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Topological recursion and Givental's formalism: Spectral curves for Gromov-Witten theories

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Abstract. We describe a way of producing local spectral curves for arbitrary semisimple cohomological field theories (and Gromov-Witten theories in particular) and global spectral curves for semisimple cohomological field theories satisfying certain conditions. By this we mean that applying the topological recursion procedure on the spectral curve reproduces the total potential of the corresponding cohomological field theory.

Present text is based on papers [*Identification of the Givental formula with the spectral curve topological recursion procedure*, Comm. Math. Phys. 328 (2014), pp. 669–700] and [*Dubrovin's superpotential as a global spectral curve*, arXiv:1509.06954] joint work with Nicolas Orantin, Sergey Shadrin and Loek Spitz and with Paul Norbury, Nicolas Orantin, Alexandr Popolitov and Sergey Shadrin, respectively.

Contents

1. Introduction

The contents of the present text were presented as a mini-course at the 2016 AMS von Neumann Symposium on Topological Recursion and its Influence in Analysis, Geometry, and Topology. It is based on papers [**17**] and [**18**], written in collaboration with Nicolas Orantin, Sergey Shadrin and Loek Spitz and with Nicolas Orantin, Paul Norbury, Alexandr Popolitov and Sergey Shadrin, respectively.

In the present text we describe the way of reproducing potentials of semisimple cohomological field theories (of which semisimple Gromov-Witten theories are a subset) with the help of spectral curve topological recursion.

Gromov-Witten theories originated from string theory, where they appeared as part of the "toy model" of topological string theory. It turned out that they

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are of particular interest in mathematics, as they provided an interesting way to build invariants of complex manifolds. Kontsevich and Manin generalized Gromov-Witten theories to the so-called *cohomological field theories*. This allowed to better incorporate this area of research into the general study of moduli spaces of algebraic curves and their cohomologies. A seminal result was first suggested by Witten and later proved by Kontsevich regarding the fact that the Gromov-Witten theory of a point corresponds to a KdV tau-function.

Spectral curve topological recursion is a quite general technique which has applications in many different branches of mathematics and physics. This technique, given certain initial data, namely, a *spectral curve* (which is an algebraic curve equipped with some additional structure), produces so-called *n-point functions* on this spectral curve. It turns out that for a vast array of problems in such areas of science as algebraic geometry, mathematical physics, topology and combinatorics, these n-point functions appear to be generating functions for numbers arising in these problems, if one takes appropriate initial data. Spectral curve topological recursion is very interesting since it is one and the same procedure that for appropriate choices of initial data produces answers to many seemingly unrelated problems.

The first aim of the present text is to describe a way [**17**] of reproducing any semisimple cohomological field theory (and any semisimple Gromov-Witten theory in particular) with the help of the so-called *local* version of spectral curve topological recursion. This local version is a generalization of the spectral curve topological recursion where instead of the whole spectral curve we consider just the germs of all objects at branching points. In particular any ordinary spectral curve topological recursion gives rise to a local version, but not necessarily the other way around. This is a quite strong result, but in itself it might be too general, as local topological recursion (to the moment) lacks interesting applications or profound meaning separate from what originates from ordinary (global) spectral curve topological recursion. Thus, at least for now, it seems that the main applications of this result are related to proving spectral curve topological recursion for those CohFTs which do correspond to global spectral curves.

This is precisely the subject of the second part of the present text (which follows [**18**]). In that part we describe certain conditions under which a cohomological field theory has a corresponding spectral curve. Under these conditions we describe (with the help of a construction due to Dubrovin) an explicit way of constructing the global spectral curve using just the information of the Frobenius manifold which underlies the CohFT, and we prove that indeed the topological recursion procedure on the said curve reproduces the total potential of the CohFT. We provide several examples of this. For all these examples the corresponding Frobenius manifold belongs to the wide class of the so-called Hurwitz-Frobenius manifolds. While this is outside of the scope of the present text, we note that in [**19**] it is proved that this technique works for *all* semisimple Hurwitz-Frobenius manifolds; moreover, it turns out that this holds in the other direction as well: all global spectral curves with a mild restriction correspond via topological recursion to CohFTs based on this large class of Hurwitz-Frobenius manifolds.

1.1. Structure of the text. We start by introducing all relevant objects in Section 2.

In Section 3 (following [**17**]), we describe how to construct a local spectral curve for an arbitrary semisimple CohFT. This culminates with Theorem 3.16 which proves, with the help of celebrated Givental formula, that the potential of the CohFT is reproduced by the topological recursion procedure on this curve. We follow it up by describing how to apply this theorem to the particular example of the Gromov-Witten theory of $\mathbb{C}\mathrm{P}^1$.

Then, in Section 4 (following [**18**]), we describe a way of constructing global spectral curves for semisimple CohFTs, with the help of Dubrovin's superpotential. Theorems 4.10 and 4.13, prove that, under certain conditions on the CohFT, Dubrovin's superpotential obtained from the corresponding Frobenius manifold can serve as a global spectral curve. We follow these theorems up with examples.

2. Preliminaries

2.1. Frobenius manifolds. A Frobenius manifold is a differential-geometric structure that was introduced by Dubrovin in the early 1990's as a mathematical framework for the study of two-dimensional topological field theory in genus zero [**10**, **12**]. It has appeared to be a quite universal structure that has many naturally arising examples. In particular, Frobenius manifolds can serve as a classification tool for (dispersionless) bi-Hamiltonian hierarchies of hydrodynamic type [**14**, **15**]. Nowadays there is a number of standard textbooks on Frobenius manifolds, see [**12**, **39**, **48**].

Here we introduce Frobenius manifolds following [**12**].

Definition 2.1. An algebra A over $\mathbb{C}$ is called (commutative) *Frobenius algebra* if:

(1) It is a commutative associative $\mathbb{C}$-algebra with a unity e.
(2) It is supplied with a $\mathbb{C}$-bilinear symmetric nondegenerate inner product

$$A \times A \to \mathbb{C},\ a,\, b \mapsto (a, b)$$

being invariant in the following sense:

$$(a \cdot b, c) = (a, b \cdot c)$$

Definition 2.2. M is *Frobenius manifold* if a structure of Frobenius algebra is specified on any tangent plane T_tM at any point $t \in M$ smoothly depending on the point such that

(1) The invariant inner product $(\ ,\ )$ is a flat metric on M.
(2) The unity vector field e is covariantly constant w.r.t. the Levi-Civitá connection ∇ for the metric $(\ ,\ )$

$$\nabla e = 0$$

(3) Let

$$c(u, v, w) := (u \cdot v, w)$$

(a symmetric 3-tensor). We require the 4-tensor

$$(\nabla_z c)(u, v, w)$$

to be symmetric in the four vector fields u, v, w, z.

(4) A vector field E must be determined on M such that

$$\nabla(\nabla E) = 0$$

and that the correspondent one-parameter group of diffeomorphisms acts by conformal transformations of the metric $(\ ,\ )$ and by rescalings on the Frobenius algebras T_tM.

Locally Frobenius manifold can be defined in terms of a *Frobenius prepotential* $F(t)$ as follows.

DEFINITION 2.3. Function $F = F(t)$, $t = (t^1, \dots, t^n)$ is a *Frobenius prepotential* if its third derivatives

$$c_{\alpha\beta\gamma}(t) := \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\gamma}$$

obey the following equations

(1) Normalization:

$$\eta_{\alpha\beta} := c_{1\alpha\beta}(t)$$

is a constant nondegenerate matrix. Let

$$(\eta^{\alpha\beta}) := (\eta_{\alpha\beta})^{-1}.$$

We will use the matrices $(\eta_{\alpha\beta})$ and $(\eta^{\alpha\beta})$ for raising and lowering indices.

(2) Associativity: the functions

$$c^\gamma_{\alpha\beta}(t) := \eta^{\gamma\epsilon} c_{\epsilon\alpha\beta}(t)$$

(summation over repeated indices is assumed here) for any t must define in the n-dimensional space with a basis e_1, ..., e_n a structure of an associative algebra A_t

$$e_\alpha \cdot e_\beta = c^\gamma_{\alpha\beta}(t) e_\gamma.$$

Note that the vector e_1 will be the unity for all the algebras A_t:

$$c^\beta_{1\alpha}(t) = \delta^\beta_\alpha.$$

(3) $F(t)$ must be quasihomogeneous function of its variables:

$$F(c^{d_1} t^1, \dots, c^{d_n} t^n) = c^{d_F} F(t^1, \dots, t^n) \tag{1}$$

for any nonzero c and for some numbers d_1, ..., d_n, d_F.

