}, B_{TR}^{univ}, C_{TR}^{univ}, \varepsilon_{TR}^{univ})$ induce a pre-Airy structure on V . We denote the corresponding tensors by $(A_{TR}^{(1)}, B_{TR}^{(1)}, C_{TR}^{(1)}, \varepsilon_{TR}^{(1)})$. Below we will calculate these tensors explicitly.

LEMMA 3.5.2. *Tensor $A_{TR}^{(1)}$ is equal to $\frac{1}{4} \sum_{\alpha} e_1^{\alpha} \otimes e_1^{\alpha} \otimes e_1^{\alpha}$.*

PROOF. The definition of tensor A_{TR}^{univ} by the formula (9) gives

$$\begin{aligned} A_{TR}^{univ} &= -\frac{1}{4} \sum_{\alpha} Res_z \left(e_1^{\alpha} \frac{1}{z dz} + \dots \right) \otimes (e_1^{\alpha} dz + \dots) \otimes (-e_1^{\alpha} dz + \dots) \\ &= \frac{1}{4} \sum_{\alpha} e_1^{\alpha} \otimes e_1^{\alpha} \otimes e_1^{\alpha}. \end{aligned}$$

In particular, we see that A_{TR}^{univ} is a *symmetric* tensor. □

LEMMA 3.5.3. *Vector $\varepsilon_{TR}^{(1)}$ is equal to the sum of two terms*

$$\sum_{\alpha} \left(\frac{e_3^{\alpha}}{16} + \frac{P_{1,1}^{\alpha,\alpha} e_1^{\alpha}}{4} \right).$$

PROOF. By definition epsilon-formula general we have

$$\varepsilon_{TR}^{(1)} = -\frac{1}{4} \sum_{\alpha; n \geq 1 \text{ odd}} e_n^{\alpha} \times Res_z \left(\frac{z^{n-2}}{dz} Coeff_{[\alpha] \otimes [\alpha]}(\omega_{0,2}(z, -z)) \right).$$

For each value of $\alpha \in Ram$ the canonical pole part of $\omega_{0,2}$ gives $e_3^{\alpha}/16$, and the regular part $\omega_{0,2}^{reg}$ gives $P_{1,1}^{\alpha,\alpha} e_1^{\alpha}/4$. □

LEMMA 3.5.4. *Operator $B_{TR}^{(1)}$ maps e_n^{α} to the sum of two terms*

$$\frac{1}{4} \sum_{n_1, n_2 \geq 1; n_1 + n_2 = n + 3; n_1 \text{ odd}} (-1)^{n_2 + 1} n_2 e_{n_1}^{\alpha} \otimes e_{n_2}^{\alpha} + \frac{1}{4} \sum_{\alpha'} P_{1,n}^{\alpha', \alpha} e_1^{\alpha'} \otimes e_1^{\alpha'}.$$

PROOF. For any $\alpha' \in Ram, n' \in \mathbf{Z}_{\neq 0}$ tensor $B_{TR}^{univ}([\alpha'] \otimes z^{n'} \frac{dz}{z}) \in V \otimes V$ is given (using (11)) by formula

$$(25) \quad -\frac{1}{4} \sum_{n_1 \geq 1 \text{ odd}} \sum_{n_2 \geq 1} e_{n_1}^{\alpha'} \otimes e_{n_2}^{\alpha'} \cdot Res \left(\frac{z^{n_1-2}}{dz} \frac{(-1)^{n_2}}{n_2} z^{n_2-1} dz \cdot z^{n'} \frac{dz}{z} \right).$$

In particular, we see that this expression vanishes if $n' \geq 2$. Hence, in the calculation of $B_{TR}^{(1)}(e_n^{\alpha})$ we should use only the polar term ($n' = -n$)

$$e_n^{\alpha} = [\alpha] \otimes z^{-n} \frac{dz}{z} + \dots$$

(giving the main part of the formula), and constant terms ($n' = 1$)

$$e_n^\alpha = \dots + \sum_{\alpha'} P_{1,n}^{\alpha',\alpha}[\alpha'] \otimes z^1 \frac{dz}{z} + \dots$$

contributing in (25) only for $n_1 = n_2 = 1$. □

LEMMA 3.5.5. *Operator $C_{TR}^{(1)}$ maps $e_{n_1}^{\alpha_1} \otimes e_{n_2}^{\alpha_2}$ to the sum of four terms:*

- (1) $\frac{1}{4}(-1)^{n_2+1} e_{n_1+n_2+3}^{\alpha_1}$ if $\alpha_1 = \alpha_2$ and $n_1 + n_2$ even,
- (2) $\frac{1}{4} \sum_{n'_2 \geq 1; 2+n'_2-n_1 \in 2\mathbf{Z}_{\geq 0}} \frac{(-1)^{n'_2+1}}{n'_2} P_{n'_2, n_2}^{\alpha_1, \alpha_2} e_{3+n_1-n'_2}^{\alpha_1}$,
- (3) $\frac{1}{4} \sum_{n'_2 \geq 1; 2+n'_2-n_1 \in 2\mathbf{Z}_{\geq 0}} \frac{(-1)^{n'_2+1}}{n'_2} P_{n'_2, n_2}^{\alpha_1, \alpha_2} e_{3+n_1-n'_2}^{\alpha_1}$,
- (4) $\frac{1}{4} \sum_{\alpha} P_{1, n_1}^{\alpha, \alpha_1} P_{1, n_2}^{\alpha, \alpha_2} e_1^\alpha$.

PROOF. For any $\alpha'_1 \neq \alpha'_2$ and any $n'_1, n'_2 \in \mathbf{Z}_{\neq 0}$ we have

$$C_{TR}^{univ} \left(([\alpha'_1] \otimes z^{n'_1} \frac{dz}{z}) \otimes ([\alpha'_2] \otimes z^{n'_2} \frac{dz}{z}) \right) = 0,$$

while for coinciding ramification points $r_{\alpha'_1} = r_{\alpha'_2}$, $\alpha'_1 = \alpha'_2 =: \alpha'$ we have (using (12)):

$$\begin{aligned} & C_{TR}^{univ} \left(([\alpha'] \otimes z^{n'_1} \frac{dz}{z}) \otimes ([\alpha'] \otimes z^{n'_1} \frac{dz}{z}) \right) \\ &= -\frac{1}{4} \sum_{n' \geq 1 \text{ odd}} e_n^{\alpha'} \cdot \text{Res} \left(\frac{z^{n'-2}}{dz} z^{n'-1} dz (-1)^{n'_2+1} z^{n'_2-1} dz \right). \end{aligned}$$

Therefore, we get a general formula

$$(26) \quad \begin{aligned} & C_{TR}^{(1)} \left(([\alpha'_1] \otimes z^{n'_1} \frac{dz}{z}) \otimes ([\alpha'_2] \otimes z^{n'_2} \frac{dz}{z}) \right) \\ &= \begin{cases} \frac{(-1)^{n'_2}}{4} e_{3-n'_1-n'_2}^{\alpha'_1} & \text{if } \alpha'_1 = \alpha'_2, 2 - n'_1 - n'_2 \in 2\mathbf{Z}_{\geq 0} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Using (22) write $e_{n_1}^{\alpha_1} \otimes e_{n_2}^{\alpha_2}$, $n_1, n_2 \geq 1$ as

$$(27) \quad + ([\alpha_1] \otimes z^{-n_1} \frac{dz}{z}) \otimes ([\alpha_2] \otimes z^{-n_2} \frac{dz}{z})$$

$$(28) \quad + \sum_{\alpha'_2; n'_2 \geq 1} \frac{1}{n'_2} P_{n'_2, n_2}^{\alpha'_2, \alpha_2} ([\alpha_1] \otimes z^{-n_1} \frac{dz}{z}) \otimes ([\alpha'_2] \otimes z^{n'_2} \frac{dz}{z})$$

$$(29) \quad + \sum_{\alpha'_1; n'_1 \geq 1} \frac{1}{n'_1} P_{n'_1, n_1}^{\alpha'_1, \alpha_1} ([\alpha'_1] \otimes z^{n'_1} \frac{dz}{z}) \otimes ([\alpha_2] \otimes z^{-n_2} \frac{dz}{z})$$

$$(30) \quad + \sum_{\alpha'_1 \alpha'_2; n'_1, n'_2 \geq 1} \frac{1}{n'_1 n'_2} P_{n'_1, n_1}^{\alpha'_1, \alpha_1} P_{n'_2, n_2}^{\alpha'_2, \alpha_2} ([\alpha'_1] \otimes z^{n'_1} \frac{dz}{z}) \otimes ([\alpha'_2] \otimes z^{n'_2} \frac{dz}{z}).$$

Applying (26) to four lines (27),(28),(29),(30) we obtain the statement of the lemma (notice that for term (30) the only relevant part will be $\alpha'_1 = \alpha'_2 =: \alpha$ and $n'_1 = n'_2 = 1$). □

Let us consider the quantum pre-Airy structure $(A_{TR}^{(0)}, B_{TR}^{(0)}, C_{TR}^{(0)}, \varepsilon_{TR}^{(0)})$ on the vector space W_- which is equal to the direct sum of copies labeled by set Ram of the canonical quantum pre-Airy structure from Section 3.2. The vector space W_+ is the tangent space at 0 to the germ of the associated Lagrangian subvariety. We have shown in Lemma 3.5.1 that a choice of the formal Bergman kernel corresponds to the choice of a complementary Lagrangian subspace V to W_+ . Hence it gives a way to modify the pre-structure $(A_{TR}^{(0)}, B_{TR}^{(0)}, C_{TR}^{(0)}, \varepsilon_{TR}^{(0)})$ by a gauge equivalence (see formulas (2) and (4) from Section 2). The role of symmetric matrix $s = (s^{ij})$ in these formulas is played by $\omega_{0,2}^{reg} \in \widehat{Sym}^2 W_+$ understood as an infinite symmetric matrix:

$$P := (P_{n_1, n_2}^{\alpha_1, \alpha_2})_{\alpha_1, \alpha_2 \in Ram; n_1, n_2 > 0}, \quad \text{index set} = Ram \times \mathbf{Z}_{>0}.$$

We identify vector spaces W_- and V via

$$W_- \xleftarrow{\sim} W/W_+ \xrightarrow{\sim} V.$$

THEOREM 3.5.6. *The gauge transformation of $(A_{TR}^{(0)}, B_{TR}^{(0)}, C_{TR}^{(0)}, \varepsilon_{TR}^{(0)})$ as described above, coincides with $(A_{TR}^{(1)}, B_{TR}^{(1)}, C_{TR}^{(1)}, \varepsilon_{TR}^{(1)})$.*

PROOF. Just read formulas from Lemmas 3.5.2, 3.5.3, 3.5.4, 3.5.5, as well as formulas (13), (14), (15), (16) for the standard pre-Airy structure, and compare with corresponding expressions (2) and (4). □

COROLLARY 3.5.7. *Pre-Airy structure $(A_{TR}^{(1)}, B_{TR}^{(1)}, C_{TR}^{(1)}, \varepsilon_{TR}^{(1)})$ contains a primitive quantum Airy substructure on the subspace V_{odd} .*

PROOF. Here it is useful to recall (essentially trivial) Remark 2.6.3. The quantum pre-Airy structure $(A_{TR}^{(0)}, B_{TR}^{(0)}, C_{TR}^{(0)}, \varepsilon_{TR}^{(0)})$ on V contains the quantum Airy substructure given by the restriction of all tensors to V_{odd} . The gauge transformation from the above Theorem transforms this Airy substructure to the Airy substructure of the pre-Airy structure $(A_{TR}^{(1)}, B_{TR}^{(1)}, C_{TR}^{(1)}, \varepsilon_{TR}^{(1)})$. This proves the result. □

There is a variation of the above theorem. Namely, consider the quantum Airy structure $(A^{(0)}, B^{(0)}, C^{(0)}, \varepsilon^{(0)})$ on W_- which is the direct sum of copies labeled by set Ram of the canonical quantum Airy structure from Section 3.3. Also, denote by $(A^{(1)}, B^{(1)}, C^{(1)}, \varepsilon^{(1)})$ the quantum pre-Airy structure on V given by the modified recursion as in Section 3.4.

THEOREM 3.5.8. *Pre-Airy structure $(A^{(1)}, B^{(1)}, C^{(1)}, \varepsilon^{(1)})$ is in fact an Airy structure, and it is obtained from $(A^{(0)}, B^{(0)}, C^{(0)}, \varepsilon^{(0)})$ by the gauge transformation associated with $\omega_{0,2}^{reg} \in \widehat{Sym}^2 W_+$. Moreover, this structure has the primitive substructure on subspace V_{odd} , and this substructure coincides with the one from Theorem 3.5.6.*

PROOF. The fact that as a pre-structure $(A^{(0)}, B^{(0)}, C^{(0)}, \varepsilon^{(0)})$ transforms to $(A^{(1)}, B^{(1)}, C^{(1)}, \varepsilon^{(1)})$ follows from the calculations almost identical to those in the proof of Theorem 3.5.6, with the only correction that factors $(-1)^{n_2+1}, (-1)^{n'_2+1}$ are removed. The rest of statements of the Theorem follows immediately. □

4. Spectral curves

4.1. Poisson surfaces with foliation. Any complex analytic Poisson surface which does not consist entirely of 0-dimensional symplectic leaves, contains an open dense symplectic leaf. The complement to the symplectic leaf is the set of zeros of the Poisson structure. Hence it is an analytic divisor, which consists of 0-dimensional symplectic leaves.

DEFINITION 4.1.1. A **good foliated Poisson surface** is a complex Poisson surface P endowed with an analytic 1-dimensional foliation \mathcal{F} , such that the Poisson structure is non-degenerate on an open dense symplectic leaf $S \subset P$, the effective divisor D of zeros of the Poisson structure is the disjoint union of smooth components with positive multiplicities, and \mathcal{F} is tangent to D .

The inverse to the non-degenerate Poisson structure on S is the symplectic form. Hence we can (and will) denote the data from the above Definition by (P, ω, \mathcal{F}) , where ω is the symplectic form on $S = P - D$ understood as a meromorphic 2-form on P with poles at D .

DEFINITION 4.1.2. Let (P, ω, \mathcal{F}) be a good foliated Poisson surface. A smooth complex curve $\Sigma \subset P$ is called a **D -transversal spectral curve** if it is a smooth connected compact complex curve which intersects D at finitely many points $\{p_\beta\}_{\beta \in Pol}$, and the intersection is transversal.

Such curve is called **(\mathcal{F}, D) -transversal**¹⁰ if Σ is tangent to leaves of \mathcal{F} at finitely many points $\{r_\alpha\}_{\alpha \in Ram}$, with the tangency order 1 (i.e. locally near the tangency point it looks like the curve $x = y^2$ tangent to the line $x = 0$). Here Pol (resp. Ram) is the notation for the above finite set of “poles” (resp. “ramification points”).¹¹

We denote by \mathcal{B}_0 (resp. \mathcal{B}_1) the moduli space of D -transversal (resp. (\mathcal{F}, D) -transversal) curves. The moduli spaces are defined in terms of complex analytic families over complex analytic bases in the usual way. Notice that the families of sets $Pol = Pol(\Sigma)$ and $Ram = Ram(\Sigma)$ are locally constant over \mathcal{B}_0 or \mathcal{B}_1 (here we speak about D -transversal or resp. (\mathcal{F}, D) -transversal spectral curves depending on the class of curves Σ).

For any D -transversal spectral curve Σ we denote the corresponding point by $[\Sigma] \in \mathcal{B}_0$ (and, sometimes, simply by Σ in order to alleviate the notation).

Obviously, \mathcal{B}_1 is an open subset of \mathcal{B}_0 (dense in typical situation). We will see below (see Lemma 4.2.1) that \mathcal{B}_0 (and hence \mathcal{B}_1) is smooth.