Coordinates t such as above on the Frobenius manifold are called the *flat coordinates.*

It will be convenient to rewrite the quasihomogeneity condition in the infinitesimal form introducing the *Euler vector field*

$$E = E^\alpha(t)\partial_\alpha$$

as

$$\mathcal{L}_E F(t) := E^\alpha(t)\partial_\alpha F(t) = d_F \cdot F(t)$$

$E(t)$ is a linear vector field

$$E = \sum_\alpha d_\alpha t^\alpha \partial_\alpha$$

generating the scaling transformations (1).

Let us denote $3 - d_F$ by d.

DEFINITION 2.4. Point of a Frobenius manifold is called *semisimple* if at this point the Frobenius algebra structure is nondegenerate, i.e. there is no tangent vector that squares to zero.

Near a semisimple point of a Frobenius manifold one can introduce the so-called *canonical coordinates*, which are defined as coordinates u^i that have the property

$$\frac{\partial}{\partial u^i} \cdot \frac{\partial}{\partial u^j} = \delta_{ij} \frac{\partial}{\partial u^j}, \tag{2}$$

where δ_{ij} is the Kronecker delta.

Define $\Delta_i := 1/(\partial_i, \partial_i)$ to be the inverse of the square of the length of the i^{th} canonical basis element, and call $\{\partial/\partial v^i := \Delta_i^{1/2} \partial/\partial u^i\}$ the normalized canonical basis in the tangent space.

Let U be the matrix of canonical coordinates $U = \operatorname{diag}(u^1, \ldots, u^n)$ and denote by Ψ the transition matrix from the flat to the normalized canonical bases. That is, denoting $\mathrm{d}t = (\mathrm{d}t^1, \ldots, \mathrm{d}t^n)^{\mathrm{T}}$ and $\mathrm{d}u = (\mathrm{d}u^1, \ldots, \mathrm{d}u^n)^{\mathrm{T}}$, one has

$$\Delta^{-1/2} \mathrm{d}u = \Psi \mathrm{d}t, \tag{3}$$

where $\Delta = \operatorname{diag}(\Delta_1, \ldots, \Delta_n)$.

Define

$$V_i := (\partial_{u_i} \Psi) \cdot \Psi^{-1}$$

and

$$V := \sum_i u_i V_i.$$

The second metric η' is defined in the following way: for any two vector fields ∂' and ∂'' we define

$$\eta'(\partial', \partial'') := \eta(\partial', E \cdot \partial'').$$

2.2. Superpotential. A convenient way to describe a Frobenius structure is in terms of a so-called Landau-Ginzburg superpotential. We recall the definition from [**12, 13**]. A *superpotential* is a holomorphic function $\lambda(p, u_1, \ldots, u_n)$ of a variable $p \in \mathcal{D}$ in some domain $\mathcal{D}$ (which is an open domain on a Riemann surface realized as a branched covering over complex plane with finite number of sheets) that depends on points $(u_1, \ldots, u_n) \in B_0 \subset B^{ss}$ in a ball in the semisimple part of the Frobenius manifold (Riemann surface which contains domain $\mathcal{D}$ may depend on u, but the projection to the complex plane should remain fixed), and satisfies the following properties:

(1) The critical values of λ as a function on $\mathcal{D}$ are $u_1, \ldots, u_n$.
(2) The critical points are non-degenerate.
(3) If there are several critical points in the inverse image $\lambda^{-1}(u_i)$, then the Hessians of λ at these points must coincide.
(4) For any choice $p_1, \ldots, p_n \in \mathcal{D}$ of the critical preimages of $u_1, \ldots, u_n$ (that is, $\lambda(p_i, u_1, \ldots, u_n) = u_i$) and for any choice of the vector fields ∂', ∂'',

and ∂''' on B_0 we have:

$$\eta(\partial', \partial'') = -\sum_{i=1}^{n} \operatorname*{Res}_{p\to p_i} \frac{\partial'(\lambda dp)\partial''(\lambda dp)}{d_p\lambda}; \tag{4}$$

$$\eta'(\partial', \partial'') = -\sum_{i=1}^{n} \operatorname*{Res}_{p\to p_i} \frac{\partial'(\log\lambda dp)\partial''(\log\lambda dp)}{d_p\log\lambda}; \tag{5}$$

$$\eta(\partial'\cdot\partial'', \partial''') = -\sum_{i=1}^{n} \operatorname*{Res}_{p\to p_i} \frac{\partial'(\lambda dp)\partial''(\lambda dp)\partial'''(\lambda dp)}{dp\, d_p\lambda} \tag{6}$$

where $\partial'(\lambda dp)$ gives the action of the vector field by derivation in the parameters u_i. In particular, the map $\partial' \mapsto \partial'(\lambda dp)$ from vector fields on B_0 to meromorphic differentials on $\mathcal{D}$ quotiented out by $d_p\lambda$ is injective.

(5) There exist some cycles $Z_1, \ldots, Z_n$ in $\mathcal{D}$ such that the integrals

$$\frac{1}{\sqrt{z}}\int_{Z_\alpha} e^{z\lambda}dp, \qquad \alpha = 1, \ldots, n \tag{7}$$

converge and give a non-degenerate system of flat coordinates for $\tilde{\nabla}$.

In these terms, the identity vector field ∂_0 of the Frobenius manifold is represented by dp, i.e. $\partial_0(\lambda dp) = dp$. Indeed, since $\eta(\partial_0\cdot\partial, \partial') = \eta(\partial, \partial')$ for all vector fields ∂, ∂', then non-degeneracy of η implies that $\partial_0 \cdot \partial = \partial$ for all ∂. The Euler vector field is represented in these terms by λdp, i.e. $E(\lambda dp) = \lambda dp$.

2.3. Dubrovin's superpotential. In this Section we recall a construction of a particular Landau-Ginzburg superpotential due to Dubrovin [**13**].

Given a manifold M equipped with a flat metric, a locally defined function t is a *flat coordinate* at $p \in M$, if

(i) $dt(p) \neq 0$ and

(ii) dt is covariantly constant with respect to the Levi-Civita connection.

Condition (i) guarantees that t is a local coordinate, i.e. we can find a coordinate system $(t^1, ..., t^n)$ with $t^1 = t$ and an open neighbourhood $B \subset M$ of p such that $(t^1, ..., t^n) : B \to B_0 \subset \mathbb{R}^n$ is a homeomorphism onto an open set B_0 of $\mathbb{R}^n$. Condition (ii), which uses the induced connection on the cotangent bundle, guarantees that $(t^1, ..., t^n)$ can be chosen so that the metric is represented by a constant matrix with respect to $(t^1, ..., t^n)$.

We now consider a flat coordinate $\rho(\lambda, u)$ with respect to the pencil of metrics $\eta' - \lambda\eta$. We study covariant constancy of $d\rho$ via its gradient vector field $\phi(\lambda, u) = \nabla\rho(\lambda, u)$ defined by

$$(\eta' - \lambda\eta)(\phi, \cdot) = d\rho.$$

The Levi-Civita connection of η' with respect to flat coordinates (for η) is given in [**13**, Equation (5.5)]). This leads to the following system of equations for vector fields ϕ expressed in canonical coordinates on a Frobenius manifold (the extended Gauss-Manin system [**13**, Equations (5.31) and (5.32)]):

$$\mathrm{d}\phi = -(U-\lambda)^{-1}\mathrm{d}(U-\lambda)\left(\frac{1}{2} + V\right)\phi + \mathrm{d}\Psi\cdot\Psi^{-1}\phi. \tag{8}$$

Here $\mathrm{d} = \mathrm{d}_\lambda + \mathrm{d}_u$ is the total de Rham differential; by writing ϕ we understand the column of components of vector field ϕ in the normalized canonical basis; $U = \operatorname{diag}(u_1, ..., u_n)$ and V and Ψ are naturally associated to a Frobenius manifold as defined in Section 2.1. Abusing notation, we use λ for the matrix of multiplication by λ. So (8) encodes the system of PDEs giving covariant constancy of $\phi(\lambda,u) = \nabla\rho(\lambda,u)$ in directions $\partial/\partial\lambda, \partial/\partial u_i$.

One can retrieve ρ from its gradient vector field via

$$\rho(\lambda,u) = \frac{\sqrt{2}}{1-d}\phi^T(U-\lambda)\Psi\,\mathbb{1}. \tag{9}$$

This is proved in [**11**, Section 2].