Notice that for a (\mathcal{F}, D) -transversal curve, we can choose near each tangent point r_α local analytic coordinates (x, y) on S such that the point r_α has coordinates $x = 0, y = 0$, the symplectic form on S is equal to $\omega = dx \wedge dy$, the leaves of foliation \mathcal{F} are given by the equations $x = const$ and the spectral curve Σ is given by the equation $x = y^2$. Such coordinates are unique modulo analytic automorphism $x \mapsto \xi, y \mapsto \xi^{-1}y$, where $\xi = e^{2\pi i/3}$ is the primitive cubic root of 1. We will call such coordinates *canonical*.

EXAMPLE 4.1.3. Let C be a smooth projective curve of genus $g \geq 2$. Let $\pi : T^*C \rightarrow C$ be the standard projection, and $\Sigma \subset T^*C$ be a smooth compact

¹⁰Probably a better name would be \mathcal{F} -Morse.

¹¹There are further possibilities for generalizations here. In particular we can consider the case of immersed rather than embedded curves Σ .

connected curve. Take $P = S := T^*C$ and \mathcal{F} to be the foliation by cotangent fibers. Then $Pol = \emptyset$ and Σ is an D -transversal spectral curve. For any integer $d \geq 1$, the connected component of \mathcal{B}_0 corresponding to smooth spectral curves Σ such that $\pi_{|\Sigma} : \Sigma \rightarrow C$ has degree d , is a Zariski open part of the base of Hitchin fibration for the group $GL(d)$. Recall that this base is

$$\mathcal{B} = \oplus_{i=1}^d \Gamma(C, (\Omega_C^1)^{\otimes i}) \simeq \mathbf{C}^{d^2(g-1)+1}$$

and it can be interpreted as the linear system of compact effective divisors in T^*C of degree d over C . The complement $\mathcal{B} - \mathcal{B}_0$ is the discriminant hypersurface parametrizing singular spectral curves. Zariski open subvariety $\mathcal{B}_1 \subset \mathcal{B}_0$ parametrizes smooth Σ such that $\pi_{|\Sigma}$ has only double ramification points.

This example suggests that in general the moduli space \mathcal{B}_0 should be considered as an open subspace of a larger space \mathcal{B} parametrizing compact effective divisors in P . It is not clear whether the unobstructedness result (Lemma 4.2.1 from the next section) extends to the larger context of effective divisors.

EXAMPLE 4.1.4. More generally, let C be a smooth projective curve endowed with irregular data in the sense of [KS14]. The irregular data consist of a collection of marked points, and Puiseux polynomials (called singular terms in the loc. cit.) with multiplicities. They encode the irregular singular behavior of Higgs fields or connections at the marked points. In the case of Higgs bundles the irregular data modify the definition of the spectral curve from the previous example.

By [KS14] one can construct canonically a Poisson surface P , which is obtained by a sequence of blow-ups of the fiberwise compactification of T^*C .

Then there are two possibilities for the choice of P and the symplectic leaf $S \subset P$. Both possibilities are described in the loc. cit. :

1) If one does not fix eigenvalues of the Higgs field (by definition, the residues of the restriction of the Liouville 1-form to Σ) then there is a choice of P which has an open dense symplectic leaf S isomorphic to T^*C . Then the spectral curve Σ is defined as the compact closure in P of a curve in S . Eigenvalues of the Higgs field correspond to intersections of Σ with $P - S$.

2) If one fixes the eigenvalues of the Higgs field, then one makes additional blow-ups in 1) at the intersection points of Σ with the divisor $P - S$. In this case P contains an open symplectic leaf S which is strictly bigger than T^*C . The spectral curve $\Sigma \subset S$ is compact.

The Poisson surface P endowed with the mermorphic foliation \mathcal{F} induced from the foliation along fibers $T^*C \rightarrow C$ is not in general a good foliated Poisson surface in the sense of definition 4.1.1. One has to remove various types of “bad” points: double points of divisor $D = P - S$, undeterminacy points of \mathcal{F} , components of D which are generically transversal to \mathcal{F} . If we are fortunate, the closure Σ of a smooth spectral curve in T^*C will not meet bad points and will be a smooth compact curve transversal to D . In this case we will obtain a point in \mathcal{B}_0 .

Notice that in [KS14] we considered compactified spectral curves which intersect divisor D only along the components with multiplicity 1, while here we allow transversal intersections of Σ with components D of arbitrary multiplicity. One can make a sequence of blow-ups at the transversal intersection point with a component of multiplicity $k > 1$ and arrive to the situation when the multiplicity of the intersection is equal to one. Nevertheless, this procedure will produce *different* moduli space of spectral curves. The conclusion is that the framework of the

present paper is indeed more general than the one in [KS14], even for the case of cotangent bundles.

EXAMPLE 4.1.5. One can consider a “multiplicative” version of the previous example. Namely, let us start with $S = \mathbf{C}^* \times \mathbf{C}^*$ endowed with the symplectic form $\frac{dx}{x} \wedge \frac{dy}{y}$ and endow with it with the foliation \mathcal{F} with leaves $x = \text{const}$. Let us compactify our symplectic surface to a toric Poisson surface P and make a chain of Poisson blow-ups similar to those mentioned in the previous example. After removing bad points as in the previous example, we obtain a good foliated Poisson surface with all components of D having multiplicity one. The corresponding spectral curves are naturally related to q -difference equations. Similarly to the previous example they could happen to be smooth and D -transversal, hence giving points in \mathcal{B}_0 .

4.2. Deformation theory of spectral curves. We are interested in the deformation theory of Σ in the class of D -transversal spectral curves. The deformation complex is $\mathbf{R}\Gamma(\Sigma, N_\Sigma)$, where $N_\Sigma = (T_P)_{|\Sigma}/T_\Sigma$ is the normal bundle to Σ in P . Using the symplectic structure on $S \subset P$ we identify

$$N_\Sigma \simeq \Omega_\Sigma^1(D_\Sigma), \quad D_\Sigma := D \cap \Sigma = \sum_{\beta \in Pol} k_\beta p_\beta,$$

where Ω_Σ^1 is the sheaf of holomorphic 1-forms on Σ , and $k_\beta \geq 1$ is the local multiplicity of D at p_β , i.e. the order of pole of ω at divisor D near the point p_β .

LEMMA 4.2.1. *The deformation theory of Σ is unobstructed.*

PROOF. In the case when $Pol \neq \emptyset$ we have $H^1(\Sigma, \Omega_\Sigma^1(D_\Sigma)) = 0$ by Serre duality. Hence the deformation theory of Σ is unobstructed.

In the more classical case when $Pol = \emptyset$ the unobstructedness is presumably known. Alternatively, one can adapt the proof of Proposition 8.2.1 in [KS14], where a similar statement (unobstructedness) was proven for compact complex Lagrangian submanifolds in quasi-projective varieties. In the present framework we do not require S to be quasi-projective, it is just an analytic surface. Nevertheless, the proof from the loc. cit. still works in our case, because the degeneration of the Hodge-to-de Rham spectral sequence (see Step 2 of the proof in the loc.cit.) holds for a family of compact complex analytic curves over the spectrum of a local Artin algebra. The Lemma is proved. □

By Lemma 4.2.1 the complex analytic moduli space space \mathcal{B}_0 of D -transversal curves is a disjoint union of complex analytic manifolds. The dimension of the the germ $\mathcal{B}_{[\Sigma]}$ of \mathcal{B}_0 at point $[\Sigma]$ is

$$\dim_{\mathbf{C}} \mathcal{B}_{[\Sigma]} = \text{rk } H^0(\Sigma, \Omega_\Sigma^1(D_\Sigma)) = \begin{cases} g & \text{if } Pol = \emptyset \\ g + \sum_{\beta \in Pol} k_\beta - 1 & \text{if } Pol \neq \emptyset \end{cases}.$$

5. Affine symplectic connection and local embedding of the moduli space of spectral curves

5.1. Generalities on local Lagrangian embeddings. Let us recall the notion of an affine flat connection on a vector bundle.

DEFINITION 5.1.1. For a complex analytic manifold X and an analytic vector bundle \mathcal{H} on X , an **affine flat connection** on \mathcal{H} is a pair (∇, ϕ) where ∇ is an analytic flat linear connection, and $\phi \in \Gamma(X, \Omega_X^1 \otimes \mathcal{H})$ is such that $\nabla(\phi) = 0$.

An affine flat connection is the same as a flat connection $\nabla + \phi$ with values in the Lie algebra of affine transformations. Hence it gives an identification of fibers \mathcal{H}_x as *affine spaces* for all points $x \in X$ close to any reference point x_0 . Under this identification the zero point $0_x \in \mathcal{H}_x$ is mapped to a certain point in \mathcal{H}_{x_0} . Therefore, we obtain a germ of an analytic map

$$\text{germ of } X \text{ at } x_0 \mapsto \text{germ of } \mathcal{H}_{x_0} \text{ at } 0.$$

This map is an immersion if and only if ϕ is a monomorphism (if it is understood as a map of vector bundles $T_X \rightarrow \mathcal{H}$). Finally, let us assume that the bundle \mathcal{H} carries a non-degenerate skew-symmetric pairing $\langle \bullet, \bullet \rangle$ which is covariantly constant, and map $\phi : T_X \rightarrow \mathcal{H}$ is a linear Lagrangian embedding. Then (∇, ϕ) induces a Lagrangian embedding from the germ of X at x_0 to the symplectic affine space \mathcal{H}_{x_0} . Moreover, these embeddings form a formally flat family, as we vary the reference point x_0 .

REMARK 5.1.2. Suppose that with any point $x_0 \in X$ and any pair of points x_1, x_2 sufficiently close to x_0 , we associate a vector $v(x_1, x_2)_{x_0} \in \mathcal{H}_{x_0}$, which is covariantly constant if we vary x_0 (and keep points x_1, x_2 fixed), and satisfying the cocycle condition

$$v(x_1, x_2)_{x_0} + v(x_2, x_3)_{x_0} = v(x_1, x_3)_{x_0}.$$

It is easy to see that such data are equivalent to a *closed* 1-form ϕ with values in a flat vector bundle \mathcal{H} on X . More precisely, suppose we have a linear flat connection ∇ and ∇ -covariant 1-form ϕ . Given a triple of points (x_0, x_1, x_2) which are sufficiently close to each other we can find a contractible open subset $U \subset X$ containing all of them. Then on $U^3 \subset X^3$ we have a trivial bundle $pr^*_{(x_0, x_1, x_2) \mapsto x_0}(\mathcal{H})$, endowed with the pull-backs of ∇ and ϕ . It can be trivialized, such that the fiber of the bundle is identified with \mathcal{H}_{x_0} . Then for a short path γ_{x_1, x_2} joining x_1 and x_2 we define $v(x_1, x_2)_{x_0} = \int_{\gamma_{x_1, x_2}} \phi \in \mathcal{H}_{x_0}$. Conversely, one can reconstruct ϕ from v by taking the first derivative of $v(x_1, x_2)_{x_0}$ w.r.t. x_2 at $x_2 = x_1$.

5.2. Local Lagrangian embedding of \mathcal{B}_0 . Our goal is to apply the general construction from the previous subsection to the case $X = \mathcal{B}_0$ and obtain for every smooth spectral curve Σ the corresponding germ of the Lagrangian embedding. The construction will depend on the foliation \mathcal{F} , but only through the collection of its $(2k - 1)$ -germs near components of D at which ω has pole of order $k \geq 1$.

First, we define a symplectic vector bundle such that its fiber at $[\Sigma] \in \mathcal{B}_0$ is the hypercohomology

$$(31) \quad \mathcal{H}_\Sigma = \mathbb{H}^0(\Sigma, Cone(d_{dR} : \mathcal{O}_\Sigma(-D_\Sigma) \rightarrow \Omega_\Sigma^1(D_\Sigma))),$$

where $D_\Sigma := \sum_{\beta \in Pol} k_\beta p_\beta$ is an effective divisor on Σ as in Section 4.2. Symplectic pairing $\langle \bullet, \bullet \rangle$ on \mathcal{H}_Σ is given by the Serre duality. It is easy to see that

$$\dim \mathcal{H}_\Sigma = \begin{cases} 2g & \text{if } Pol = \emptyset \\ 2g + 2 \sum_{\beta \in Pol} k_\beta - 2 & \text{if } Pol \neq \emptyset \end{cases}.$$

The 1-form ϕ interpreted as a morphism $T_{\mathcal{B}_0} \rightarrow \mathcal{H}$ is given by the linear map

$$(32) \quad T_{[\Sigma]} \mathcal{B}_0 = H^0(\Sigma, \Omega_\Sigma^1(D_\Sigma)) \rightarrow \mathcal{H}_\Sigma$$

which is the boundary map in the standard long exact sequence. Equivalently, it comes from the exact sequence of morphisms of cones

$$Cone(0 \rightarrow \Omega_\Sigma^1(D_\Sigma)) \rightarrow Cone(\mathcal{O}_\Sigma(-D_\Sigma) \rightarrow \Omega_\Sigma^1(D_\Sigma)).$$

It is easy to see that ϕ always gives a Lagrangian embedding.

It is left to construct a linear flat symplectic connection on the bundle \mathcal{H} and prove that ϕ is covariant. In the case $Pol = \emptyset$ there is an obvious candidate, namely, the Gauss-Manin connection. For $Pol \neq \emptyset$ the story is more complicated.

Let us use analytic topology, and consider a disjoint union of open discs

$$\sqcup_{\beta \in Pol} \mathbb{D}_{p_\beta} \subset \Sigma, \quad p_\beta \in \mathbb{D}_{p_\beta} \quad \forall \beta$$

containing points p_β . Let us choose a coordinate $x = x_{(\beta)}$ on each disc \mathbb{D}_{p_β} for each β , such that $x_{(\beta)}$ vanishes at the point p_β . Using these local coordinates we will construct an isomorphism

$$(33) \quad \psi : \mathcal{H}_\Sigma \xrightarrow{\sim} \left(H^1(\Sigma - \{p_\beta\}_{\beta \in Pol}, i_\epsilon(Pol); \mathbf{Z}) \otimes \mathbf{C} \right) \oplus \bigoplus_{\beta \in Pol} \left(\bigoplus_{i=-k_\beta}^{k_\beta-2} \mathbf{C} \cdot x^i dx / \mathbf{C} x^{-1} dx \right),$$

where $i_\epsilon : Pol \hookrightarrow \Sigma - \{p_\beta\}_{\beta \in Pol}$ is the embedding which associates with β the point in \mathbb{D}_{p_β} with $x_{(\beta)}$ -coordinate equal to ϵ , $0 < \epsilon \ll 1$. The definition makes sense as for small ϵ the relative cohomology groups are canonically identified.

In order to describe the morphism ψ it will be convenient to use the following explicit presentation of \mathcal{H}_Σ in terms of global differential forms with arbitrary poles at a finite *non-empty* collection of distinct points $(r_i \in \Sigma)_{i \in I}$, which are also distinct from points $(p_\beta)_{\beta \in Pol}$ ¹²:

$$(34) \quad \mathcal{H}_\Sigma \simeq \{ \eta \in \Gamma(\Sigma, \Omega_\Sigma^1(D_\Sigma + \sum_i \infty r_i) \mid \forall i \text{ Res}_{r_i} \eta = 0) / d\Gamma(\Sigma, \mathcal{O}_\Sigma(-D_\Sigma + \sum_i \infty r_i)) \}.$$

For a class $[\eta] \in \mathcal{H}_\Sigma$ represented by meromorphic 1-form η , we define $\psi([\eta])$ as the sum of two terms

$$\psi([\eta]) = \psi_1(\eta) + \psi_2(\eta)$$

described such as follows.

The term $\psi_2(\eta)$ is defined as the sum over $\beta \in Pol$ of the image of the formal power series decomposition of η in the coordinate x at the point $p_\beta \in \mathbb{D}_{p_\beta}$, under the natural map

$$x^{-k_\beta} \mathbf{C}[[x]] dx \rightarrow \bigoplus_{i=-k_\beta}^{k_\beta-2} \mathbf{C} \cdot x^i dx / \mathbf{C} x^{-1} dx, \quad x = x_{(\beta)}.$$

The term $\psi_1(\eta) \in H^1(\Sigma - \{p_\beta\}_{\beta \in Pol}, i_\epsilon(Pol); \mathbf{Z}) \otimes \mathbf{C}$ is defined via the pairing with relative homology classes such as follows. Let $\gamma = \gamma(\epsilon)$ be a 1-chain on $\Sigma - \{p_\beta\}_{\beta \in Pol}$ with the boundary lying in the finite set $i_\epsilon(Pol)$ and depending continuously on the parameter ϵ (and hence representing the same relative homology class). Let us

¹²The points r_i at this time have nothing to do with ramification points r_α . But our notation is suggestive: we will later choose the set $(r_i \in \Sigma)_{i \in I}$ in such a way that it will coincide with the set $(r_\alpha)_{\alpha \in Ram}$.

assume that this chain avoids the set $\{r_i\}_{i \in I}$. Then the integral $\int_{\gamma(\epsilon)} \eta$ considered as a function of ϵ has an expansion as $\epsilon \rightarrow 0$ of the form:

$$(35) \quad \int_{\gamma(\epsilon)} \eta = \sum_{i=-N}^{\infty} c_i \epsilon^i + c_{log} \cdot \log(\epsilon), \quad N = \max_{\beta} (k_{\beta}) - 1.$$

Then we define the cohomology class $\psi_1(\eta)$ by the formula

$$(36) \quad \psi_1(\eta)([\gamma]) := \text{coefficient } c_0 \text{ in the above expansion.}$$

PROPOSITION 5.2.1. *Cohomology class $\psi_1(\eta)$ does not depend on the choice of representatives $\gamma(\epsilon)$. Furthermore it depends only on the class $[\eta]$ in \mathcal{H}_{Σ} . Also, $\psi_1(\eta)$ depends only on the class $[\eta]$ in \mathcal{H}_{Σ} .*

The proof is a straightforward check, and is left as an exercise to the reader. The above Proposition implies that the map ψ is well-defined.

THEOREM 5.2.2. *Map ψ defined in (33) is an isomorphism.*

PROOF. The dimensions of the RHS and LHS in (33) are equal to each other, hence it is sufficient to check that $\text{Ker } \psi = 0$. Let us assume that $\psi([\eta]) = 0$ for some meromorphic 1-form η representing the cohomology class $[\eta]$ via (34). Then η should have zero residue at each point p_{β} (otherwise it will produce a non-zero class in $H^1(\Sigma - \{p_{\beta}\}_{\beta \in Pol}, i_{\epsilon}(Pol); \mathbf{C})$). Vanishing of other Taylor coefficients (guaranteed by the condition $\psi_2(\eta) = 0$) implies that

$$\eta \in \Gamma(\Sigma, \Omega_{\Sigma}^1(-\sum_{\beta \in Pol} (k_{\beta} - 1)p_{\beta} + \sum_i \infty r_i)), \quad \text{Res}_{r_i}(\eta) = 0 \quad \forall i.$$

We see that η is non-singular at all points $p_{\beta}, \beta \in Pol$. Therefore in the formula (35) there are no terms $c_i, i < 0$ and no term c_{log} . Hence we can define c_0 as the limit as $\epsilon \rightarrow 0$ of $\int_{\gamma(\epsilon)} \eta$. The condition $\psi_1(\eta) = 0$ implies that η represents zero cohomology class in $H^1(\Sigma - \{r_i\}_{i \in I}, \{p_{\beta}\}_{\beta \in Pol}; \mathbf{C})$, hence it is equal to df for some function f with poles at $\{r_i\}_{i \in I}$ and vanishing at $\{p_{\beta}\}_{\beta \in Pol}$. Moreover, vanishing of $\psi_2(\eta)$ implies that for every $\beta \in Pol$ the function f vanishes at p_{β} with multiplicity $\geq k_{\beta}$. Therefore $[\eta] = 0 \in \mathcal{H}_{\Sigma}$. □

Now we are ready to define the desired flat connection ∇ on the vector bundle \mathcal{H} on the moduli space \mathcal{B}_0 of smooth D -transversal spectral curves. For a given curve $[\Sigma] \in \mathcal{B}_0$ and every $\beta \in Pol := Pol(\Sigma)$ let us choose a local coordinate $x_{(\beta)}$ on a small disc parametrizing leaves of the foliation \mathcal{F} which are close to p_{β} . We also assume that $x_{(\beta)}$ vanishes on the leaf passing through p_{β} . This local coordinate defines a local coordinate on the disc $\mathbb{D}'_{p_{\beta}} \subset \Sigma'$ associated to $\beta \in Pol(\Sigma) = Pol(\Sigma')$ for any spectral curve Σ' which is sufficiently close to Σ . We identify $\psi(\mathcal{H}_{\Sigma})$ and $\psi(\mathcal{H}_{\Sigma'})$ using homotopy equivalence or surfaces with marked points (Gauss-Manin connection)

$$H^1(\Sigma - \{p_{\beta}\}_{\beta \in Pol}, i_{\epsilon}(Pol); \mathbf{Z}) \simeq H^1(\Sigma' - \{p'_{\beta}\}_{\beta \in Pol}, i'_{\epsilon}(Pol); \mathbf{Z}),$$

and using the identity isomorphism on spaces $\oplus_{i=-k_{\beta}}^{k_{\beta}-2} \mathbf{C} \cdot x^i dx / \mathbf{C}x^{-1} dx, x = x_{(\beta)}$ for each β .

THEOREM 5.2.3. *The above construction gives a flat connection ∇ on the bundle \mathcal{H} which does not depend on the choices of local coordinates $x_{(\beta)}$, and also on the choice of auxiliary punctures $(r_i)_{i \in I}$. Moreover, the connection depends only on the*

collection of $(2k-1)$ -jets of foliation \mathcal{F} near components of divisor $D = P - S$ where symplectic form ω has pole of order k . Furthermore, the connection ∇ preserves the natural symplectic pairing on \mathcal{H} , and the \mathcal{H} -valued 1-form ϕ defined in (32) is closed.

PROOF. The independence of the choice of the auxiliary finite nonempty set $\{r_i\}_{i \in I} \subset \Sigma - \{p_\beta\}_{\beta \in Pol}$ is obvious, as one can pass from this subset to arbitrary larger finite subset. The independence of the choice of local coordinates $\{x_{(\beta)}\}_{\beta \in Pol}$ follows from the fact that a change of coordinates $x_{(\beta)} \rightarrow \tilde{x}_{(\beta)}$ modifies the isomorphism $\psi = \psi_1 + \psi_2$ in certain universal way which is independent of a spectral curve Σ' close to Σ .

The covariance of the symplectic pairing follows from the description of connection in terms of Hamiltonian reduction in Section 7. Notice that considerations of Section 7 are given for $\Sigma \in \mathcal{B}_1$. On the other hand, as one can show that \mathcal{B}_1 is dense in \mathcal{B}_0 , it follows that the symplectic pairing is covariant by the argument of continuity. Alternatively, one can describe the symplectic structure on \mathcal{H}_Σ in terms of the RHS of the formula (33) and deduce the covariance from that description.

The fact that the 1-form ϕ is closed, follows from Remark 5.1.2 as well as the alternative description of the local embedding of the germ of \mathcal{B}_0 at $[\Sigma]$ to the affine space \mathcal{H}_Σ given in the next subsection. The symplectic pairing is explained in Section 7, see Remark 7.1.5. Technically considerations there deal with the open dense subspace $\mathcal{B}_1 \subset \mathcal{B}_0$. Continuity arguments show that the result holds for \mathcal{B}_0 as well. □

REMARK 5.2.4. In Section 8 we will use a slightly different approach which gives an alternative (and maybe more natural) proof of Theorem 5.2.3.

5.3. Connection and 1-form via regularized integrals. In this subsection we will use Remark 5.1.2 in order to prove that $\nabla(\phi) = 0$.

For that let us take three points $x_0 = \Sigma, x_1 = \Sigma_0, x_2 = \Sigma_1 \in \mathcal{B}_0$ which are sufficiently close to each other and consider an analytic path $\Sigma(\lambda), \lambda \in [0, 1]$ such that $\Sigma(0) = \Sigma_0, \Sigma(1) = \Sigma_1$.

Following Remark 5.1.2 we would like to define the corresponding vector $v(\Sigma_0, \Sigma_1)_\Sigma \in \mathcal{H}_\Sigma$ which satisfies the cocycle condition. As we see the curve Σ plays an auxiliary role in the considerations, so we can rename Σ_0 to Σ and consider just a pair of sufficiently close spectral curves. The path $\Sigma(\lambda)$ can be thought of as an analytic map $f : T \rightarrow S$ of the total space T of the analytic family of spectral curves $\Sigma(\lambda)$ with the above boundary conditions as $\lambda = 0, 1$. Then $\eta(\lambda) := \frac{f^*(\omega)}{d\lambda}$ is a family of meromorphic 1-forms on $\Sigma(\lambda)$.

Recall that the RHS of the formula (33) consists of two summands. Let us consider the first one, which denote by H_{top} . Relative cohomology classes in that formula are dual to the corresponding relative homology classes.

Let $T' := T - \cup_{\beta \in Pol} \{(\lambda, p_\beta(\lambda))\}, \lambda \in [0, 1]$. We denote by Q_ε the subset of T' given by $(0, \Sigma(0)) \cup (1, \Sigma(1)) \cup (\lambda, i_\varepsilon(\beta)(\lambda))$, where $\lambda \in [0, 1]$. Here we use the self-explained notation like $(\lambda, p), \lambda \in [0, 1], p \in \Sigma(\lambda)$ for points of T , There is a natural isomorphism $g : H_1(\Sigma - \{p_\beta\}_{\beta \in Pol}, i_\varepsilon(Pol); \mathbf{Z}) \simeq H_2(T', Q_\varepsilon; \mathbf{Z})$ which follows from the observation that families of points $p_\beta(\lambda), i_\varepsilon(\beta)(\lambda), \lambda \in [0, 1]$ is a union of $|Pol|$ closed intervals, and for the interval $[0, 1]$ we have the natural isomorphism $H_1([0, 1], \{0\} \cup \{1\}; \mathbf{Z}) \simeq \mathbf{Z}$.

Let $[\delta(\varepsilon)] = g([\gamma(\varepsilon)])$ be the class of a relative 2-chain corresponding to the class of the 1-chain $\gamma(\varepsilon)$. Then $\int_{f^*([\delta(\varepsilon)])} f^*(\omega)$ has the asymptotic expansion (as $\varepsilon \rightarrow 0$) given by the RHS of the formula (35). Take the constant terms of this expansion (i.e. the coefficient c_0). Then we obtain the functional $c_0 = c_0(\Sigma_0, \Sigma_1)$ on H_{top} .

In order to take care about the second summand in the RHS of the isomorphism (33) let us use as before the coordinate $x(\lambda) = x_{(\beta)}(\lambda)$ on the disc $\mathbb{D}_{p_\beta}(\lambda)$ containing $p_\beta(\lambda)$, and identify all coordinates $x(\lambda)$ for $\lambda \in [0, 1]$. Let us denote this coordinate by x . Then we take the image of the Laurent expansion in x of $\int_{[0,1]} \eta(\lambda)d\lambda$ in $\bigoplus_{i=-k_\beta}^{k_\beta-2} \mathbf{C} \cdot x^i dx / \mathbf{C}x^{-1}dx$ and after that take the direct sum over all $\beta \in Pol$.

Let us denote the sum of the above two summands by $v(\Sigma_0, \Sigma_1)_\Sigma$. It is clear from the construction that for a triple of spectral curves $\Sigma_0, \Sigma_1, \Sigma_2$ we have $v(\Sigma_0, \Sigma_1)_\Sigma + v(\Sigma_1, \Sigma_2)_\Sigma = v(\Sigma_0, \Sigma_2)_\Sigma$. Indeed having 2-chains adjacent one to another one we naturally compose them into a single 2-chain, which gives the additivity of the corresponding integrals over the chains. Similarly the Laurent expansions are added because the corresponding integrals of 1-forms $\eta(\lambda)$ are added.

PROPOSITION 5.3.1. *The map which assigns $\eta(0)$ to the tangent vector at $\lambda = 0$ to the path $\Sigma(\lambda)$ coincides with ψ (after the above identification with the RHS of the formula (33)).*

PROOF. The map ψ assigns to a tangent vector from $T_{[\Sigma]} \mathcal{B}_0$ the corresponding 1-form on Σ . By construction $\eta(0)$ can be identified with the same. □

This concludes the alternative description of the connection and the 1-form as well as the proof of the Theorem 5.2.3.

REMARK 5.3.2. In the simplest case $Pol = \emptyset$ considerations of this subsection have simple (and well-known) meaning. Indeed, the family of the first homology groups $H_1(\Sigma', \mathbf{Z})$ form a local system over a neighborhood of $[\Sigma] \in \mathcal{B}_0$. The map $\kappa : l \mapsto \int_l \omega, l \in H_1(\Sigma', \mathbf{Z}) \simeq H_1(\Sigma, \mathbf{Z})$ defines a local embedding of \mathcal{B}_0 into the symplectic vector space $H^1(\Sigma, \mathbf{C})$. The integer homology groups of the spectral curve carry a linear flat connection along \mathcal{B}_0 , since they do not change when we deform Σ . The closed 1-form κ defines an embedding of the germ \mathcal{B}_Σ into $\mathcal{H}_\Sigma \simeq H^1(\Sigma, \mathbf{C})$.

6. Formal discs and universal Airy structure

6.1. Space of formal discs. Let (S, ω) be a complex analytic symplectic surface endowed with an analytic 1-dimensional foliation \mathcal{F} (i.e. we consider only the symplectic leaf in a good foliated Poisson surface, and endow the leaf with the induced foliation).

For every $n \geq 2$ denote by $Discs^{(n)}$ the space of n -jets of embedded formal discs in S which are tangent to a leaf of foliation \mathcal{F} with the tangency order equal to one. Technically speaking, a point of $Discs^{(n)}$ is given by a point $s \in S$, an epimorphism from the local ring \mathcal{O}_s (germs of analytic functions at s) to a local Artin algebra R isomorphic to the algebra of truncated polynomials $\mathbf{C}[z]/(z^{n+1})$, such that the image of the local coordinate on the space of leaves belongs to $\mathfrak{m}_R^2 - \mathfrak{m}_R^3$, where $\mathfrak{m}_R \subset R$ is the maximal ideal.

The space $Discs^{(n)}$ is a smooth variety fibered over S with fibers isomorphic to $\mathbf{C}^* \times \mathbf{C}^{n-2}$. Also, we have an obvious projective system of fibrations

$$Discs^{(2)} \leftarrow Discs^{(3)} \leftarrow \dots$$

We define $Discs$ to be the projective limit, understood e.g. as a functor on finite-dimensional analytic spaces. In the algebro-geometric framework we assume that S is an algebraic surface and line subbundle $T_{\mathcal{F}} \subset T_S$ is algebraic as well. Then $Discs$ is a scheme of infinite type.

Our goal in this section will be to construct germs of Lagrangian embeddings of $Discs$ into infinite-dimensional affine symplectic spaces, in terms of the general formalism from Section 5.1.