This equation has poles at $\lambda = u_1,\dots,u_n$ on the λ-plane, so we choose parallel straight cuts $L_1,\dots,L_n$ from the points u_i to infinity (we assume that $u_j \notin L_i$ for $i \neq j$). On $\mathbb{C}\setminus\cup_{i=1}^n L_i$ we choose branches of functions $\sqrt{u_i-\lambda}$, $i=1,\dots,n$. We denote by $\mathcal{R}_i$ the monodromy of the space of solutions of Equation (8) corresponding to following a small loop around u_1.

Dubrovin proves that there exist a unique system of solutions $\phi^{(1)},\dots,\phi^{(n)}$ to equation (8) satisfying the following properties:

$$\mathcal{R}_j\phi^{(j)} = -\phi^{(j)}, \qquad j=1,\dots,n; \tag{10}$$

$$\phi_j^{(j)} = \frac{1}{\sqrt{u_j-\lambda}} + O(\sqrt{u_j-\lambda}) \text{ for } \lambda\to u_j, \qquad j=1,\dots,n; \tag{11}$$

$$\phi_a^{(j)} = \sqrt{u_j-\lambda}\cdot O(1) \text{ for } \lambda\to u_j, \qquad a\neq j;\ a,j=1,\dots,n; \tag{12}$$

$$\mathcal{R}_j\phi^{(i)} = \phi^{(i)} - 2G^{ij}\phi^{(j)}, \qquad i,j=1,\dots,n; \tag{13}$$

where $G^{ij} := (\phi^{(i)})^T(U-\lambda)\phi^{(j)}$ is a bilinear form that doesn't depend on λ and $u_1,\dots,u_n$.

Assume that G^{ij} is non-degenerate and denote by G_{ij} the inverse matrix. Note that non-degeneracy of G^{ij} is a property of the Frobenius manifold M which holds generically. In fact the proof of Theorem 4.1 does not require the non-degeneracy of G^{ij}—see Remark 4.2. Consider a special solution of Equation (8) given by $\phi := \sum_{i,j=1}^n G_{ij}\phi^{(j)}$. The main property of this solution is that ϕ has the local behavior

$$\phi_j = \frac{1}{\sqrt{u_j-\lambda}} + O(1) \text{ for } \lambda\to u_j, \qquad j=1,\dots,n; \tag{14}$$

$$\phi_a = \sqrt{u_j-\lambda}\cdot O(1) \text{ for } \lambda\to u_j, \qquad a\neq j; a,j=1,\dots,n. \tag{15}$$

We consider the function $p = p(\lambda,u)$ given by the formula

$$p(\lambda,u) := \frac{\sqrt{2}}{1-d}\phi^T(U-\lambda)\Psi\,\mathbb{1}. \tag{16}$$

This function is analytic in $\mathbb{C}\setminus\cup_{i=1}^n L_i$, with a regular singularity at infinity, and its local behavior for $\lambda\to u_i$ is given by

$$p(\lambda,u) = p(u_i,u) + \Psi_{i,\mathbb{1}}\sqrt{2(u_i-\lambda)} + O(u_i-\lambda), \qquad i=1,\dots,n. \tag{17}$$

The 1-form $d_\lambda p$ has at most a finite number of zeros. We denote them by $r_1, \dots, r_N$ and we assume that they do not belong to the cuts L_i, $i = 1, \dots, n$. Let D be the image of $\mathbb{C} \setminus \cup_{i=1}^n L_i$ under the map $p(\lambda, u)$. This domain has a boundary given by the unfolding of the cuts L_i, $i = 1, \dots, n$. The inverse function $\lambda = \lambda(p, u)$ is a multivalued function on D. Consider the points $p(r_c, u)$, $c = 1, \dots, N$. We glue a finite number of copies of D along the cuts from the points $p(r_c, u)$ to infinity, $c = 1, \dots, N$. In this way we obtain a domain $\hat{D}$, where the function λ is single-valued. This is possible since $p(\lambda, u)$ is analytic as a function of $\lambda \in \mathbb{C} \setminus \cup_{i=1}^n L_i$ with a regular singularity at infinity. In other words, the resulting surface is essentially the surface $\{(\lambda, p) \in (\mathbb{C} \setminus \cup_{i=1}^n L_i) \times \mathbb{C} \mid p = p(\lambda, u)\}$.

We analytically continue the function λ on $\hat{D}$ beyond the boundary. This essentially corresponds to gluing several copies of $\hat{D}$ along the boundaries that are the images of the same cuts on the λ-plane. This is always a correctly defined procedure as this corresponds to simply crossing the cut on the λ-plane. This procedure is not unique as this gluing can happen in various different ways. In any case, we can perform this construction uniformly over a small ball in the space of parameters $u_1, \dots, u_n$. This way we obtain a (not necessarily compact) Riemann surface $\mathcal{D}$, with a function $\lambda = \lambda(\tilde{p}, u) \colon \mathcal{D} \to \mathbb{C}$ (by $\tilde{p}$ we denote some local coordinate on $\mathcal{D}$).

Dubrovin proves in [**13**] that the family of functions $\lambda(\tilde{p}, u)$ defined this way is a superpotential of the Frobenius manifold which was the input of this construction.

2.4. Gromov-Witten theory. Let us introduce Gromov–Witten theory, following, e.g. [**57**].

First, let us introduce moduli spaces of algebraic curves. The moduli space of curves $\mathcal{M}_{g,n}$, $g \geq 0$, $n \geq 0$, $2g - 2 + n > 0$, parametrizes smooth complex curves of genus g with n ordered marked points. It is a smooth complex orbifold of dimension $3g - 3 + n$.

The space $\overline{\mathcal{M}}_{g,n}$ is a compactification of $\mathcal{M}_{g,n}$. It parametrizes stable curves of genus g with n ordered marked points. A stable curve is a possibly reducible curve with possible nodes, such that the order of its automorphism group is finite. Genus of a stable curve is the arithmetic genus, namely, the genus of the smooth curve that we get if the replace each node (given locally by the equation $xy = 0$) with a cylinder (given locally by the equation $xy = \epsilon$). The space $\overline{\mathcal{M}}_{g,n}$ is a smooth compact complex orbifold.

There is a number of natural mappings between the moduli spaces of curves.

First, there are projections $\pi \colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ that forget the last marked point. Note that there is a subtlety related to the fact that when we forget a marked point a stable curve can become unstable.

Second, there is a 2-to-1 mapping $\sigma \colon \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$ whose image is the boundary divisor of irreducible curves with one node.

Third, there are mappings $\rho \colon \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g,n}$, $g_1 + g_2 = g$, $n_1 + n_2 = n$, whose images are the other irreducible boundary divisors of the compactification of $\overline{\mathcal{M}}_{g,n}$.

Let L_i the line bundle over $\overline{\mathcal{M}}_{g,n}$, whose fiber over a point $x \in \overline{\mathcal{M}}_{g,n}$ represented by a curve C_g with marked points $x_1, \dots, x_n$ is equal to $T^*_{x_i} C_g$. Denote by $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n})$ the first Chern class of L_i. ψ_i are referred to as *psi-classes*.

Given a complex manifold X, one can consider $[X_{g,k,\mathrm{deg}}]$, the moduli space of degree deg stable maps to X of genus-g curves with k marked points. There is a natural projection from $[X_{g,k,\deg}]$ to $\overline{\mathcal{M}}_{g,k}$ which consists of forgetting the map to X. Note that there is a subtlety related to the fact that an unstable curve can be mapped to X in a stable way. One can consider pullback of psi-classes to $[X_{g,k,\deg}]$ with respect to this projection. We will denote them as ψ_i too.

Gromov–Witten theory studies the so-called Gromov–Witten invariants (also called Gromov–Witten correlators), defined as follows (left hand side is just a notation):

$$\langle \tau_{d_1}(e_{i_1})\tau_{d_2}(e_{i_2})\cdots\tau_{d_k}(e_{i_k})\rangle_g := \sum_{\deg}\int_{[X_{g,k,\deg}]^{\mathrm{vir}}} ev_1^*(e_{i_1})\psi_1^{d_1}\, ev_2^*(e_{i_2})\psi_2^{d_2}\cdots ev_k^*(e_{i_k})\psi_k^{d_k}, \tag{18}$$

where , ev_i is the evaluation map at the i^{th}.

Gromov–Witten *potential* $\mathcal{F}_g$ of genus g is the following generating function for the correlators ($v^{d,i}$ are formal variables):

$$\mathcal{F}_g = \sum \frac{\langle \tau_{d_1}(e_{i_1})\tau_{d_2}(e_{i_2})\cdots\tau_{d_k}(e_{i_k})\rangle_g}{|\operatorname{Aut}((i_m,d_m)_{m=1}^k)|} v^{d_1,i_1}\cdots v^{d_k,i_k}, \tag{19}$$

where $|\operatorname{Aut}((i_m,d_m)_{m=1}^k)|$ denotes the number of automorphisms of the collection of multi-indices (i_m,d_m) and where the sum is such that it includes each monomial $v^{d_1,i_1}\cdots v^{d_k,i_k}$ exactly once.

It turns out [**48**] that *the genus zero potential without descendants*, i.e.

$$F = \mathcal{F}_0\big|_{v^{d,i}=0,\ d>0} \tag{20}$$

is a Frobenius prepotential.

Partition function of the Gromov–Witten theory is defined as follows:

$$Z = \exp\left(\sum_{g\geq 0}\hbar^{g-1}\mathcal{F}_g\right), \tag{21}$$

where $\hbar$ is a formal parameter.

Celebrated result of Kontsevich–Witten states that the partition function for the Gromov-Witten theory of a point (i.e. for $X=\{\mathrm{pt}\}$) is the tau-function of the KdV integrable hierarchy. We will denote this partition function as Z_{KdV}.

2.5. Cohomological field theory. Here we introduce cohomological field theory, again following [**57**].

Cohomological field theory is a generalization of Gromov-Witten theory where one drops the target complex manifold and takes just some vector space V in place of its cohomology. Roughly speaking, a CohFT is a system of cohomology classes on the moduli spaces of curves with the values in the tensor powers of V, compatible with all natural mappings between the moduli spaces.

The formal definition is the following. We fix a vector space $V=\langle e_1,\dots,e_s\rangle$ (e_1 will play a special role) with a non-degenerate scalar product η. A cohomological field theory is a system of cohomology classes $\alpha_{g,n}\in H^*(\overline{\mathcal{M}}_{g,n},V^{\otimes n})$ satisfying the properties:

(1) $\alpha_{g,n}$ is equivariant with respect to the action of S_n on the labels of marked points and components of $V^{\otimes n}$.

(2) $\sigma^*\alpha_{g,n} = \left(\alpha_{g-1,n+2}, \eta^{-1}\right)$; $\rho^*\alpha_{g,n} = \left(\alpha_{g_1,n_1+1} \cdot \alpha_{g_2,n_2+1}, \eta^{-1}\right)$ (in both cases we contract with the scalar product the two components of V corresponding to the two points in the preimage of the node under normalization).

(3) $(\alpha_{0,3}, e_1 \otimes e_i \otimes e_j) = \eta_{ij}$, $\pi^*\alpha_{g,n} = (\alpha_{g,n+1}, e_1)$ (again, we contract the component of V corresponding to the last marked point with e_1).

Then *correlators* in cohomological field theory are defined as follows:

$$\langle \tau_{d_1}(e_{i_1}) \cdots \tau_{d_n}(e_{i_n})\rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \prod_{j=1}^{n} \psi_j^{d_j} \cdot \left(\alpha_{g,n}, \otimes_{j=1}^{n} e_{i_j}\right) \tag{22}$$

Genus-g potentials and partition function are defined for cohomological field theories in exactly the same way as for Gromov–Witten theories above.

The statement about the genus zero potential without the descendants being a Frobenius prepotential is also true for any cohomological field theory.

2.6. Givental theory. Givental theory [**35**, **36**, **38**] is one of the most important tools in the study of Gromov-Witten invariants of target varieties and general cohomological field theories. It allows, in particular, to obtain explicit relations between the partition functions of different theories, reconstruct higher genera correlators from the genus 0 data, and establish general properties of semi-simple theories.

The core of the theory is Givental's formula that gives a formal Gromov-Witten potential associated to a semi-simple Frobenius structure. Teleman proves [**60**] that the formal Gromov-Witten potential associated to the Frobenius structure of a target variety with semi-simple quantum cohomology coincides with the actual Gromov-Witten potential in all genera.

In order to write Givental formula, let us construct an operator series $R(z) = \sum_{k\geq 0} R_k z^k$ in the following way.

Recursively define the off-diagonal entries of R_k in normalized canonical coordinates by solving the equation

$$\Psi^{-1}\mathrm{d}(\Psi R_{k-1}) = [\mathrm{d}U, R_k]. \tag{23}$$

using $R_0 = \mathbf{I}$ as a base case. Construct the diagonal entries of R_k by integrating the next equation

$$\Psi^{-1}\mathrm{d}(\Psi R_k) = [\mathrm{d}U, R_{k+1}] \tag{24}$$

using the fact that the diagonal entries of $[\mathrm{d}U, R_{k+1}]$ are equal to zero. To fix the integration constant, use the Euler equation

$$R_k = -(i_E \mathrm{d}R_k)/k, \tag{25}$$

where $E = \sum u^i\partial_i$ is the Euler field.

This procedure recursively defines R_k for all k.

Let us consider the following reexpansion:

$$R(z) = \sum_{l=0}^{\infty} R_l z^l = \exp\left(\sum_{l=1}^{\infty} r_l z^l\right). \tag{26}$$

Then we denote by $(r_l z^l)\hat{}$ the following differential operator:

$$(r_l z^l)\hat{} := -(r_l)^i_{\mathbf{1}} \frac{\partial}{\partial v^{l+1,i}} + \sum_{d=0}^{\infty} v^{d,i} (r_l)^j_i \frac{\partial}{\partial v^{d+l,j}} + \frac{\hbar}{2} \sum_{m=0}^{l-1} (-1)^{m+1} (r_l)^{i,j} \frac{\partial^2}{\partial v^{m,i} \partial v^{l-1-m,j}}. \tag{27}$$

Here the indices $i, j \in \{1, \ldots, N\}$ on r_l correspond to the basis $\{e_1, \ldots, e_N\}$ of V, and the index $\mathbf{1}$ corresponds to the unit vector $e_{\mathbf{1}}$. When we write r_l with two upper-indices we mean as usual that we raise one of the indices using the scalar product η.

The quantization $\hat{R}$ of series $R(z)$ is then defined by

$$\hat{R} = \exp\left(\sum_{l=1}^{\infty} \left((-1)^l r_l z^l \right)\hat{} \right). \tag{28}$$

Givental formula then allows to reproduce all ancestor correlators of the given (homogenous) cohomological field theory in the following way:

$$Z = \hat{\Psi} \hat{R} \hat{\Delta} \mathcal{T}, \tag{29}$$

where

$$\mathcal{T} := Z_{\mathrm{KdV}}(\{u^{d,1}\}) \cdots Z_{\mathrm{KdV}}(\{u^{d,N}\});$$

$\hat{\Delta}$ replaces the variables of i^{th} KdV τ-function according to $u^{d,i} = \Delta_i^{1/2} v^{d,i}$ and replaces $\hbar$ with $\Delta_i \hbar$, while $\hat{\Psi}$ is the change of variables $v^{d,i} = \Psi^i_\nu t^{d,\nu}$.

2.7. Spectral curve topological recursion. The theory developed by Eynard and Orantin (see [**8**, **27**, **29**]), is a procedure, called spectral curve topological recursion, that takes the following objects as input. First, a particular Riemann surface, which is called the *spectral curve*. Second, two functions x and y on this surface, and, third, a choice of a bi-differential on this surface, which we will call the *two-point function* (which also is often called Bergman kernel). And, occasionally, a particular extra choice of a coordinate on an open part of the Riemann surface. The output of the topological recursion is a set of n-forms $\omega_{g,n}$, whose expansion in this additional coordinate generates interesting numbers.

In some cases these numbers are correlators of a matrix model (that was the original motivation for introducing the topological recursion; it is a natural generalization of the reconstruction procedure for the correlators of a certain class of matrix models, see, e.g. [**3**]), in some other cases they appear to be related to Gromov-Witten theory and to various intersection numbers on the moduli space of curves.

Note that this spectral curve topological recursion is unrelated to the topological recursion occurring in the theory of moduli spaces of curves.

One of the ways to think about the input data of the topological recursion theory is to say that the $(g, n) = (0, 1)$ part of a partition function in some geometrically motivated theory determines the spectral curve; the $(g, n) = (0, 2)$ part of a partition function determines the two-point function, and the rest of the correlators can be reconstructed from these two via topological recursion, in terms of a proper expansion of $\omega_{g,n}$ (see [**16**]).

The topological recursion theory is often used to reproduce known partition functions, extracts from them some higher genus correlators which were up to now unreachable and gives new non-trivial relations for the correlators, see e. g. [**31**].

Local version of the spectral curve topological recursion is defined as follows.