For a point $t \in Discs$ we denote by $\widehat{\mathbb{D}}_t$ the corresponding formal disc embedded in S . Algebra of function $\mathcal{O}(\widehat{\mathbb{D}}_t)$ is isomorphic *non-canonically* to $\mathbf{C}[[z]]$. Denote by $UnivDiscs$ the total space of the fibration over $Discs$ whose fiber at t is formal scheme \mathbb{D}_t . We have following universal maps:

- (i) fibration $UnivDiscs \rightarrow Discs$;
- (ii) section $Discs \rightarrow UnivDiscs$ of the above fibration, associating to the disc its unique tangency point to \mathcal{F} ;
- (iii) $UnivDiscs \rightarrow S$, the universal embedding $\widehat{\mathbb{D}}_t \hookrightarrow S$ for $t \in Discs$.

As in the case of D -transversal spectral curves, we have a natural isomorphism

$$(37) \quad T_t Discs = \Gamma(\widehat{\mathbb{D}}_t, (T_S)_{|\widehat{\mathbb{D}}_t} / T_{\widehat{\mathbb{D}}_t}) \simeq \Omega^1(\widehat{\mathbb{D}}_t).$$

Denote by $UnivDiscs^*$ the fibration over $Discs$ with fiber $\widehat{\mathbb{D}}_t^*$ at $t \in Discs$ given by the *punctured* formal disc, i.e. $\widehat{\mathbb{D}}_t$ with the removed tangency point. Algebra of functions on $\widehat{\mathbb{D}}_t^*$ is isomorphic to $\mathbf{C}((z))$.

We will define a canonical non-linear flat connection on the fibration $UnivDiscs^* \rightarrow Discs$, using the foliation \mathcal{F} (and *not* using the symplectic form ω). In general, in the case of *non-punctured* discs, if one wants to make a non-linear connection on $UnivDiscs \rightarrow Discs$ along a tangent vector $v \in T_t Discs$, one has to make a lift

$$v \in \Gamma(\widehat{\mathbb{D}}_t, (T_S)_{|\widehat{\mathbb{D}}_t} / T_{\widehat{\mathbb{D}}_t}) \rightsquigarrow \tilde{v} \in \Gamma(\widehat{\mathbb{D}}_t, (T_S)_{|\widehat{\mathbb{D}}_t}).$$

In the case of punctured discs, one should find a lift in $\Gamma(\widehat{\mathbb{D}}_t^*, (T_S)_{|\widehat{\mathbb{D}}_t^*})$. We define the **canonical** lift to be a unique lift which is the section of the subbundle $T_{\mathcal{F}} \subset T_S$. Such a lift exists and unique because \mathcal{F} is transversal to $\widehat{\mathbb{D}}_t^*$. As a result we obtain a universal non-linear connection ∇_{univ} on the fibration $UnivDiscs^* \rightarrow Discs$.

EXAMPLE 6.1.1. Let us consider in local coordinates (x, y) on S such that $T_{\mathcal{F}} = \mathbf{C}\partial_y$ (leaves of \mathcal{F} are given by $x = const$). Consider 1-parameter family of embeddings $(f_\lambda)_{\lambda \in \mathbf{C}}$ of the standard coordinate formal disc into S with the first order tangency to \mathcal{F} :

$$\text{parameter } \lambda \in \mathbf{C} \mapsto (f_\lambda : Spf(\mathbf{C}[[z]]) \rightarrow Spec(\mathbf{C}[x, y]), z \mapsto (x = z^2 + \lambda, y = z)).$$

In this way we obtain an analytic path $\mathbf{C} \rightarrow Discs$, $\lambda \mapsto t(\lambda)$ is in $Discs$. The tangent vector ∂_λ to the path can be lifted naively using identification of standard coordinates z on discs $\widehat{\mathbb{D}}_{t(\lambda)}$. The corresponding section of $(T_S)_{|\widehat{\mathbb{D}}_{t(\lambda)}}$ is given by ∂_x . The *canonical* lift of ∂_λ is different, it is equal to $-\frac{1}{2z}\partial_y$. The difference between the

canonical and the naive lift is a meromorphic section of $(T_S)_{\widehat{\mathbb{D}}_{t(\lambda)}}$ which is tangent to $\widehat{\mathbb{D}}_{t(\lambda)}$, and corresponds to the vector field $-\frac{1}{2z}\partial_z$ on $\widehat{\mathbb{D}}_{t(\lambda)}^*$:

$$T_{f_\lambda} : -\frac{1}{2z}\partial_z \mapsto -\frac{1}{2z}\partial_y - \partial_x.$$

PROPOSITION 6.1.2. *The non-linear connection ∇_{univ} defined above is flat.*

This statement can be proven by direct calculation. If one replaces formal punctured discs by actual embedded annuli transversal to the foliation, an analogous statement will be almost a tautology.

Let us return to the formal case. Lie algebra of vector fields on a punctured formal disc $\widehat{\mathbb{D}}_t$ naturally acts on the symplectic vector space

$$\mathcal{H}_t := \{\gamma \in \Omega^1(\widehat{\mathbb{D}}_t^*) \mid \text{Res}(\gamma) = 0\},$$

where the symplectic form is given by (17) (and is coordinate-independent). Therefore, the non-linear connection ∇_{univ} gives rise to a flat linear symplectic connection on the infinite-dimensional¹³ bundle \mathcal{H} with fibers defined above.

Define a \mathcal{H} -valued 1-form ϕ on *Discs* as the natural embedding (here we use only the symplectic form ω , and not the foliation):

$$T_t \text{Discs} = \Omega^1(\widehat{\mathbb{D}}_t) \hookrightarrow \text{Ker} \left(\text{Res} : \Omega^1(\widehat{\mathbb{D}}_t^*) \rightarrow \mathbf{C} \right) = \mathcal{H}_t.$$

THEOREM 6.1.3. *Form ϕ defined above is closed under the covariant derivative, and defines a Lagrangian embedding of the tangent bundle T_{Discs} into the symplectic bundle \mathcal{H} .*

PROOF. The fact that ϕ gives a Lagrangian embedding is obvious, as in local coordinates $\mathbf{C}[[z]]dz$ is a Lagrangian subspace in $\text{Ker}(\text{Res} : \mathbf{C}((z))dz \rightarrow \mathbf{C})$. The fact that ϕ is closed can be checked by direct calculation.

Alternatively, it can be explained by direct construction of a germ of Lagrangian embedding in the non-formal case and the Remark 5.1.2. Namely, consider a germ of a finite-dimensional family of embedded analytic discs $\mathbb{D}_{t(\lambda)}$ where $\lambda \in \Lambda$ (parameter space) is close to λ_0 . then one can pick an open domain $\mathcal{U} \subset S$ close to the tangency point of $\mathbb{D}_{t(\lambda_0)}$ such that $\mathcal{U} \cap \mathbb{D}_{t(\lambda_0)}$ is an annulus $A \simeq \{z \in \mathbf{C} \mid \epsilon_1 < |z| < \epsilon_2\}$ for $0 < \epsilon_1 < \epsilon_2 \ll 1$, and \mathcal{U} is fibered over A with fibers being leaves of \mathcal{F} . Symplectic structure ω gives an affine structure on fibers $\mathcal{U} \rightarrow A$, and moreover makes fibers to be parallel to the cotangent bundle T^*A ¹⁴. Then for any λ, λ' close to λ_0 we have two sections $\mathcal{U} \cap \mathbb{D}_{t(\lambda)}$ and $\mathcal{U} \cap \mathbb{D}_{t(\lambda')}$ of the fibration $\mathcal{U} \rightarrow A$. Their difference is a 1-form $\eta_{\lambda_1, \lambda_2}$ on A (here we use affine structure along leaves of \mathcal{F}).

LEMMA 6.1.4. *The contour integral of this 1-form over a non-trivial loop in A is equal to zero.*

Proof. Indeed, one can choose locally near the tangency point of $\mathbb{D}_{t(\lambda_0)}$ a local system of coordinates (x, y) (canonical coordinates) such that $\omega = dx \wedge dy$, and the foliation is given by $x = \text{const}$. The contour integral of ydx over a loop in $\mathbb{D}_{t(\lambda)}$ vanishes for any λ close to λ_0 , because this loop bounds a disc. Therefore, the countour integral over a loop in A of the pullback ξ_λ of ydx via the map

¹³Technically, one should consider \mathcal{H} as an inductive limit of projective limits of finite-dimensional vector bundles.

¹⁴This is a general feature of Lagrangian foliations in arbitrary dimensions.

$A \rightarrow \mathcal{U} \cap \mathbb{D}_{t(\lambda)}$ (inverse to the projection) also vanishes. Finally, the 1-form $\eta_{\lambda_1, \lambda_2}$ is equal to the difference $\xi_{\lambda_1} - \xi_{\lambda_2}$. Therefore its contour integral vanishes as well. The Lemma is proved. \square

We see that in the analytic situation we obtain a flat family of local embeddings of the space of discs into infinite-dimensional affine space. The cocycle condition 5.1.2 for the closedness of our vector-valued 1-form follows from the trivial identity

$$\eta_{\lambda_1, \lambda_2} + \eta_{\lambda_2, \lambda_3} = \eta_{\lambda_1, \lambda_3}.$$

This explains that $\nabla\phi = 0$. \square

6.2. Relation to the universal classical Airy structure. For a given symplectic surface (S, ω) endowed with (automatically Lagrangian) one-dimensional foliation \mathcal{F} , define an infinite-dimensional space *Coord* to be the space of tuples (s, x, y) where $s \in S$ is a point, (x, y) is a formal coordinate system at s such that

- (i) symplectic form ω is equal to $dx \wedge dy$;
- (ii) tangent bundle $T_{\mathcal{F}}$ to the foliation \mathcal{F} is spanned by ∂_y .

This space is projective limit of a tower of fibrations of finite-dimensional smooth varieties (like the space *Discs* in the previous subsection).

The tangent space to *Coord* at any point is canonically isomorphic to the Lie algebra \mathfrak{g} where:

$$\mathfrak{g} := \{ \xi = a(x, y) \partial_x + b(x, y) \partial_y \mid a, b \in \mathbf{C}[[x, y]], \xi \text{ preserves } \omega \text{ and } T_{\mathcal{F}} \}.$$

As in the formal differential geometry of Gelfand and Kazhdan (see [GelKaz]), we have a *free transitive action* of \mathfrak{g} on *Coord*. An easy calculation shows that \mathfrak{g} is canonically isomorphic to the quotient of the Lie algebra of differential operators of order ≤ 1 on $\mathbf{C}[[x]]$ by the one-dimensional center consisting of constants, i.e. $\mathfrak{g} \simeq \mathbf{C}[[x]]\partial_x \times (\mathbf{C}[[x]]/\mathbf{C})$, where ¹⁵

$$f(x) \partial_x + g(x)/const \mapsto f(x) \partial_x - yf'(x) \partial_y - g'(x) \partial_y = \{ f(x)y + g(x), \cdot \}.$$

Consider the map $\pi : \textit{Coord} \rightarrow \textit{Discs}$ given by

$$(s, x, y) \mapsto \text{formal disc with coordinate } z \text{ embedded via } (z \mapsto (x = z^2, y = z)).$$

Map π is a 3 : 1 cover. Indeed $(s, x' = \xi^2 x, y' = \xi)$ gives the same embedded formal disc as (s, x, y) for any $\xi \in \mathbf{C}, \xi^3 = 1$.

For any point $(s, x, y) \in \textit{Coord}$ and the corresponding point $t = \pi(s, x, y) \in \textit{Discs}$, the local transitive free action of Lie algebra \mathfrak{g} on *Coord* gives an isomorphism

$$\mathfrak{g} \simeq T_{(s, x, y)}\textit{Coord} \simeq T_t\textit{Discs} \simeq \Omega^1(\widehat{\mathbb{D}}_t) \simeq \mathbf{C}[[z]]dz.$$

PROPOSITION 6.2.1. *The above isomorphism does not depend on $(s, x, y) \in \textit{Coord}$, and is given by the formula*

$$f(x) \partial_x + g(x)/const \mapsto d(f(z^2)z + g(z^2)) \in \mathbf{C}[[z]]dz.$$

The proof is given by direct calculation.

The pullback $\pi^*\mathcal{H}$ is a canonically trivialized symplectic vector bundle, each fiber is identified with the space $W := W_{\textit{Airy}} := \text{Ker}(\text{Res} : \mathbf{C}((z))dz \rightarrow \mathbf{C})$ from Section 3.3.

¹⁵Here $g(x)/const$ denotes the image of the element $g = g(x)$ of $\mathbf{C}[[x]]$ in the quotient space $\mathbf{C}[[x]]/\mathbf{C}$.

THEOREM 6.2.2. *Consider the affine flat connection on the trivial bundle with fiber W over $Coord$ given by the homomorphism*

$$\mathfrak{g} \rightarrow spAff(W) = sp(W) \ltimes W$$

which is associated with the standard classical Airy structure from Section 3.3. Then, after the above identifications, it coincides with the affine flat connection constructed in the Section 6.1.

PROOF. We have to check first that the linear connections match. Let us calculate the linear part of the flat affine connection from Section 6.1 following the definitions. We obtain it as the composition of two homomorphisms

$$\mathfrak{g} \rightarrow \mathbf{C}((z)) \partial_z \rightarrow sp(W)$$

where the second maps is the natural action by coordinate changes. The first map comes from the geometric description of the non-linear flat connection ∇_{univ} via foliation. Namely, with any element $f(x)\partial_x + g(x)/const$ we associate the corresponding Hamiltonian vector field v in $Spf(\mathbf{C}[[x, y]])$. Let us consider restriction of v to the parabola $x = y^2$, and find another meromorphic vector field v' on parabola which is vertical and such that $u = v - v'$ is tangent to the parabola. Then u expanded in coordinate z gives an element in $\mathbf{C}((z)) \partial_z$ corresponding to $v \in \mathfrak{g}$. The direct calculation gives

$$v_{\widehat{\mathbb{D}}_t} = f(z^2) \partial_x - z f'(z^2) \partial_y - g'(z^2) \partial_y .$$

The map from vector fields tangent to parabola in coordinate z , to section of the pullback T_S to the parabola is given by

$$h(z) \partial_z \mapsto 2zh(z) \partial_x + h(z) \partial_y .$$

Therefore, the vector field tangent to parabola which differ from v by a purely vertical field is

$$u = \frac{f(z^2)}{2z} \partial_z .$$

This gives a homomorphism $\mathfrak{g} \rightarrow sp(W)$. Direct calculation shows that it coincides with the linear part of the homomorphism from Section 3.3. Similarly, the shift part (i.e. 1-form ϕ with values in the trivial bundle with the fiber W) matches, as we identify the tangent space to $Coord$ with the space of 1-forms $\mathbf{C}[[z]] dz$ as in Proposition 6.2.1. □

6.3. Geometric meaning of residue constraints. Let $t \in Discs$ and $\widehat{\mathbb{D}}_t$ the corresponding formal disc. We endow it with canonical coordinates (x, y) , so that the foliation \mathcal{F} is given by the equation $x = const$ and the symplectic form $\omega = dx \wedge dy$. Disc $\widehat{\mathbb{D}}_t$ is endowed with coordinate z such that the embedding into S is $z \mapsto (x = z^2, y = z)$.

Tate vector space $W = W_{Airy} = Ker(Res_z : \mathbf{C}((z))dz \rightarrow \mathbf{C})$ is a hyperplane in $W' := \mathbf{C}((z))dz$. The formal germ \widehat{W}' of W' at 0 controls deformations of the punctured formal disc $\widehat{\mathbb{D}}_t^*$. Its formal subgerm \widehat{W} is determined by the condition that the residue of the pull-back of the 1-form ydx to the deformed punctured formal disc is trivial. The formal germ \widehat{W} contains a formal germ of $t \in Discs$ corresponding to deformations of the non-punctured formal disc \mathbb{D}_t .¹⁶ The latter

¹⁶We understand all submanifolds as functors on commutative local Artin algebras, see Remark 3.3.1.

germ is *contained* in the formal germ of the Lagrangian submanifold L which by definition is singled out in \widehat{W} by the system of equations

$$Res_z(x^k y dx) = 0, k \geq 1, Res_z(x^k y^2 dx) = 0, k \geq 0.$$

These are exactly the residue constraints from Section 3.3. They hold on the formal completion of $Discs$ since the residue of a holomorphic form is trivial.

Notice that the tangent space at zero $T_0L \simeq \mathbf{C}[[z]]dz$ and hence it coincides with the tangent space to $Discs$. Hence L coincides the formal neighborhood of $t \in Discs$ embedded in \widehat{W} . This explains the residue constraints from Section 3.3.

We will also explain below the invariant meaning the residue constraints.

Let U_1 be the vector space of 1-forms on the formal completion of S at $(x, y) = (0, 0) \in \mathbb{D}_t$ which are equal to zero on the leaves of the foliation \mathcal{F} with the order ≤ 2 (the latter notion makes sense even globally because of the intrinsically defined affine structure on the leaves).

In the canonical coordinates (x, y) an element of U_1 can be written as $F(x, y)dx$, where $F(x, y)$ is a polynomial of degree ≤ 2 in y for a fixed x . Let $U_2 \subset U_1$ be a vector subspace consisting of such 1-forms μ that $d\mu = c \cdot dx \wedge dy, c \in \mathbf{C}$.

Spaces U_1, U_2 have topological bases

$$\{x^{\geq 0} dx, x^{\geq 0} y dx, x^{\geq 0} y^2 dx\}, \quad \{x^{\geq 0} dx, y dx\}$$

respectively. Therefore, the quotient U_1/U_2 has the topological basis

$$\{x^{\geq 1} y dx, x^{\geq 0} y^2 dx\}.$$

The pull-back of $\mu \in U_2$ to a deformed \mathbb{D}_t^* has trivial residue (we deform the disc inside of $\widehat{W} \subset \widehat{W}'$). Hence U_1/U_2 is naturally mapped to the space of formal functions on the formal neighborhood of $0 \in \widehat{W}$. Vanishing of these functions is the same as the residue constraints from Section 3.3.

7. Hamiltonian reduction and Holomorphic Anomaly Equation

7.1. Global Hamiltonian reduction. In this subsection we explain how to treat the germ \mathcal{B}_Σ as a Hamiltonian reduction of a germ of an infinite-dimensional Lagrangian submanifold of a Tate affine symplectic space (see Remark 3.3.1) that we understand Tate spaces as functors on Artin algebras.

We start with a finite-dimensional model. Let M be a finite-dimensional symplectic manifold, $G \subset M$ a coisotropic submanifold and $L \subset M$ a Lagrangian submanifold. Let $x \in L \cap G$ be a point satisfying the following equivalent properties:

- 1) $T_x L + T_x G = T_x M,$
- 2) $T_x L \cap (T_x G)^\perp = 0,$

where \perp means the skew-orthogonal with respect to the symplectic form.

Notice that the family of vector spaces $(T_x G)^\perp$ defines an integrable distribution of planes. Hence we have the corresponding foliation of G with (locally) symplectic quotient. The following proposition is well-known and is easy to prove.

PROPOSITION 7.1.1. *The image \mathcal{L}_x of the germ of $L_x \cap G_x$ at x is a germ of Lagrangian submanifold in the (germ of the) above symplectic quotient.*

This Proposition can be generalized to the case of Tate spaces and formal germs.

Let now Σ be a (\mathcal{F}, D) -transversal curve Σ in a good foliated Poisson surface. Let us choose local coordinates (x, y) on S and z on Σ near each ramification point so we get locally a standard parabola, as in the previous section. The germ \mathcal{B}_Σ of the moduli space \mathcal{B}_1 at $[\Sigma]$ ¹⁷ maps to the product $Discs^{Ram}$ of copies of the space $Discs$ labeled by ramification points. We denote by $\widehat{\mathcal{B}}_\Sigma$ the formal germ of \mathcal{B}_Σ at the point $[\Sigma] \in \mathcal{B}_1$. Thus, we obtain a local immersion of the finite-dimensional formal germ $\widehat{\mathcal{B}}_\Sigma$ into an infinite-dimensional affine vector space $W := W_{Airy}^{Ram}$. On the other side, we have a natural germ L of infinite-dimensional Lagrangian subvariety in W_{Airy}^{Ram} , given by the formal neighborhood of a point in $Discs^{Ram}$. Clearly $L = L_{Airy}^{Ram}$ is the product of copies of formal germs of infinite-dimensional Lagrangian submanifolds L_{Airy} from Section 3.3.

In order to simplify the notation we will denote by $\Delta = \sum_{\alpha \in Ram} r_\alpha$ the ramification divisor, and as before we will denote by $D_\Sigma = \sum_{\beta \in Pol} k_\beta p_\beta$ the pole divisor.

PROPOSITION 7.1.2. *The finite-dimensional germ $\widehat{\mathcal{B}}_\Sigma$ is equal to the intersection of the infinite-dimensional germ of $Discs^{Ram}$ with the coisotropic affine subspace $G_\Sigma \subset W_{Airy}^{Ram}$ defined as the space of 1-forms*

$$(38) \quad G_\Sigma := \{ \eta \in \Gamma(\Sigma, \Omega_\Sigma^1(D_\Sigma + \sum_{\alpha \in Ram} \infty r_\alpha)) \mid Res_{r_\alpha}(\eta) = 0 \}$$

expanded in canonical local coordinates z near each ramification point r_α (compare with (34)). Moreover, the subbundle with the fiber G_Σ of the trivial bundle with the fiber W_{Airy}^{Ram} , is covariantly constant as an affine subbundle.

PROOF. We will explain below both statements in the complex analytic setting. There is a similar but less transparent proof in the framework of Tate spaces.

Notice that by our assumptions, the foliation \mathcal{F} is transversal to Σ everywhere except of ramification points. Therefore, there exists an open domain $\mathcal{U} \subset P$ such that

$$\mathcal{U} \cap \Sigma = \Sigma^0 := \Sigma - \sqcup_{\alpha \in Ram} \mathbb{D}_{\alpha, \epsilon_1}$$

where $\mathbb{D}_{\alpha, \epsilon_1}$ is a closed disc of radius ϵ_1 in the standard coordinate $z = z^{(\alpha)}$ near r_α , $\forall \alpha \in Ram$, and $\epsilon_1, 0 < \epsilon_1 \ll 1$ is sufficiently small. Moreover, we may assume that the projection to the space of leaves of \mathcal{F} gives a fibration of \mathcal{U} over Σ^0 . Any spectral curve Σ' which is sufficiently close to Σ gives a section $\Sigma' \cap \mathcal{U}$ of the fibration $\mathcal{U} \rightarrow \Sigma^0$, which differs from the section given by Σ by a 1-form on Σ^0 with poles of order $\leq k_\beta$ at $p_\beta, \beta \in Pol$. This follows from the fact that fibers of the fibration $\mathcal{U} \rightarrow \Sigma^0$ are naturally affine lines parallel to the cotangent bundle to Σ^0 , twisted by divisor D_Σ (here we use the symplectic structure to ensure that fibers of a Lagrangian foliation carry affine structure). This 1-form has contour integrals over all annuli $\epsilon_1 < |z^{(\alpha)}| < \epsilon_2 \ll 1$ equal to zero (because these annuli are contained in discs). This gives an element in the (analytic version of) G_Σ .

We see that spectral curves Σ' can be described as deformations of annuli $\epsilon_1 < |z^{(\alpha)}| < \epsilon_2$ along the foliation \mathcal{F} satisfying the vanishing of residue condition as well as two other constraints:

- each annulus is contained in a holomorphic disc (geometrically it means that we land into the product of infinite-dimensional Lagrangian subvarieties given by quadratic equations of the standard classical Airy structure)

¹⁷We will use the notation \mathcal{B}_Σ and $\mathcal{B}_{[\Sigma]}$ for this germ on the equal footing.

- the collection of 1-forms on annuli extends to a global 1-form from the analytic version of G_Σ , equivalently the deformed annuli lie in a shifted section of the fibration $\mathcal{U} \rightarrow \Sigma^0$.

This description identifies the formal germ $\widehat{\mathcal{B}}_\Sigma$ with the intersection of a Lagrangian subvariety coming from the classical Airy structure and an affine coisotropic subvariety. □

Next, observe that the 2-term complex of sheaves $\mathcal{O}_\Sigma(-D_\Sigma) \rightarrow \Omega_\Sigma^1(D_\Sigma)$ endowed with de Rham differential is quasi-isomorphic to the following complex of sheaves:

$$\mathcal{O}_\Sigma(-D_\Sigma + *\Delta) \rightarrow \Omega_\Sigma^1(D_\Sigma + *\Delta) \rightarrow \bigoplus_{r_\alpha \in Ram} \mathbf{C} \rightarrow 0,$$

where $\Delta = \sum_{\alpha \in Ram} r_\alpha$, first two terms are sheaves of functions/ forms having poles/zeros at the points p_β bounded by k_β as well as and poles of arbitrary order at the points r_α (the map is the de Rham differential), and second (surjective) arrow is the map which assigns to a differential form the sum of residues at all points r_α . Each sheaf in the above complex has trivial cohomology, except of (possibly) in degree zero. It follows that $\mathcal{H}_\Sigma \simeq G_\Sigma/d(\mathcal{O}_\Sigma(-D_\Sigma + *\Delta))$. Thus we have the natural map $G_\Sigma \rightarrow \mathcal{H}_\Sigma$.

PROPOSITION 7.1.3. *We have the following isomorphism of Tate spaces:*

$$d(H^0(\mathcal{O}_\Sigma(-D_\Sigma + *\Delta))) = G_\Sigma^\perp,$$

and hence $G_\Sigma/G_\Sigma^\perp \simeq \mathcal{H}_\Sigma$.

PROOF. It suffices to check that $d(H^0(\mathcal{O}_\Sigma(-D_\Sigma + *\Delta)))^\perp = G_\Sigma$.

By definition the LHS consists of collections of forms $(\mu_\alpha)_{\alpha \in Ram}$ such that $\mu_\alpha \in \mathbf{C}((z_{(\alpha)}))dz_{(\alpha)}$ (here $z_{(\alpha)}$ is the canonical local parameter at r_α), satisfying the conditions

- $Res(\mu_\alpha) = 0 \forall \alpha$,
- $\sum_\alpha Res(f\mu_\alpha) = 0$ for any $f \in H^0(\mathcal{O}_\Sigma(-D_\Sigma + *\Delta))$.

Let us introduce the Tate vector space G'_Σ as the space of collections of 1-forms $\mu_\alpha, \alpha \in Ram$ as above, but without the condition $Res(\mu_\alpha) = 0 \forall \alpha$.

Before proceeding further, let us recall the following well-known result about Tate spaces associated with vector bundles on curves (Serre duality for curves formulated in terms of local fields, see [Tat68], Corollary to Theorem 5).

Let Σ be a smooth projective curve over a field \mathbf{k} , and \mathcal{E} be a vector bundle over Σ . Let $\{r_i\}_{i \in I}$ be a finite subset of $\Sigma(\mathbf{k})$. Let us denote by \mathcal{E}' the vector bundle $\mathcal{E}^\vee \otimes \Omega_\Sigma^1$. For each r_i denote by K_{r_i} the completed local field of r_i . Each K_{r_i} is isomorphic to the field $\mathbf{k}((z_{(i)}))$, where $z_{(i)}$ is a local coordinate at r_i . For each $i \in I$ we have the finite-dimensional K_{r_i} -vector spaces $\widehat{\mathcal{E}}_{r_i}$ and $\widehat{\mathcal{E}}'_{r_i}$ of sections over the formal punctured discs at r_i . They are Tate \mathbf{k} -vector spaces endowed with the topological perfect duality $(v, w) \mapsto Res_{r_i}(v, w)$. Direct sum $\bigoplus_{i \in I} \widehat{\mathcal{E}}_{r_i}$ contains the subspace $\Gamma(\Sigma - \{r_i\}, \mathcal{E})$, and similarly $\bigoplus_{i \in I} \widehat{\mathcal{E}}'_{r_i}$ contains the subspace $\Gamma(\Sigma - \{r_i\}, \mathcal{E}')$. The direct sums themselves are naturally dual to each other.

LEMMA 7.1.4. *The duality between above Tate spaces makes the subspaces orthogonal to each other.*

We will apply the Lemma to $\mathcal{E} = \mathcal{O}_\Sigma(-D_\Sigma), \mathcal{E}' = \Omega_\Sigma^1(D_\Sigma), \{r_i\}_{i \in I} = \{r_\alpha\}_{\alpha \in Ram}$.

Then the lemma implies that

$$G'_\Sigma \simeq \{ \eta \in \Gamma(\Sigma, \Omega^1_\Sigma(D_\Sigma + \sum_{\alpha \in Ram} \infty r_\alpha)) \}.$$

Imposing the condition $Res_{r_\alpha}(\eta) = 0$ on elements of G'_Σ we obtain G_Σ . This proves the Proposition. □

REMARK 7.1.5. The Proposition gives also an alternative explanation to the fact that the hypercohomology is a symplectic vector space.

The Proposition 7.1.3 gives rise to a formal symplectic vector space and formal connection only, since in the framework of Tate spaces we understand varieties and bundles only as functors on Artin algebras.

Recall that we have an embedding of the germ \mathcal{B}_1 (in fact even a bigger space \mathcal{B}_0) into \mathcal{H}_Σ derived from the Theorem 5.2.2. Let us compare these two embeddings.

PROPOSITION 7.1.6. *The formal Lagrangian submanifold of \mathcal{H}_Σ defined in this subsection is a formal completion at zero of the germ of the Lagrangian submanifold defined in Section 5.2.*

PROOF. Recall the RHS of the formula (33) which consists of two summands. Recall also the global description of \mathcal{H}_Σ given in the formula (34). Consider a complex manifold with boundary obtained from Σ by removing small open discs $\mathbb{D}_{r_\alpha}, \alpha \in Ram$ which contain ramification points r_α . We have the corresponding vector space of holomorphic 1-forms, which are analytic on the boundary, and which have trivial integrals over each boundary component. We can use this space of 1-forms in order to obtain an presentation for \mathcal{H}_Σ analogous to (34):

$$(39) \quad \mathcal{H}_\Sigma = \{ \eta \in \Gamma(\Sigma - \cup_\alpha \mathbb{D}_{r_\alpha}, \Omega^1_\Sigma(D_\Sigma)) \mid \int_{\partial \mathbb{D}_{r_\alpha}} \eta = 0 \} / d\Gamma(\Sigma - \cup_\alpha \mathbb{D}_{r_\alpha}, \mathcal{O}_\Sigma(-D_\Sigma)).$$

Using the foliation \mathcal{F} we can identify as abstract complex manifolds with boundary

$$\Sigma - \cup_\alpha \mathbb{D}_{r_\alpha} \simeq \Sigma' - \cup_\alpha \mathbb{D}'_{r_\alpha},$$

for any Σ' which is sufficiently close to Σ and the corresponding \mathbb{D}'_{r_α} . Hence the RHS of the formula (39) is invariant with respect to the deformations of Σ . As a result we obtain a linear flat symplectic connection described in terms of spectral curves with small discs removed.

Notice the RHS of the formulas (39) and (34) are the only cohomologies in degree 1 of a 2-term complexes. There is a natural quasi-isomorphism of these complexes induced by the embedding $\Sigma - \cup_\alpha \mathbb{D}_{r_\alpha} \rightarrow \Sigma - \cup_\alpha \{r_\alpha\}$. Moreover the isomorphism ψ from formula (33) can be generalized to the description of \mathcal{H}_Σ given in the formula (39).