For $N \in \mathbb{N}^*$, we call *times* a set of N families of complex numbers $\left\{h_k^i\right\}_{k\in\mathbb{N}}$ for $i = 1,\dots,N$ and *jumps* another set of $N \times N$ infinite families of complex numbers $\left\{B_{k,l}^{i,j}\right\}_{(k,l)\in\mathbb{N}^2}$ for $i,j = 1,\dots,N$. We finally define a set of canonical coordinates $\{a_i\}_{i=1}^N \in \mathbb{C}^N$ subject to $a_i \neq a_j$ for $i \neq j$.

Definition 2.5. For all $i,j \in \{1,\dots,N\}$, we define the following set of analytic functions and differential forms in a neighborhood of $0 \in \mathbb{C}$:

$$x^i(z) := z^2 + a_i\,, \quad y^i(z) := \sum_{k=0}^{\infty} h_k^i z^k \tag{30}$$

and

$$B^{i,j}(z,z') = \delta_{i,j}\frac{dz\otimes dz'}{(z-z')^2} + \sum_{k,l=0}^{\infty} B_{k,l}^{i,j} z^k z'^l dz\otimes dz'. \tag{31}$$

For $2g-2+n > 0$, we define the genus g, n-point correlation functions $\omega_{g,n}^{i_1,\dots,i_n}(z_1,\dots z_n)$ recursively by

$$\omega_{g,n+1}^{i_0,i_1,\dots,i_n}(z_0,z_1,\dots,z_n) := \sum_{j=1}^{N} \operatorname*{Res}_{z\to 0} \frac{\int_{-z}^{z} B^{i_0,j}(z_0,\cdot)}{2\left(y^j(z)-y^j(-z)\right)\mathrm{d}x^j(z)} \times \tag{32}$$
$$\left(\omega_{g-1,n+2}^{j,j,i_1,\dots,i_n}(z,-z,z_1,\dots,z_n) + \sum_{A\cup B=\{1,\dots,n\}} \sum_{h=0}^{g} \omega_{h,|A|+1}^{j,\mathbf{i}_A}(z,\mathbf{z}_A)\omega_{g-h,|B|+1}^{j,\mathbf{i}_B}(-z,\mathbf{z}_B)\right),$$

where for any set A, we denote by $\mathbf{z}_A$ (resp., $\mathbf{i}_A$) the set $\{z_k\}_{k\in A}$ (resp., $\{i_k\}_{k\in A}$), and where the base of the recursion is given by

$$\omega_{0,1}^i(z) := 0; \qquad \omega_{0,2}^{i,j}(z,z') := B^{i,j}(z,z'). \tag{33}$$

For the global version we have the following.

Consider a Riemann surface Σ with meromorphic functions $x,y\colon \Sigma\to\mathbb{C}$ such that x has a finite number of critical points, $c_1,\dots,c_n$, and y is holomorphic near these points with a non-vanishing derivative. Let B be a symmetric bi-differential on $\Sigma\times\Sigma$, with a double pole on the diagonal, the double residue equal to 1, and no further singularities.

We define a sequence of symmetric n-forms $\omega_{g,k}(z_1,\dots,z_k)$ on $\Sigma^{\times k}$, known as *correlation differentials* for the spectral curve, by the following recursion:

$$\omega_{0,1}(z) := y(z)dx(z); \tag{34}$$

$$\omega_{0,2}(z_1,z_2) := B(z_1,z_2); \tag{35}$$

$$\omega_{g,k+1}(z_0,z_1,\dots,z_k) := \tag{36}$$
$$\sum_{i=1}^{n} \operatorname*{Res}_{z\to c_i} \frac{\int_z^{\sigma_i(z)} \omega_{0,2}(\bullet,z_0)}{2(\omega_{0,1}(\sigma_i(z)) - \omega_{0,1}(z))} \tilde\omega_{g,2|k}(z,\sigma_i(z)|z_1,\dots,z_k),$$

where σ_i is the deck transformation for the function x near the point c_i, $i = 1, \ldots, n$, and $\tilde{\omega}_{g,2|k}$ is defined by the following formula:

$$\tilde{\omega}_{g,2|k}(z', z''|z_1, \ldots, z_k) := \omega_{g-1,k+2}(z', z'', z_1, \ldots, z_k) + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{1,\ldots,k\} \\ 2g_1-1+|I_1|\geq 0 \\ 2g_2-1+|I_2|\geq 0}} \omega_{g_1,|I_1|+1}(z', z_{I_1})\omega_{g_2,|I_2|+1}(z'', z_{I_2}). \tag{37}$$

Here we denote by z_I the sequence $z_{i_1}, \ldots, z_{i_{|I|}}$ for $I = \{i_1, \ldots, i_{|I|}\}$.

REMARK 2.6. In the global recursion we also allow y to be the (multivalued) primitive of a differential ω on Σ. The ambiguity in y consists of periods and residues of ω and hence the ambiguity is locally constant. Since y appears in the recursion only via $y(\sigma_i(z)) - y(z)$ (and there are no poles of ω at the zeros of dx) the locally constant ambiguity disappears and the recursion is well-defined.

3. Local spectral curves for cohomological field theories

In this section we establish the identification of Givental formula and the local spectral curve topological recursion procedure with an appropriate choice of initial data. Then, as a corollary, we prove the Norbury–Scott conjecture on the spectral curve topological recursion for the Gromov–Witten theory of $\mathbb{C}\mathrm{P}^1$.

This section deals with the Givental theory and the spectral curve topological recursion theory; for more information on both theories we refer to [**29**, **46**, **57**] as possible sources. In Section 3.1 we recall the Givental theory, and present the Givental formula as a sum over graphs. In Section 3.3 we do the same for the topological recursion. In Section 3.4 we prove the theorem on identification of the two theories and provide a corresponding dictionary. In Section 3.5 we provide the computations showing that this identification works for the spectral curve proposed by Norbury and Scott for the Gromov-Witten theory of $\mathbb{C}\mathrm{P}^1$.

3.1. Givental group action as a sum over graphs. In this section we review the Givental group action and we remind the reader how it can be used to write the partition function of an N-dimensional semi-simple cohomological field theory as an operator acting on the product of N KdV τ-functions. Using this, we write the partition function for such a cohomological field theory as a sum over decorated graphs.

3.1.1. *Givental group action.* We remind the reader of the original formulation, due to Y.-P. Lee, of the infinitesimal Givental group action in terms of differential operators [**43**–**45**].

Consider the space of partition functions for N-dimensional cohomological field theories

$$Z = \exp\left(\sum_{g\geq 0} \hbar^{g-1}\mathcal{F}_g\right) \tag{38}$$

in variables $v^{d,i}$, $d \geq 0$, $i = 1, \ldots, N$. There is a fixed scalar product $\eta_{ij} = \delta_{ij}$ on the vector space $V := \langle e_1, \ldots, e_N\rangle$ of primary fields corresponding to the indices $i = 1, \ldots, N$. Furthermore, we will denote by $e_{\mathbf{1}}$ the vector in V that plays the role of the unit.

Later on we will also use the so-called *correlators*

$$\langle \tau_{d_1}(e_{i_1})\tau_{d_2}(e_{i_2})\cdots\tau_{d_k}(e_{i_k})\rangle_g$$

which correspond to the coefficients of formal power series $\mathcal{F}_g$ in the following way:

$$\mathcal{F}_g = \sum \frac{\langle \tau_{d_1}(e_{i_1})\tau_{d_2}(e_{i_2})\cdots\tau_{d_k}(e_{i_k})\rangle_g}{|\operatorname{Aut}((i_m,d_m)_{m=1}^k)|} v^{d_1,i_1}\cdots v^{d_k,i_k}, \tag{39}$$

where $|\operatorname{Aut}((i_m,d_m)_{m=1}^k)|$ denotes the number of automorphisms of the collection of multi-indices (i_m,d_m) and where the sum is such that it includes each monomial $v^{d_1,i_1}\cdots v^{d_k,i_k}$ exactly once. Note that in the special case of a Gromov-Witten theory for some manifold X, these correlators carry the following meaning:
(40)

$$\langle \tau_{d_1}(e_{i_1})\tau_{d_2}(e_{i_2})\cdots\tau_{d_k}(e_{i_k})\rangle_g = \sum_{\deg}\int_{[X_{g,k,\deg}]} ev_1^*(e_{i_1})\psi_1^{d_1}\, ev_2^*(e_{i_2})\psi_2^{d_2}\cdots ev_k^*(e_{i_k})\psi_k^{d_k},$$

where $[X_{g,k,\deg}]$ is the moduli space of degree deg stable maps to X of genus-g curves with k marked points, ev_i is the evaluation map at the i^{th} point and ψ correspond to ψ-classes.