This ensures equivalence of both descriptions of \mathcal{H}_Σ , which in turn ensures equivalence of two descriptions of the affine symplectic connection on \mathcal{H}_Σ .

Notice that the Hamiltonian reduction in the framework of Tate spaces should be understood in terms of functors on Artin algebras (see Remark 3.3.1). Then we see that the embedding of the moduli space \mathcal{B}_1 obtained from the Proposition 7.1.3 coincides (as a functor on Artin algebras) with the description given in Section 5. This proves the statement. □

7.2. Quantum Airy structures and quantum Hamiltonian reduction of the wave function. In the case $Pol = \emptyset$ let us choose a Lagrangian complement $\mathcal{V}_\Sigma \subset H^1(\Sigma, \mathbf{C})$ to the Lagrangian subspace given by the first term of the Hodge filtration $F^1 H^1_{dR}(\Sigma, \mathbf{C}) := H^{1,0}(\Sigma) = T_\Sigma \mathcal{B} \subset H^1(\Sigma, \mathbf{C}) = \mathcal{H}_\Sigma$ (a splitting of the Hodge filtration is equivalent to a choice of Bergman kernel in the traditional language of topological recursion).

Let us consider the infinite-dimensional vector space $V_\Sigma \subset G_\Sigma$ consisting of holomorphic 1-forms $\gamma \in \Omega^1(\Sigma - \cup_{\alpha \in Ram} \{r_\alpha\})$, which are meromorphic on Σ , have trivial residue at all r_α , and such that $[\gamma] \in \mathcal{V}_\Sigma$. It is easy to see that V_Σ is a Lagrangian complement to the tangent space to the Lagrangian submanifold $L = L^{Ram}_{Airy} \subset W = W^{Ram}_{Airy}$.

The story is similar in the case $Pol \neq \emptyset$. In this case we choose a transversal Lagrangian complement \mathcal{V}_Σ in \mathcal{H}_Σ to the Lagrangian subspace $H^0(\Sigma, \Omega^1_\Sigma(D_\Sigma))$. Its preimage $V_\Sigma := (G_\Sigma \rightarrow \mathcal{H}_\Sigma)^{-1} \mathcal{V}_\Sigma$ is a Lagrangian complement to $T_0 L$. Thus, we have in general the following result.

COROLLARY 7.2.1. *The Lagrangian complement V_Σ carries both classical and quantum Airy structures.*

It follows that we have the wave function $\psi = \exp(\sum_{g \geq 0} S_g \hbar^{g-1})$ associated with the cyclic DQ -module corresponding to L and the Lagrangian complement V_Σ . If we don't choose the Lagrangian complement then we still have a cyclic DQ -module with a cyclic vector, i.e. a left ideal in the completed Weyl algebra. The choice of Lagrangian complement allows us to encode the left ideal in terms of the wave function.

REMARK 7.2.2. Notice that in the global version of the story the space $V_{\Sigma, odd}$ is the subspace of the previously defined "local" space V_{odd} which consists of such meromorphic forms γ on $\Sigma - \cup_\alpha \{r_\alpha\}$ that $\gamma + \sigma^*(\gamma)$ has no poles at each r_α . Here σ is the local involution from Section 3.1. The subspace $V_{\Sigma, odd} \subset V_\Sigma$ carries a quantum Airy substructure.

Let us explain how the Hamiltonian reduction gives us the cyclic DQ -module \mathcal{E}_Σ with a cyclic vector quantizing the Lagrangian germ $\mathcal{B}_\Sigma \subset \mathcal{H}_\Sigma$. This is a part of the following more general story.

Suppose we are given an affine symplectic vector space M an affine coisotropic subspace $G \subset M$, a point $x \in G$ and a formal germ of Lagrangian submanifold L_x at x satisfying the conditions 1) or 2) from Section 7.1. Let us work for simplicity over the field of complex numbers \mathbf{C} , although it is not necessary.

Notice that with the point x we can associate the completed Weyl algebra $Weyl_x$ of the tangent space $T_x M$. It is a topologically free algebra over $\mathbf{C}[[\hbar]]$. Suppose we are given a cyclic module \mathcal{E} over $Weyl_x$ with the cyclic vector $e \in \mathcal{E}$. Assume that modulo \hbar we have $\mathcal{E} \simeq \mathcal{O}(L_x)$ and $e = 1 \in \mathcal{O}(L_x)$. By Proposition 7.1.1 we have a symplectic affine space $M' = G/G^\perp$ and the Hamiltonian reduction $L'_x \subset M'$ of the germ L_x . We also have the completed Weyl algebra $Weyl'_x$ associated with M' . Then G is naturally embedded an affine Lagrangian subspace in the symplectic affine space $M' \times \overline{M}$, where \overline{M} denote M with the opposite symplectic form. By general theory we have a **canonical** cyclic DQ modules \mathcal{E}_G^{can} with the canonical cyclic vector e_G^{can} quantizing G .

We treat \mathcal{E}_G^{can} as a $Weyl'_x - Weyl_x$ -bimodule.

DEFINITION 7.2.3. Quantum Hamiltonian reduction of (\mathcal{E}, e) is the $Weyl'_x$ -module $\mathcal{E}' = \mathcal{E}_G^{can} \otimes \mathcal{E}$ endowed with the cyclic vector $e' = e_G^{can} \otimes e$.

Clearly modulo \hbar we obtain the Hamiltonian reduction L'_x of L_x .

The above construction works for our infinite-dimensional affine spaces and germs of formal Lagrangian manifolds. As a result we obtain a cyclic DQ -module \mathcal{E}_Σ associated with the spectral curve Σ . In general a choice of Lagrangian complement to $T_x(L_x)$ allows us to translate the story from the language of cyclic DQ -modules to the language of wave functions. In the case of spectral curves this translation looks as follows.

Recall that we have a “local” symplectic vector space W^{Ram} together with a germ of formal Lagrangian submanifold $L \subset W^{Ram}$, both depending only¹⁸ on the set Ram of ramification points. We have also defined a coisotropic vector space $G_\Sigma \subset W^{Ram}$ depending on Σ .

Then the quantization procedure give us a cyclic DQ -module \mathcal{E}_Σ with the cyclic vector ψ_L . Let us split coordinates in each of the symplectic spaces, so that the dual coordinates in W^{Ram} will be denoted by (q, p) and in \mathcal{H}_Σ by (q', p') in such a way that $\psi_L = \psi_L(q)$ is the wave function corresponding to the quantized Airy structure.

We have the wave function $\psi_{G_\Sigma} = \psi_{G_\Sigma}(q, q') = e^{Q_2(q, q')/\hbar}$ corresponding to the coisotropic subspace G_Σ , where Q_2 is the quadratic form such that $graph(dQ_2) = G_\Sigma$. Then the Hamiltonian reduction of $L \subset W^{Ram}$ to $\hat{B}_\Sigma \subset \mathcal{H}_\Sigma$ at the level of wave functions becomes the integral operator

$$\psi_L \mapsto \psi_{\hat{B}} := \int e^{Q_2(q, q')/\hbar} \psi_L(q) dq$$

understood as a formal integration with respect to the Gaussian measure.

7.3. Holomorphic Anomaly Equation. Let us recall the framework and notation from Section 5.1. Thus we have a tuple $(X, \mathcal{H}, \langle \bullet, \bullet \rangle, \nabla, \phi)$ consisting of an analytic manifold X , analytic vector bundle \mathcal{H} over X , linear flat connection ∇ on \mathcal{H} , \mathcal{H} -valued ∇ -flat 1-form ϕ , and a ∇ -covariant symplectic structure $\langle \bullet, \bullet \rangle$ on \mathcal{H} .

Assume that for any point $x \in X$ and assume that the embedding of the germ of X_x of X at x into \mathcal{H}_x defined in Section 5.1 is Lagrangian (i.e. the image is a germ L_x of Lagrangian submanifold in \mathcal{H}_x). Notice that the affine flat connection $\nabla + \phi$ induces a flat connection $\widehat{\nabla}$ on the vector bundle over X with the fiber $Weyl_x$ (completed at zero Weyl algebra of \mathcal{H}_x). This comes from the fact that the group of affine symplectic transformations acts on the non-completed Weyl algebra.

DEFINITION 7.3.1. A **quantization** of the tuple $(X, \mathcal{H}, \langle \bullet, \bullet \rangle, \nabla, \phi)$ is given by a $\widehat{\nabla}$ -covariantly constant family of closed left ideals $J_x \subset Weyl_x$ such that the cyclic module $Weyl_x/J_x$ is a quantization the Lagrangian germ $L_x := \phi(X_x)$ (i.e. modulo \hbar it gives the algebra of function $\mathcal{O}(L_x)$, and the cyclic vector $1 \in Weyl_x/J_x$ corresponds to $1 \in \mathcal{O}(L_x)$).

Let us choose a ∇ -covariantly constant subspace V_x which is transversal to L_x for all $x \in X$. Then the condition that J_x form a covariantly constant family can be translated into a system of differential equations on the components $(S_{g,n})_x \in$

¹⁸Here we choose a canonical coordinate near each ramification point, determined a priori only up to multiplication by a third root of 1.

$Sym^n(V_x)$ of the cyclic vector $\psi_x = \exp(\sum_{g,n} \hbar^{g-1}(S_{g,n})_x)$. The summation is over pairs (g, n) with either $g = 0, n \geq 3$ or $g \geq 1, n \geq 1$. This system of equations is called (quantum) *Holomorphic anomaly equation* (HAE for short).

REMARK 7.3.2. Originally HAE was formulated in the context of *compact* Calabi-Yau 3-folds. In this case X (probably should be better denoted by \mathcal{L} , as it is locally a Lagrangian cone) is a \mathbf{C}^* -bundle over a connected component of the moduli space \mathcal{M} of projective algebraic 3-dimensional varieties Y with $H^1(Y, \mathbf{C}) = 0$ and trivial canonical class. Space $X = \mathcal{L}$ parametrizes pairs $(Y, \Omega_Y^{3,0})$ where $\Omega_Y^{3,0} \in \Gamma(Y, \Omega_Y^3) - \{0\}$ a non-zero holomorphic volume form, and it is embedded locally as holomorphic Lagrangian cone into symplectic vector space $H^3(Y, \mathbf{C})$ via

$$(Y, \Omega_Y^{3,0}) \mapsto [\Omega_Y^{3,0}] \in H^3(Y, \mathbf{C}).$$

It was predicted in [BCOV94]¹⁹ that this Lagrangian cone has a **canonical** quantization, equivariant with respect to \mathbf{C}^* -action acting by rescaling

$$[\Omega_Y^{3,0}] \mapsto \lambda[\Omega_Y^{3,0}], \quad \hbar \mapsto \lambda^2 \hbar, \quad \lambda \in \mathbf{C}^*.$$

The rigorous construction of this quantization was proposed by K. Costello in [Cos]. Our framework here is more general, and is applicable e.g. to the moduli spaces of non-compact Calabi-Yau 3-folds, as in [KS14].

Let us now set $X = \mathcal{B}_1$. We have constructed the symplectic vector bundle \mathcal{H} with the fiber \mathcal{H}_Σ previously, same for the 1-form ϕ . It follows from considerations of the Sections 7.1, 7.2 that we have a tuple $(X, \mathcal{H}, \langle \bullet, \bullet \rangle, \nabla, \phi)$ and its quantization.

Notice that the larger space $\mathcal{B}_0 \supset \mathcal{B}_1$ does not depend on the choice of foliation \mathcal{F} . Its local Lagrangian embeddings into symplectic affine spaces depend only on finite jets of \mathcal{F} at D_Σ .

QUESTION 7.3.3. Can this quantization be extended to \mathcal{B}_0 ?

We expect that the answer is positive. Notice that since \mathcal{B}_0 does not depend on the foliation then the corollary would be symplectic invariance in topological recursion which does not have so far a conceptual explanation (see [Eyn0r2]).

QUESTION 7.3.4. If the answer to the previous question is positive, does the quantization for \mathcal{B}_0 depend only on the collection of $(2 \cdot k_\beta - 1)$ -jets of \mathcal{F} at D near points $p_\beta, \beta \in Pol$ (see Section 5.2)?

In the next section we will ask a stronger question 8.2.4, positive answer to which implies independence of the quantization of the choice of finite jets of \mathcal{F} .

Notice that although consideration of the Sections 7.1, 7.2 ensures that we have formal germs and quantizations only, the condition to be covariantly constant guarantees that everything is analytic (e.g. $S_{g,n}$ are analytic germs at each point $x \in X$).

REMARK 7.3.5. Our definition 7.3.1 of quantization can be improved further. Namely, for any covariantly constant family of left ideals $J_x \subset Weyl_x$ one can look for cyclic vectors (after choosing a Lagrangian complement to $T_x X \subset \mathcal{H}_x$) of the form $\exp(\sum_{g,n} \hbar^{g-1}(S_{g,n})_x)$ where the constraint $n > 0$ is omitted. The

¹⁹In [BCOV94] at each point $x = [(Y, \Omega_Y^{3,0})] \in \mathcal{L}$ the Lagrangian complement to the tangent space $T_x \mathcal{L} = H^{3,0} \oplus H^{2,1} \subset H^3(Y, \mathbf{C})$ was secretly chosen as the complex conjugate $H^{1,2} \oplus H^{0,3}$. E. Witten in [Wit] explained the invariant meaning of HAE as a quantization.

ambiguity for the lift is a torsor over the abelian group $\hbar^{-1}\mathbf{C}[[\hbar]]$ (or, equivalently, the multiplicative group of formal expressions $\exp(\sum_{g \geq 0} (S_{g,0})_x \hbar^{g-1})$ where $(S_{0,0})_x, (S_{1,0})_x, \dots \in \mathbf{C}$ is a sequence of constants). If we ignore terms $(S_{0,0})_x$ and $(S_{1,1})_x$, the rest does not depend on the choice of Lagrangian splitting, and one can speak about covariantly constant formal wave functions modulo the factor $\exp(c_0 \hbar^{-1} + c_1)$. Therefore, for each $g \geq 2$ we obtain a canonical \mathbf{C} -torsor with flat connection over X . The monodromy gives for every $g \geq 2$ a homomorphism $\pi_1(X) \rightarrow \mathbf{C}$. Combining them together we obtain an “obstruction” class in $H^1(X, \hbar \mathbf{C}[[\hbar]])$.

DEFINITION 7.3.6. We call an **enhanced** quantization a choice of flat sections of the above torsors for all $g \geq 2$

The enhanced quantization exists if and only if the obstruction class vanishes. In the case of a conical quantization, like for compact Calabi-Yau 3-folds, there is a canonical conical flat trivialization of all above torsors for $g \geq 2$, hence a canonical enhancement.

QUESTION 7.3.7. Does our quantization of the moduli space of spectral curves \mathcal{B}_1 extend (in some canonical way) to an enhanced quantization?

REMARK 7.3.8. The datum of a (non-enhanced) quantization corresponding to $g = 0$ is canonically determined by the tuple $(X, \mathcal{H}, \langle \bullet, \bullet \rangle, \nabla, \phi)$, hence carry no more additional information. It is well-known that $g = 1$ part of the quantization is equivalent to a choice of flat connection on virtual line bundle $(\Omega_X^{\dim X})^{\otimes (1/2)}$ (well-defined locally up to involution $\times(-1)$). Equivalently, by taking the square, one gets a flat connection on the honest line bundle $\Omega_X^{\dim X}$, but it is still preferable to speak about its square root as the wave function is invariantly a formal section of $(\Omega_X^{\dim X})^{\otimes (1/2)}$. In the example of compact Calabi-Yau 3-folds the monodromy of the induced flat connection on $\Omega_X^{\dim X}$ is non-trivial and is equal to multiplication by $\exp(2\pi i \chi(Y)/24)$ along the loop in \mathbf{C}^* in the conical direction. One may ask the question whether there exists a canonical flat section well-defined up to $\sqrt[24]{1}$ (as an analog of enhanced quantization for $g = 1$).