Consider a sequence of operators $r_l \in \operatorname{Hom}(V,V)$ for $l \geq 1$, such that the operators with odd (resp., even) indices are symmetric (resp., skew-symmetric). Then we denote by $(r_l z^l)\hat{}$ the following differential operator:

$$(r_l z^l)\hat{} := -(r_l)_{\mathbf{1}}^i \frac{\partial}{\partial v^{l+1,i}} + \sum_{d=0}^{\infty} v^{d,i}(r_l)_i^j \frac{\partial}{\partial v^{d+l,j}} + \frac{\hbar}{2}\sum_{m=0}^{l-1}(-1)^{m+1}(r_l)^{i,j}\frac{\partial^2}{\partial v^{m,i}\partial v^{l-1-m,j}}.$$

Here the indices $i,j \in \{1,\dots,N\}$ on r_l correspond to the basis $\{e_1,\dots,e_N\}$ of V, and the index $\mathbf{1}$ corresponds to the unit vector $e_{\mathbf{1}}$. When we write r_l with two upper-indices we mean as usual that we raise one of the indices using the scalar product η.

Given such a sequence of operators r_l, we define an operator series $R(z)$ in the following way

$$R(z) = \sum_{l=0}^{\infty} R_l z^l := \exp\left(\sum_{l=1}^{\infty} r_l z^l\right). \tag{41}$$

The quantization $\hat{R}$ of this series is defined by

$$\hat{R} = \exp\left(\sum_{l=1}^{\infty}\left((-1)^l r_l z^l\right)\hat{}\,\right). \tag{42}$$

Givental observed that the action of such operators $\hat{R}$ on formal power series Z for which the number of ψ-classes (given by the first index of $v^{d,\mu}$) at any monomial of degree n is no more than $3g-3+n$, is well-defined. The main theorem of [**32**] states that this action preserves the property that Z is a generating function of the correlators of a cohomological field theory with target space (V,η) (see also [**40**,**60**]).

Remark 3.1. Note that in order to agree with the conventions of Givental, we label in a matrix by the upper index the column and by the lower index the row.

3.1.2. *Givental operator for a Frobenius manifold.* Let $Z(\{t^{d,\mu}\})$ be the partition function of some N-dimensional semi-simple conformal cohomological field theory. We recall the construction (due to Givental [**35**, **36**, **38**], see also Dubrovin [**13**]) of an operator series $R(z)$ as in the previous section whose quantization takes the product of N KdV τ-functions to Z.

Let F be the restriction of $\log(Z)$ to the genus zero part without descendants. Denote $t^\mu := t^{0,\mu}$. Then F can be interpreted as a formal Frobenius manifold with metric

$$\eta_{\alpha\beta} = \frac{\partial^3 F}{\partial t^{\mathbf{1}} \partial t^\alpha \partial t^\beta} \tag{43}$$

and Frobenius algebra structure $c_{\alpha\beta}^\gamma$

$$c_{\alpha\beta\gamma} = \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma}. \tag{44}$$

We can assume that $\eta_{\alpha\beta} = \delta_{\alpha+\beta,n+1}$ and $e_{\mathbf{1}} = e_1$. According to [**12**] it is always possible by an appropriate choice of these flat coordinates t^μ.

3.1.3. *Canonical coordinates.* Another set of coordinates is given by the *canonical coordinates* $\{u^i\}$ which can be found as solutions to Equation (3.54) from [**12**], and have the property that $\{\partial_i := \partial/\partial u^i\}$ forms a basis of canonical idempotents of the Frobenius algebra product. In these coordinates the metric is diagonal and the unit vector field is given by $e_{\mathbf{1}} = \partial_1 + \cdots + \partial_N$.

Define $\Delta_i := 1/(\partial_i, \partial_i)$ to be the inverse of the square of the length of the i^{th} canonical basis element, and call $\{\partial/\partial v^i := \Delta_i^{1/2} \partial/\partial u^i\}$ the normalized canonical basis in the tangent space. We denote the coordinates corresponding to this basis by v^i, and the formal variables corresponding to these coordinates by $v^{d,i}$. They are precisely the formal variables $v^{d,i}$ appearing in the previous section.

Let U be the matrix of canonical coordinates $U = \operatorname{diag}(u^1, \ldots, u^N)$ and denote by Ψ the transition matrix from the flat to the normalized canonical bases. That is, denoting $\mathrm{d}t = (\mathrm{d}t^1, \ldots, \mathrm{d}t^N)^{\mathrm{T}}$ and $\mathrm{d}u = (\mathrm{d}u^1, \ldots, \mathrm{d}u^N)^{\mathrm{T}}$, one has

$$\Delta^{-1/2} \mathrm{d}u = \Psi \mathrm{d}t, \tag{45}$$

where $\Delta = \operatorname{diag}(\Delta_1, \ldots, \Delta_N)$.

Remark 3.2. Note that Ψ obtained with the help of the definition above depends on the point p of the Frobenius manifold.

3.1.4. *Recursion.* Construct an operator series $R(z) = \sum_{k\geq 0} R_k z^k$ as in the previous section in the following way.

Recursively define the off-diagonal entries of R_k in normalized canonical coordinates by solving the equation

$$\Psi^{-1} \mathrm{d}(\Psi R_{k-1}) = [\mathrm{d}U, R_k]. \tag{46}$$

using $R_0 = \mathbf{I}$ as a base case. Construct the diagonal entries of R_k by integrating the next equation

$$\Psi^{-1} \mathrm{d}(\Psi R_k) = [\mathrm{d}U, R_{k+1}] \tag{47}$$

using the fact that the diagonal entries of $[\mathrm{d}U, R_{k+1}]$ are equal to zero. To fix the integration constant, use the Euler equation

$$R_k = -(i_E \mathrm{d}R_k)/k, \tag{48}$$

where $E = \sum u^i \partial_i$ is the Euler field (here we use the fact that we started with a conformal cohomological field theory).

This procedure recursively defines R_k for all k. The following proposition is essentially proved in Givental's papers [**35**, **36**].

PROPOSITION 3.3. *Let F be a local N-dimensional Frobenius manifold structure, semisimple at the origin, and let (R_k) be the series of operators constructed from this F by the recursive procedure described above, at the origin. Let Ψ and Δ be as above, taken at the origin as well. Then we have the following formula:*

$$\mathcal{F}_0 = \operatorname*{Res}_{\hbar=0} d\hbar \cdot \log \hat{\Psi}\hat{R}\hat{\Delta}\mathcal{T}. \tag{49}$$

Here $\mathcal{F}_0 = \mathcal{F}_0(\{t^{d,\mu}\})$ is the genus 0 descendant potential of cohomological field theory associated to F; $\mathcal{T}$ is the product of N KdV tau-functions,

$$\mathcal{T} := Z_{\mathrm{KdV}}(\{u^{d,1}\}) \cdots Z_{\mathrm{KdV}}(\{u^{d,N}\});$$

$\hat{\Delta}$ replaces the variables of i^{th} KdV τ-function according to $u^{d,i} = \Delta_i^{1/2} v^{d,i}$ and replaces $\hbar$ with $\Delta_i \hbar$, while $\hat{\Psi}$ is the change of variables $v^{d,i} = \Psi^i_\nu t^{d,\nu}$. The unit for the R-action is given by $(\Psi^1_1, \ldots, \Psi^N_1)$.

REMARK 3.4. In fact, using Teleman's result in [**60**], one has a refined version of Equation (49):

$$Z = \hat{\Psi}\hat{R}\hat{\Delta}\mathcal{T}. \tag{50}$$

Note that it holds for cohomological field theories. In the Gromov-Witten case, when quadratic terms in the potential cannot be neglected, there appears an additional complication, see the next remark below.