8. Semi-affine Lagrangian embeddings

The goal of this section is to define a generalization of local Lagrangian embedding

$$\text{germ at } [\Sigma] \text{ of } \mathcal{B}_0 \hookrightarrow \text{germ at } 0 \text{ of } \mathcal{H}_\Sigma$$

which *does not* use foliation \mathcal{F} , and formulate questions about the irrelevance of the choice of \mathcal{F} for the canonical quantization of locally embedded \mathcal{B}_1 discussed in the previous section.

8.1. Semi-affine symplectic structures.

DEFINITION 8.1.1. A **semi-affine structure** on a complex analytic symplectic manifold $(M, \omega = \omega_M)$ is given by a coisotropic foliation \mathcal{F} and an affine structure on leaves of \mathcal{F} such that near each point there exists a system of local coordinates

$$(40) \quad x_1, \dots, x_n, y^1, \dots, y^n; z_1, \dots, z_{2m}, \omega = \sum_{i=1}^n dx_i \wedge dy^i + \sum_{j=1}^m dz_j \wedge dz_{j+m},$$

the leaves of \mathcal{F} are given by $x_\bullet = const$ and the affine structure on leaves is given by coordinates y^\bullet, z_\bullet .

There are two particular cases:

- (1) $n = 0$, then M is the only leaf of the foliation and carries an affine symplectic structure,
- (2) $m = 0$, then M is endowed with a Lagrangian foliation, and the affine structure on leaves is automatic (e.g. $M = T^*X$ is a cotangent bundle endowed with the cotangent foliation).

In general, local coordinates in the above definition are determined up to the action of the infinitesimal group of transformations whose Lie algebra $\mathfrak{g}_{m,n}$ is given by Poisson brackets with analytic functions of the form

$$F^{(1)}(x_\bullet) + \sum_{i=1}^n F_i^{(2)}(x_\bullet)y^i + \sum_{j=1}^m F_j^{(3)}(x_\bullet)z_j + \sum_{j_1=1}^m \sum_{j_2=1}^m F_{j_1 j_2}^{(4)}(x_\bullet)z_{j_1}z_{j_2}.$$

In what follows we will use an alternative description of germs of semi-affine symplectic structures:

LEMMA 8.1.2. *A semi-affine structure on a germ at $m \in M$ of symplectic manifold M is equivalent to a Poisson fibration*

$$\pi : \text{germ of } M \text{ at } m \rightarrow \text{germ of Poisson manifold } P \text{ at } p = \pi(m)$$

such that P is regular (i.e. the rank of Poisson structure is constant), and symplectic leaves of P near p are endowed with affine structures compatible with the natural symplectic structures (i.e. the symplectic structure is constant in affine coordinates).

PROOF. Our assumptions (without the use of affine structures) imply that one can choose local coordinates on M at m as in (40) such that $\pi : M \rightarrow P$ is the coordinate projection

$$\pi : (x_\bullet, y^\bullet, z_\bullet) \mapsto (x_\bullet, z_\bullet).$$

The natural map π_1 from the germ of P at p to the “center” of P (the germ of the space of symplectic leaves in P) is given by the coordinate projection

$$\pi_1 : (x_\bullet, z_\bullet) \mapsto (z_\bullet).$$

The Lie algebra of germs of vector field at m preserving ω_M and acting on $\pi : M \rightarrow P$ is given by Poisson brackets with analytic functions of the form

$$(41) \quad \sum_i f_i(x_\bullet)y^i + G(x_\bullet, z_\bullet).$$

In particular, it maps *epimorphically* to the Lie algebra of germs of Poisson vector fields on P at p . The latter contains as a Lie subalgebra germs of symplectic vector fields in coordinates z_\bullet depending arbitrarily on center coordinates x_\bullet . Therefore, we can assume that the affine symplectic structures on symplectic leaves of P near p (see the Lemma) are the *standard* ones in coordinates z_\bullet . It is straightforward to see that Poisson brackets with functions from (41) act as affine transformations in z_\bullet if and only if $G = G(x_\bullet, z_\bullet)$ is at most quadratic in z_\bullet . Hence we obtain exactly the Lie algebra $\mathfrak{g}_{m,n}$ of local symmetries of a semi-affine symplectic structure, where n is number of x_\bullet -coordinates, and $2m$ is number of z_\bullet -coordinates. □

DEFINITION 8.1.3. A **flat family of semi-affine symplectic germs** on a complex manifold X is a holomorphically depending on $x \in X$ family of germs of semi-affine symplectic manifolds (M_x, m_x) (with symplectic forms ω_x , and germs of Poisson quotients (P_x, p_x) as in Lemma 8.1.2), endowed with a non-linear flat connection preserving semi-affine symplectic structures, but not necessary based points $(m_x)_{x \in X}$.

Such a family gives for any $x_0 \in X$ a map from the germ of X at x_0 to the germ M_{x_0} at m_{x_0} given by $x \mapsto m_x$ for x sufficiently close to x_0 , where we identify locally fibers with M_{x_0} via the flat connection.

DEFINITION 8.1.4. A flat family of semi-affine symplectic germs is said to give a **flat family of local Lagrangian embeddings** if the above-defined maps of germs gives a Lagrangian embedding

$$\text{germ of } X \text{ at } x_0 \hookrightarrow M_{x_0}$$

for any $x_0 \in X$.

This definition generalizes tuples $(X, \mathcal{H}, \langle \bullet, \bullet \rangle, \nabla, \phi)$ from Sections 5.1, 7.3 which give local Lagrangian embeddings into affine symplectic spaces. One can also generalize straightforwardly the previous definitions to the case when germs of semi-affine manifolds are replaced by **formal** germs.

REMARK 8.1.5. Omitting semi-affine symplectic structures in the definition 8.1.3 we obtain a more basic notion of a **flat family of germs** (or formal germs) on a manifold X . For example, if X is endowed with a foliation \mathcal{F} , then on X we have a flat family of germs of the “space of leaves” of \mathcal{F} .

There is an embedding of the Lie algebra $\mathfrak{g}_{m,n}$ into the Lie algebra of \hbar -differential operators in $(x_i)_{1 \leq i \leq n}, (z_j)_{1 \leq j \leq m}$ under which $y^j \mapsto \hbar \partial / \partial x_j, z_{l+m} \mapsto \hbar \partial / \partial z_l, 1 \leq j \leq n, 1 \leq l \leq m$, and the rest of variables are mapped to themselves. This give rise to a *canonical* completed Weyl algebra associated with a formal germ of a semi-affine symplectic manifold.

For any flat family of formal semi-affine symplectic germs we obtain a bundle of completed Weyl algebras endowed with a flat connection. Similarly to Section 7.3 we define **quantization** and **enhanced quantization** of a flat family of formal local Lagrangian embeddings.

A particular example is the one of a global Lagrangian submanifold $X = L$ in a cotangent bundle $M = T^*Y$ endowed with the natural semi-affine structure given by the foliation by cotangent fibers. Taking formal germs of M at points of L we obtain a flat family of local Lagrangian embeddings. The quantization of this structure in our sense is the same as a cyclic DQ -module on T^*Y with the classical limit \mathcal{O}_L endowed with cyclic vector 1, i.e. a WKB-module.

DEFINITION 8.1.6. For a given symplectic manifold (M, ω) and two semi-affine structures given by coisotropic foliations $\mathcal{F}, \mathcal{F}'$ with affine structures on leaves, we say that \mathcal{F} is **weaker** than \mathcal{F}' if leaves of \mathcal{F} are contained in leaves of \mathcal{F}' and the inclusion of any leaf is an affine map.

It is easy to see that in such a case the flat bundles of completed Weyl algebras associated with two semi-affine structures are *canonically* isomorphic. For example, the Weyl algebra associated with the cotangent space T^*V to an affine space V with the natural semi-affine structure of the cotangent bundle is canonically isomorphic

to the Weyl algebra associated with T^*V considered as an affine symplectic space. Same is true for completed Weyl algebras.

8.2. Canonical semi-affine structure on \mathcal{B}_0 . The setup in this section is the following: P is a complex analytic Poisson surface, with open dense symplectic leaf $(S, \omega = \omega_S)$ and such that the effective divisor D of zeroes of Poisson structure is smooth. We denote as usual by \mathcal{B}_0 the moduli space of D -transversal smooth connected compact curves $\Sigma \subset P$. Our goal here is to construct a flat family of Lagrangian embeddings into formal semi-affine symplectic germs

$$\text{germ } \mathcal{B}_{[\Sigma]} \text{ of } \mathcal{B}_0 \text{ at } [\Sigma] \hookrightarrow \text{germ } M_{[\Sigma]}$$

as in the previous section.

For a given D -transversal spectral curve Σ we denote as before by $\{p_\beta\}_{\beta \in Pol}$ the set $D \cap \Sigma$, by $(k_\beta \geq 1)_{\beta \in Pol}$ the corresponding multiplicities, and by D_Σ the effective divisor $\sum_\beta k_\beta p_\beta$ on Σ . **We assume that $Pol \neq \emptyset$.**

In order to define a formal germ of symplectic manifold $M_{[\Sigma]}$ and a Lagrangian embedding of germ $B_{[\Sigma]}$, we proceed as follows.

DEFINITION 8.2.1. For $[\Sigma] \in \mathcal{B}_0$ define a **collection of movable loops** as a tuple consisting of

- (1) a *non-empty* finite collection of disjoint closed discs $(\mathbb{D}_i)_{i \in I}$ in $\Sigma - \{p_\beta\}_{\beta \in Pol}$ with real-analytic boundaries,
- (2) a collection of germs of holomorphic 1-forms ν_i on S defined in a neighborhood of $\mathbb{D}_i \subset S$ such that $d\nu_i = \omega$ and ν_i is everywhere non-vanishing 1-form near $\partial\mathbb{D}_i$ with the kernel *transversal* to $\partial\mathbb{D}_i$.

We denote by $ML(\Sigma)$ the set of such collections. We endow $ML(\Sigma)$ with the partial order:

$$\sigma_1 \prec \sigma_2,$$

if σ_2 is obtained from σ_1 by adding new discs and new germs of 1-forms satisfying the above conditions.

Obviously, the nerve of this poset is *contractible*. The reason for that is the simple fact that for any finite collection of closed discs (maybe intersecting each other) in the non-compact (since $Pol \neq \emptyset$) surface $\Sigma - \{p_\beta\}_{\beta \in Pol}$, there exists another disc in $\Sigma - \{p_\beta\}_{\beta \in Pol}$ disjoint from our collection.

For every collection of movable loops $\sigma \in ML(\Sigma)$ we define a *formal germ of infinite-dimensional manifold* G_σ by considering formal deformations Σ'_0 of the complex curve

$$\Sigma_0 := \Sigma - \sqcup_i \text{int}(\mathbb{D}_i) \subset P$$

with real-analytic boundaries, such that the boundary is locally a shift of $\partial\mathbb{D}_i$ along the foliation $\text{Ker}(\nu_i)$, and the integral of ν_i over the corresponding boundary component vanishes²⁰.

The germ G_σ is naturally embedded into the infinite-dimensional vector space

$$W_\sigma = \left\{ (\eta_i)_{i \in I} \mid \eta_i \text{ is } \mathbf{C}\text{-valued real analytic 1-form on } \partial\mathbb{D}_i, \text{ and } \forall i \int_{\partial\mathbb{D}_i} \eta_i = 0 \right\}$$

²⁰The germ G_σ can be thought of as a non-linear analog of the coisotropic subspace G_Σ defined in (38).

endowed with the usual skew-symmetric pairing:

$$\langle (\eta_i^{(1)})_{i \in I}, (\eta_i^{(2)})_{i \in I} \rangle := \sum_i \int_{\partial \mathbb{D}_i} f_i^{(1)} \eta_i^{(2)}, \quad df_i^{(1)} = \eta_i^{(1)}.$$

Therefore, G_σ carries a closed 2-form ω_σ , which is the pullback of the translation-invariant closed 2-form on W_σ associated with the above pairing.

Let us describe the tangent space to G_σ and its skew-orthogonal. The tangent space to G_σ at any point Σ'_0 close to Σ_0 ²¹ is given by formula (compare with the proof of Proposition 7.1.2)

$$(42) \quad T_{\Sigma'_0} G_\sigma = \left\{ \eta \in \Gamma(\Sigma'_0, \Omega^1_{\Sigma'_0}(D_{\Sigma'_0})) \mid \int_{\partial_i \Sigma'_0} \eta = 0, \forall i \in I \right\}.$$

It is naturally embedded into W_σ , using foliations $Ker(\nu_i)$. The skew orthogonal to $T_{\Sigma'_0} G_\sigma$ is the subspace

$$(43) \quad (T_{\Sigma'_0} G_\sigma)^\perp = d\Gamma(\Sigma'_0, \mathcal{O}(-D_{\Sigma'_0})) \subset T_{\Sigma'_0} G_\sigma.$$

The quotient space $T_{\Sigma'_0} G_\sigma / (T_{\Sigma'_0} G_\sigma)^\perp$ is a finite-dimensional symplectic vector space which we denote by $\mathcal{H}_{\Sigma'_0, D_{\Sigma'_0}}$. This space depends only on Σ'_0 (which is considered as an abstract complex curve with real-analytic boundary) and on an effective divisor $D_{\Sigma'_0}$ with the support in the interior of Σ'_0 .

Suppose we have a germ of a complex manifold endowed with a closed 2-form of constant rank. It is well-known that the subbundle of the tangent bundle given by the kernel of the form defines a foliation. The germ of the space of leaves of this foliation is naturally a germ of symplectic manifold. We apply these general considerations to the infinite-dimensional manifold $(G_\sigma, \omega_\sigma)$ and obtain a well-defined finite-dimensional formal symplectic germ, which we denote by M_σ (abusing the notation we will denote the symplectic form on M_σ also by ω_σ as the 2-form on G_σ is its pullback).

Let us fix Σ . Now we are going to identify *canonically* all symplectic germs $(M_\sigma, \omega_\sigma)$ for $\sigma \in ML(\Sigma)$. Recall the partial order \prec on $ML(\Sigma)$. Obviously, if $\sigma_1 \prec \sigma_2$, we have a natural embedding

$$i_{\sigma_1, \sigma_2} : G_{\sigma_1} \hookrightarrow G_{\sigma_2}$$

and in this way we get a functor from the poset $ML(\Sigma)$ to the category of infinite-dimensional formal germs and embeddings. One can see immediately that $i^* \omega_{\sigma_2} = \omega_{\sigma_1}$, and moreover i_{σ_1, σ_2} induces a symplectomorphism of formal germs $M_{\sigma_1} \xrightarrow{\sim} M_{\sigma_2}$. The connectedness and 1-connectedness of the nerve of poset $ML(\Sigma)$ implies that we identify all formal germs M_σ with each other.

DEFINITION 8.2.2. For any point $[\Sigma] \in \mathcal{B}_0$ define the associated symplectic formal germ $M_{[\Sigma]}$ as the result of the identifications of formal germs M_σ for $\sigma \in ML(\Sigma)$ described above.

It is obvious from our description that formal germs $(M_{[\Sigma]})_{[\Sigma] \in \mathcal{B}_0}$ form a *flat family* of formal germs, and symplectic structures on fibers are covariantly constant. Also it is straightforward to see that the flat family of germs $(\mathcal{B}_\Sigma)_{[\Sigma] \in \mathcal{B}_0}$ has natural covariantly constant Lagrangian embeddings into $(M_{[\Sigma]})_{[\Sigma] \in \mathcal{B}_0}$.

²¹Since we speak about formal germs, we mean that Σ'_0 is a deformation of Σ_0 over the spectrum of a local Artin algebra over \mathbf{C} .

Finally let us describe a natural covariantly constant family of semi-affine structures on $M_{[\Sigma]}$. For given curve Σ and $\sigma \in ML(\Sigma)$, let us consider a subbundle of finite codimension of the tangent bundle to G_σ . The subbundle is given by first order deformations such that the corresponding 1-form η with poles at $D_{\Sigma'_0}$ on Σ'_0 (which is a small deformation of Σ_0 as a curve with boundary) is *holomorphic*, i.e. has no poles. It is easy to see that this distribution is *integrable*, i.e. it defines a foliation \mathcal{F}_σ on G_σ . This follows immediately from the geometric interpretation of the above constraint on first order deformations. More precisely, it means that intersection points p_β of the moving curve with the divisor D stay constant, and moreover $(k_\beta - 1)$ -jet of Σ_0 at p_β stays constant. Also it is clear from this description that leaves of \mathcal{F}_σ contain leaves of the foliation $Ker(\omega_\sigma)$. Hence \mathcal{F}_σ induces a foliation on $M_{[\Sigma]}$, and it is easy to see that it does not depend on σ (since the maps i_{σ_1, σ_2} identify induced foliations).

Let us denote by $\mathcal{F}_{M_{[\Sigma]}}^{co}$ the resulting foliation on $M_{[\Sigma]}$. Pick any $\sigma \in ML(\Sigma)$. Since the restriction of ω_σ to $T_{\mathcal{F}_\sigma}$ vanishes, it follows that $\mathcal{F}_{M_{[\Sigma]}}^{co}$ is coisotropic. The codimension of leaves of $\mathcal{F}_{M_{[\Sigma]}}^{co}$ coincides with the codimension of leaves of \mathcal{F}_σ , and is equal to $\sum_\beta k_\beta - 1$ (the term -1 comes from the condition that the sum of residues of 1-forms η from (42) at points $p'_\beta \in \Sigma'_0$ vanish).

Notice that there is a 1-form ξ on $\mathcal{F}_{M_{[\Sigma]}}^{co}$. In order to see this, let us replace in (43) the space of functions vanishing at $p_\beta, \beta \in Pol$ by the space of all functions. Then the quotient of the tangent space to $\mathcal{F}_{M_{[\Sigma]}}^{co}$ by this bigger subspace is isomorphic to $H^1(\Sigma, \mathbf{C})$. Natural (surjective) projection to the quotient gives rise to an $H^1(\Sigma, \mathbf{C})$ -valued 1-form ξ along leaves of $\mathcal{F}_{M_{[\Sigma]}}^{co}$.

LEMMA 8.2.3. *The form ξ is closed.*

PROOF. We will check vanishing of $d\xi$ using the lift to G_σ for any given $\sigma \in ML(\Sigma)$. In order to do this, we will construct an $H^1(\Sigma, \mathbf{C})$ -valued functional on pairs of points of G_σ lying in the same leaf of \mathcal{F}_σ , and satisfying the cocycle property. The construction is similar to what we had before. We interpret $H^1(\Sigma, \mathbf{C})$ as the space of \mathbf{C} -valued linear functionals on $H_1(\Sigma'_0, \mathbf{Z})$ vanishing on classes of boundary loops $\partial_i \Sigma'_0$ (here as before, Σ'_0 is a small deformation of Σ_0). With any two deformed surfaces Σ'_0, Σ''_0 and closed 1-chains ρ', ρ'' on these surfaces representing homology classes identified by the Gauss-Manin connection, we associate a complex number by integrating $\omega = \omega_S$ along 2-chain in S with the boundary equal to $\rho'' - \rho'$. This assignment depends only on homology classes, vanishes on boundary loops, satisfies the cocycle property from Remark 5.3.2, and gives the pullback of ξ as the first derivative at the diagonal. □

Thus we have a foliation on each \mathcal{F}_σ^{co} with fibers which are fibers of the map $\xi : T_{\mathcal{F}_\sigma^{co}} \rightarrow H^1(\Sigma, \mathbf{C})$. Tangent space to a fiber is isomorphic to $(T_{\mathcal{F}_\sigma^{co}})^\perp$, which is the skew-orthogonal with respect to the symplectic form ω_σ . Thus the skew-orthogonal subbundle $(T_{\mathcal{F}_{M_{[\Sigma]}}^{co}})^\perp \subset TM_{[\Sigma]}$ gives an isotropic subfoliation. The quotient bundle

$$T_{\mathcal{F}_{M_{[\Sigma]}}^{co}} / (T_{\mathcal{F}_{M_{[\Sigma]}}^{co}})^\perp$$

is naturally isomorphic to a flat symplectic bundle of rank $2g$ (where g is the genus of Σ) with the fiber at the base point equal to $H^1(\Sigma, \mathbf{C})$.

This finishes the construction of the canonical flat family of Lagrangian embeddings of \mathcal{B}_0 into a family of semi-affine symplectic germs.

Also, we see immediately that in the case when the foliation \mathcal{F} on P exists, the semi-affine structure coming from general considerations is automatically weaker (see Definition 8.1.6) than the semi-affine structure coming from \mathcal{F} (which is in fact a symplectic affine structure on M_Σ). According to the discussion after the Definition 8.1.6 in that case we obtain isomorphic flat bundles of completed Weyl algebras. It is naturally to inquire whether the corresponding quantizations depend on chosen foliations. Hence we ask the following question.

QUESTION 8.2.4. Is it true that the flat family of quantizations of locally Lagrangian embeddings $\mathcal{B}_{[\Sigma]} \hookrightarrow M_\Sigma$ does not depend on the choice of foliation \mathcal{F} ?

Presumably the answer is positive. We expect that it is related to the deep result about symplectic invariance in TR proven by Eynard and Orantin in [EynOr2]. Also, the positive answer will imply (via finite jet approximations of the foliation) that the canonical quantization can be constructed even in the case when there is no foliation \mathcal{F} at all.

9. Two more speculations

9.1. Wave function for an individual spectral curve. One can generalize considerations of this paper to the case of immersed (rather than embedded) spectral curves which can have singularities. The moduli space \mathcal{B} of immersed singular spectral curves contains an open subspace \mathcal{B}_0 , the moduli space of immersed smooth spectral curves.

Having in mind the above remark let us make a comment about a quantization of an individual spectral curve Σ , assuming that the open symplectic leaf S is isomorphic to T^*C , where C is a compact curve. If the answer to the Question 8.2.4 is positive, then one can study the behavior of the corresponding wave function $\psi_{\mathcal{B}_{[\Sigma]}}$ near the complement $\mathcal{B} - \mathcal{B}_0$ (discriminant locus) in the bigger space \mathcal{B} .

Let T_x^* , $x \in C$ denote the cotangent fiber at $x \in C$. For generic collection of points $x_1, \dots, x_n \in C$ let us pick some pre-images $y_1, \dots, y_n \in \Sigma$. Then the abstract curve Σ_{y_1, \dots, y_n} is defined as the disjoint union $T_{x_1}^* \sqcup \dots \sqcup T_{x_n}^* \sqcup \Sigma$ with identified pairs of points of Σ and $T_{x_i}^*$, $1 \leq i \leq n$ corresponding to the same point from the collection $y_1, \dots, y_n \in \Sigma$. This singular curve is naturally immersed into the Poisson surface P , and hence gives a point in the discriminant $\mathcal{B} - \mathcal{B}_0$. We can try to deform it to an immersed smooth curve.

REMARK 9.1.1. The reason for picking *more* than 1 point in the collection $(x_i)_{1 \leq i \leq n}$ is the following. If we want to deform an immersion of the singular curve Σ_{y_1, \dots, y_n} , we need (at first order deformation) a meromorphic 1-form on Σ with first order poles at $y_1, \dots, y_n \in \Sigma$. In the case $n = 1$ such a form is automatically holomorphic, because the sum of residues vanishes. Hence there is no smoothening of the singular curve in the case $n = 1$.

It is plausible that with the Lagrangian subvarieties $\Sigma, \Sigma_{y_1, \dots, y_n} \subset S = T^*C$ one can associate cyclic DQ -modules with cyclic vectors ψ_Σ and $\psi_{\Sigma_{y_1, \dots, y_n}}$ respectively.

QUESTION 9.1.2. How to relate ψ_Σ and $\psi_{\Sigma_{y_1, \dots, y_n}}$ to $\psi_{\mathcal{B}_{[\Sigma]}}$?

Probably the possibility to extract ψ_Σ from $\psi_{\mathcal{B}}$ is encoded into some constraints which allow us to extend differential forms from $\Omega^1(\Sigma - \cup_\alpha \{r_\alpha\})$ to the whole curve Σ . Relations of ψ_Σ to the topological recursion were discussed in [DuMu].

9.2. dg-Airy structure associated with compact Calabi-Yau 3-fold. In this subsection we discuss a differential-graded version of an Airy structure associated with the moduli space of complex Calabi-Yau 3-folds. It is well-known that the analog of the above-discussed moduli space \mathcal{B} is a complex Lagrangian cone in the middle cohomology. The corresponding wave function contains all the information about Gromov-Witten invariants. Moreover, it satisfies the HAE.

Let Y be a compact complex 3-dimensional Calabi-Yau manifold (CY 3-fold for short). Deformation theory of complex structures on Y is controlled by the differential-graded Lie algebra (DGLA for short)

$$\mathfrak{g}_Y = \Gamma(Y, \oplus_{i \geq 0} \wedge^i T_Y^{1,0} \otimes \Omega_Y^{0,*})[1].$$

We split \mathfrak{g}_Y into the direct sum of \mathbf{Z} -graded subspaces:

$$\mathfrak{g}_Y = \oplus_{i \geq 0} \mathfrak{g}^i, \quad \mathfrak{g}^i = \Gamma(Y, \wedge^i T_Y^{1,0} \otimes \Omega_Y^{0,*})[1].$$

The differential is $d_{\mathfrak{g}} = \bar{\partial} + div$, where div is the divergence operator corresponding to the holomorphic volume for $\Omega^{3,0}$ (see e.g. [BK98]). The corresponding homological vector field on $\mathfrak{g}_Y[1]$ is given

$$\dot{\gamma} = (\bar{\partial} + div)(\gamma) + \frac{[\gamma, \gamma]}{2}$$

and the Maurer-Cartan equation (MC equation in short) is the equation for zeroes of this field. The space of solutions of the MC equation carries a foliation with the tangent space at γ consisting of

$$\delta(\gamma) = (\bar{\partial} + div)(\epsilon) + [\epsilon, \gamma], \quad \epsilon \in \mathfrak{g}_Y.$$

The *extended moduli space of complex structures* is the quotient of the space of solutions to the MC-equation by this foliation. The extended moduli space is a smooth finite-dimensional \mathbf{Z} -graded analytic supermanifold by a generalization of Tian-Todorov theorem (see loc.cit.).

Let us make the change of variables $\mu = e^\gamma$. Then the above homological vector field becomes

$$(44) \quad \dot{\mu} = (\bar{\partial} + div)(\mu)$$

and the foliation becomes also linear:

$$\delta(\mu) = (\bar{\partial} + div)(\epsilon), \quad \epsilon \in \mathfrak{g}_Y.$$

In the new variable μ the image of an element $\gamma_0 + \gamma_1$ belonging to the graded Lie subalgebra $\mathfrak{g}^0 \oplus \mathfrak{g}^1 \subset \mathfrak{g}_Y$ has the form $\exp(\gamma_0 + \gamma_1) = \mu_0 + \mu_1 + \mu_2 + \mu_3$ where:

$$\mu_0 = \exp(\gamma_0), \mu_1 = \exp(\gamma_0) \cdot \gamma_1, \mu_2 = \exp(\gamma_0) \cdot (\gamma_1^2/2), \mu_3 = \exp(\gamma_0)(\gamma_1^3/6).$$

Using contraction with volume element $\Omega^{3,0}$ we identify (as a $\mathbf{Z}/2$ -graded supermanifold) the vector space $\mathfrak{g}_Y[1]$ endowed with the linear homological vector field (44) with the space of forms

$$W := \Gamma(Y, \oplus_{p,q \geq 0} \Omega_Y^{p,q})[3]$$

endowed with the linear homological vector field $\dot{\alpha} = d\alpha$. The space W is an infinite-dimensional symplectic vector space endowed with the symplectic form $\langle \omega_1, \omega_2 \rangle = \int_X \omega_1 \wedge \omega_2$.

PROPOSITION 9.2.1. *Let us define $\mathcal{L} = \exp(\mathfrak{g}^0 \oplus \mathfrak{g}^1)$ considered as a subvariety of W . Then:*

- (1) \mathcal{L} is a quadratic Lagrangian subvariety in W .
- (2) The vector space $\text{Ker}(d) \subset W$ is a coisotropic vector subspace and $\text{Im}(d)$ is symplectic orthogonal to $\text{Ker}(d)$.
- (3) The natural projection (symplectic reduction) $pr : \mathcal{L} \cap \text{Im}(d) \rightarrow H^\bullet(Y) \rightarrow H^3(Y, \mathbf{C})$ maps \mathcal{L} into the Lagrangian cone $L \subset H^3(Y, \mathbf{C})$ which is the moduli space of pairs consisting of a complex structure on X and a choice of holomorphic volume form vol_Y .

PROOF. Parts 2) and 3) follow from the deformation theory of the Calabi-Yau structure on X , and can be derived in many ways (see e.g. [BK98]).

The fact that W is symplectic is general. It holds for any real compact manifold Y such that $\dim_{\mathbf{R}} Y \in 2 + 4\mathbf{Z}_{\geq 0}$ (e.g. for any odd-dimensional compact complex manifold).

Notice that the Calabi-Yau pairing implies the following isomorphisms (properly understood in the infinite-dimensional context): $\mathfrak{g}^0 \simeq (\mathfrak{g}^3)^*$, $\mathfrak{g}^1 \simeq (\mathfrak{g}^2)^*$. Quadratic equations for \mathcal{L} require passing to the coordinates $\mu_i, i = 0, 1, 2, 3$, in which \mathcal{L} can be described by the equations

$$\begin{aligned} \mu_0\mu_2 &= \frac{\mu_1^2}{2}, \\ \mu_0\mu_3 &= \frac{\mu_1\mu_2}{3}. \end{aligned}$$

The fact that \mathcal{L} is Lagrangian can be checked directly. Namely, one checks first that the dimension \mathcal{L} understood in the sense of infinite-dimensional smooth topology on \mathfrak{g} is “one half” of the dimension of \mathfrak{g} . Then one shows by a direct computation, that the algebra of functions vanishing on \mathcal{L} is closed under the Poisson bracket. Such functions have form

$$f_{a_0, a_1}(\mu_0, \dots, \mu_3) = \text{Tr} (a_0(\mu_0\mu_2 - \mu_1^2/2) + a_1(\mu_0\mu_3 - \mu_1\mu_2/3)),$$

where $a_i \in \mathfrak{g}^i, i = 0, 1$ and $\text{Tr} : \mathfrak{g}^3 \rightarrow \mathbf{C}$ is the functional given by the holomorphic volume form vol_X . Then one computes $\{f_{a_0, a_1}, f_{b_0, b_1}\}$ and sees that it gives a solvable Lie algebra structure on functions vanishing on \mathcal{L} .

This completes the proof. □

We should warn the reader that despite the existence of quadratic Lagrangian, there is no Airy structure in the compact case. The corresponding tensors A, B, C belong to the *completed* tensor products. As a result the series which should give the wave function are divergent and do not give a meaningful answer.

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IHES, 35 ROUTE DE CHARTRES, BURES-SUR-YVETTE, F-91440, FRANCE
Email address: `maxim@ihes.fr`

DEPARTMENT OF MATHEMATICS, KANSAS STATE UNIVERSITY, MANHATTAN, KS 66506
Email address: `soibel@math.ksu.edu`