REMARK 3.5. Givental's formula [**36**] for a Gromov-Witten total descendant potential (without the $(g = 1, n = 0)$-term),

$$Z = \hat{S}^{-1}\hat{\Psi}\hat{R}\hat{\Delta}\mathcal{T}, \tag{51}$$

also includes the operator $\hat{S}$, given by

$$\hat{S} = \exp\left(\sum_{l=1}^{\infty} (s_l z^{-l})\hat{}\ \right), \tag{52}$$

where the operators $(s_l z^{-l})\hat{}$ are defined in the following way (see, e. g., [**32**, Section 4.2]):

$$\begin{aligned} \sum_{l=1}^{\infty} (s_l z^{-l})\hat{} = &- (s_1)^\mu_1 \frac{\partial}{\partial t^{0,\mu}} + \frac{1}{\hbar} \sum_{d=0}^{\infty} (s_{d+2})_{1,\mu}\, t^{d,\mu} \\ &+ \sum_{\substack{d=0 \\ l=1}}^{\infty} (s_l)^\mu_\nu\, t^{d+l,\nu} \frac{\partial}{\partial t^{d,\mu}} + \frac{1}{2\hbar} \sum_{\substack{d_1,d_2 \\ \mu_2,\mu_2}} (-1)^{d_1} (s_{d_1+d_2+1})_{\mu_1,\mu_2}\, t^{d_1,\mu_1} t^{d_2,\mu_2}. \end{aligned} \tag{53}$$

Note that formula (52) for the quantization of S differs from the analogous formula (42) for R by a factor of $(-1)^l$ in the exponent, which agrees with the definition in Givental's papers [**35**, **36**].

The matrices s_k are defined through the following relation:

$$S(z) = \sum_{k=0}^{\infty} S_k z^{-k} = \exp\left(\sum_{l=0}^{\infty} s_l z^{-l}\right), \tag{54}$$

where for $S(z)$, taken at a point p of the Frobenius manifold, we have (see [**36**]), for any points a and b of the Frobenius manifold,

$$(a, b\, S_p) := (a,b) + \sum_{k=0}^{\infty} \langle \tau_0(a) \exp\left(\tau_0(p)\right) \tau_k(b)\rangle_0\, z^{-1-k}. \tag{55}$$

Here on the left hand side the brackets stand for the scalar product on the tangent space to the Frobenius manifold at p, and we used an identification of the tangent space with the whole Frobenius manifold, since in this case the Frobenius manifold is itself a vector space. If p is the origin, we have just

$$(a, b\, S) := (a,b) + \sum_{k=0}^{\infty} \langle \tau_0(a)\tau_k(b)\rangle \;\; z^{-1-k}. \tag{56}$$

Note that this S action is defined in the general case when the total descendant genus 0 potential is known. For the case when only a Frobenius potential is specified, the choice of S is then called a *calibration* of the Frobenius manifold, see [**13, 35**] for related details. In the case of cohomological field theory when we disregard quadratic terms, the S action is trivial if p is taken to be the origin.

It turns out that in most of the relevant cases, e.g. for the Gromov-Witten theory of $\mathbb{C}\mathrm{P}^1$ (see section 3.5.1 below), the only relevant term in equation (53) is

$$\sum_{\substack{d=0\\ l=1}}^{\infty} (s_l)^{\mu}_{\nu}\, t^{d+l,\nu} \frac{\partial}{\partial t^{d,\mu}},$$

since $(s_1)^{\mu}_1$ vanishes and all other terms just change the unstable terms in the potential.

This means, that in these cases $\hat{S}^{-1}$ just performs a linear change of formal variables $t^{d,\mu}$ in the following way:

$$t^{d,\mu} \mapsto \sum_{m=d}^{\infty} (S_{m-d})^{\mu}_{\nu} t^{m,\nu}. \tag{57}$$

3.2. Expressions in terms of graphs.

NOTATION 3.6. Let γ be any graph. By a half-edge we mean either a leaf or an edge together with a choice of one of the two vertices it is attached to. By $V(\gamma)$, $E(\gamma)$, $H(\gamma)$ and $L(\gamma)$ we denote the sets of vertices, edges, half-edges and leaves of γ. For any vertex v of γ, denote by $H(v)$ the set of half-edges connected to v.

Let $\widetilde{\Gamma}$ be the set of all connected graphs γ together with a choice of disjoint splitting $L(\gamma) = L^*(\gamma) \coprod L^\bullet(\gamma)$, a labelling of the vertices by pairs $(g,i) \in \mathbb{Z}_{\geq 0} \times \{1,\dots,N\}$ and a labelling of the elements of $H(\gamma)$ by non-negative integers, such that the label of a leaf in $L^\bullet$ is always greater than one. The elements of $L^*(\gamma)$ are called *ordinary leaves*, the elements of $L^\bullet$ are called *dilaton leaves*. We denote by Γ the subset of all graphs in $\widetilde{\Gamma}$ that are *stable*; that is, any vertex labelled $(0,i)$ for some i is of valence at least three.

For any graph γ denote by $\mathfrak{g}\colon V(\gamma) \to \mathbb{Z}_{\geq 0}$ and $\mathfrak{i}\colon V(\gamma) \to \{1,\dots,N\}$ the maps that associate to any vertex its first and second label respectively, and by $\mathfrak{k}\colon H(\gamma) \to \mathbb{Z}_{\geq 0}$ the map that associates to any half-edge its label. Denote by $\mathfrak{v}\colon L(\gamma) \to V(\gamma)$ the map that associates to each leaf the corresponding vertex, and by $\mathfrak{v}_1, \mathfrak{v}_2\colon E(\gamma) \to V(\gamma)$ and by $\mathfrak{h}_1, \mathfrak{h}_2\colon E(\gamma) \to H(\gamma)$ the maps that associate to an edge the first and second vertex, and the corresponding half-edges respectively.

REMARK 3.7. The labels introduced above are used to keep track of different data for the trivial cohomological field theory; g is for the genus, i for the primary field in canonical coordinates and the labelling of the marked half-edges is for the power of ψ-class.

REMARK 3.8. Edges of a graph in Γ are considered to be oriented (this allows to define the maps $\mathfrak{v}_1$ and $\mathfrak{v}_2$ unambiguously); the final result does not depend on the orientation.

Let $R(z)^i_j$ be the components of the operator series $R(z)$ in normalized canonical basis as computed in Section 3.1.4. To each part of a graph $\gamma \in \Gamma$ we assign some polynomial in formal variables $\hbar$ and $v^{d,i}$. Here $\hbar$ is used to keep track of the genus, while the first index of $v^{d,i}$ keeps track of the number of ψ-classes and the second index keeps track of the normalized canonical coordinate.

3.2.1. *Leaves.* To each ordinary leaf $l \in L^*$ marked by k attached to a vertex marked by the pair (g, i), we assign

$$(\mathcal{L}^*)^i_k(l) := [z^k] \left(\sum_{d \geq 0} \left((R(-z))^i_j v^{d,j} z^d \right) \right), \tag{58}$$

which corresponds to the second term in (41).

To a dilaton leaf $\lambda \in L^\bullet(\gamma)$ marked by k attached to a vertex marked by (g, i) we assign

$$(\mathcal{L}^\bullet)^i_k(\lambda) := [z^{k-1}] \left(-(R(-z))^i_{\mathbf{1}} \right), \tag{59}$$

which corresponds to the first term in (41), which is called the *dilaton shift.*

3.2.2. *Edges.* To an edge e connecting a vertex v_1 marked by (g_1, i_1) to a vertex v_2 marked by (g_2, i_2) and with markings k_1 and k_2 at the corresponding half-edges, we assign

$$\mathcal{E}^{i_1,i_2}_{k_1,k_2}(e) := [z^{k_1} w^{k_2}] \left(\hbar \cdot \frac{\delta^{i_1 i_2} - \sum_s (R(-z))^{i_1}_s (R(-w))^{i_2}_s}{z+w} \right). \tag{60}$$

Note that this does not depend on the choice of ordering of the vertices and that it follows from the fact that $R(z)$ can be written as $R(z) = \exp(\sum r_l z^l)$ that the numerator on the right-hand side is equal to the product of $(z + w)$ with some power series in z and w, so this definition makes sense.

3.2.3. *Vertices.* Let v be a vertex marked by (g, i) with n half-edges attached to it (this includes all ordinary and dilaton leaves and also half-edges that are parts of internal edges) labelled by $k_1, \ldots, k_n$. Then we assign to v the following expression:

$$\mathcal{V}^{(g,i)}_{\{k_1,\ldots,k_n\}}(v) := \hbar^{g-1} (\Delta_i)^{\frac{1}{2}(2g-2+n)} \int_{\bar{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n}. \tag{61}$$

3.2.4. *Z as a sum over graphs.* We recover the partition function Z of the cohomological field theory we started with as a sum over Γ:

$$(62)\quad (\hat{R}\hat{\Delta}\mathcal{T})(\{v^{d,j}\}) = \sum_{\gamma\in\Gamma} \frac{1}{|\operatorname{Aut}(\gamma)|} \prod_{v\in V(\Gamma)} \hbar^{\mathfrak{g}(v)-1} (\Delta_{\mathfrak{i}(v)})^{\frac{1}{2}(2\mathfrak{g}(v)-2+\mathrm{val}(v))} \left\langle \prod_{h\in H(v)} \tau_{\mathfrak{k}(h)} \right\rangle_{\mathfrak{g}} \prod_{e\in E(\gamma)} \mathcal{E}^{\mathfrak{i}(\mathfrak{v}_1(e)),\mathfrak{i}(\mathfrak{v}_2(e))}_{\mathfrak{k}(\mathfrak{h}_1(e)),\mathfrak{k}(\mathfrak{h}_2(e))}(e) \prod_{l\in L^*(\gamma)} (\mathcal{L}^*)^{\mathfrak{i}(\mathfrak{v}(l))}_{\mathfrak{k}(l)}(l) \prod_{\lambda\in L^\bullet(\gamma)} (\mathcal{L}^\bullet)^{\mathfrak{i}(\mathfrak{v}(l))}_{\mathfrak{k}(l)}(\lambda).$$

3.3. Topological recursion. In this section, we define a local version of the topological recursion and write the corresponding invariants as a sum over graphs, which allows us to compare it to the Givental action in the next section.

3.3.1. *Local topological recursion.* We define a local version of the topological recursion in the following way. The term local refers to the fact that the data are all defined locally around the canonical coordinates without any reference to the possible existence of a global manifold where these functions can be defined.

DEFINITION 3.9. For $N \in \mathbb{N}^*$, we call times a set of N families of complex numbers $\left\{h^i_k\right\}_{k\in\mathbb{N}}$ for $i = 1,\ldots,N$ and jumps another set of $N\times N$ infinite families of complex numbers $\left\{B^{i,j}_{k,l}\right\}_{(k,l)\in\mathbb{N}^2}$ for $i,j = 1,\ldots,N$. We finally define a set of canonical coordinates $\{a_i\}_{i=1}^N \in \mathbb{C}^N$ subject to $a_i \neq a_j$ for $i\neq j$.

For all $i,j\in\{1,\ldots,N\}$, we define the following set of analytic functions and differential forms in a neighborhood of $0\in\mathbb{C}$:

$$(63)\qquad x^i(z) := z^2 + a_i\,, \quad y^i(z) := \sum_{k=0}^{\infty} h^i_k z^k$$

and

$$(64)\qquad B^{i,j}(z,z') = \delta_{i,j}\frac{dz\otimes dz'}{(z-z')^2} + \sum_{k,l=0}^{\infty} B^{i,j}_{k,l} z^k z'^l dz\otimes dz'.$$

For $2g-2+n>0$, we define the genus g, n-point correlation functions $\omega^{i_1,\ldots,i_n}_{g,n}(z_1,\ldots z_n)$ recursively by

$$(65)\quad \omega^{i_0,i_1,\ldots,i_n}_{g,n+1}(z_0,z_1,\ldots,z_n) := \sum_{j=1}^{N} \operatorname*{Res}_{z\to 0} \frac{\int_{-z}^{z} B^{i_0,j}(z_0,\cdot)}{2\,(y^j(z)-y^j(-z))\,\mathrm{d}x^j(z)} \times \left(\omega^{j,j,i_1,\ldots,i_n}_{g-1,n+2}(z,-z,z_1,\ldots,z_n) + \sum_{A\cup B=\{1,\ldots,n\}} \sum_{h=0}^{g} \omega^{j,\mathbf{i}_A}_{h,|A|+1}(z,\mathbf{z}_A)\omega^{j,\mathbf{i}_B}_{g-h,|B|+1}(-z,\mathbf{z}_B)\right),$$

where for any set A, we denote by $\mathbf{z}_A$ (resp., $\mathbf{i}_A$) the set $\{z_k\}_{k\in A}$ (resp., $\{i_k\}_{k\in A}$), and where the base of the recursion is given by

$$(66)\qquad \omega^i_{0,1}(z) := 0; \qquad \omega^{i,j}_{0,2}(z,z') := B^{i,j}(z,z').$$

For convenience, in the sequel we denote

$$(67)\qquad K^{i,j}(z,z') = \frac{\int_{-z}^{z} B^{i,j}(z',\cdot)}{2(y^j(z)-y^j(-z))dx^j(z)}$$

and

$$\omega_{g,n}(\vec{z}) = \sum_{\vec{i}} \omega_{g,n}^{\vec{i}}(\vec{z}) \; , \tag{68}$$

where the length of $\vec{z}$ and $\vec{i}$ is n.

3.3.2. *Correlation functions and intersection numbers.* The correlation functions built by this topological recursion can actually be written in terms of intersection of ψ classes on the moduli space of Riemann surfaces. This result is a slight generalization of [**24**, **25**] to the local topological recursion.

3.3.3. *One-branch point case.* The link between the topological recursion formalism and intersection numbers on the moduli space of Riemann surfaces comes from the application of this formalism to the Airy curve. This case corresponds to $N = 1$ and:

$$x(z) = z^2 + a\,, \quad y(z) = z \quad \text{and} \quad B(z,z') = \frac{dz \otimes dz'}{(z-z')^2}. \tag{69}$$

Remark 3.10. Since there is only one branch point in this case, i.e. $N = 1$, we omit the superscript indicating which branch point we consider in the notations of this section.

For further convenience, we introduce two additional parameters by considering the curve

$$x(z) = z^2 + a\,, \quad y(z) = \alpha z \quad \text{and} \quad B(z,z') = \beta \frac{dz \otimes dz'}{(z-z')^2}, \tag{70}$$

the usual Airy curve being $\alpha = \beta = 1$. In this case, the topological recursion reads

$$\omega_{g,n+1}(z_0, z_1, \ldots, z_n) := \operatorname*{Res}_{z \to 0} \frac{\beta}{2\alpha} \frac{dz_0}{2z\, dz} \frac{1}{(z_0^2 - z^2)} \times \tag{71}$$
$$\left(\omega_{g-1,n+2}(z,-z,z_1,\ldots,z_n) + \sum_{A \cup B = \{1,\ldots,n\}} \sum_{h=0}^{g} \omega_{h,|A|+1}(z, \mathbf{z}_A) \omega_{g-h,|B|+1}(-z, \mathbf{z}_B) \right)$$

and one has

Lemma 3.11. *The correlation functions of the Airy curve can be expressed in terms of intersection numbers:*

$$\omega_{g,n}(z_1,\ldots,z_n) = \tag{72}$$
$$\left(-\frac{\beta}{2\alpha}\right)^{2g+n-2} \beta^{g+n-1} \sum_{\alpha_1,\ldots,\alpha_n \geq 0} \langle \tau_{a_1} \ldots \tau_{a_n} \rangle_{g,n} \prod_{i=1}^{n} \frac{(2\alpha_i + 1)!!\, dz_i}{z_i^{2\alpha_i+2}}.$$

This lemma was proved many times by direct computation [**23**, **24**, **29**, **62**], matching the topological recursion with the recursive definition of the intersection numbers.

As a side note, the first few correlation functions are

$$\omega_{0.3}(z_1,z_2,z_3) = -\frac{\beta^3}{2\alpha} \prod_{i=1}^{3} \frac{dz_i}{z_i^2}, \tag{73}$$

$$\omega_{0,4}(z_1,z_2,z_3,z_4) = \frac{\beta^5}{4\alpha^2}\prod_{i=1}^{4}\frac{dz_i}{z_i^2}\sum_{i=1}^{4}\frac{3}{z_i^2}, \tag{74}$$

$$\omega_{1,1}(z) = \frac{-\beta^2}{2\alpha}\frac{dz}{8z^4} \tag{75}$$

and

$$\omega_{1,2}(z_1,z_2) = \frac{\beta^4}{4\alpha^2}\frac{dz_1\,dz_2}{8z_1^2z_2^2}\left(\frac{5}{z_1^4}+\frac{5}{z_2^4}+\frac{3}{z_1^2z_2^2}\right). \tag{76}$$

REMARK 3.12. It is important to remark that there exist different conventions in the literature for defining the topological recursion, mainly differing by a change of sign of the recursion kernel. The latter can be recovered by a change of sign $\alpha \to -\alpha$.

Let us now consider a deformation of the Airy curve which we will refer to as the KdV curve in the following. It has only one branch point, $N = 1$, and reads

$$\begin{cases} x(z) = z^2 + a_i \\ y(z) = \alpha\sum_{k=1}^{\infty} h_k z^k \\ B(z,z') = \beta B_{\mathrm{KdV}}(z,z') = \beta dz\otimes dz'(z-z')^2 \end{cases}. \tag{77}$$

The corresponding correlation functions can also be expressed in terms intersection numbers as follows:

LEMMA 3.13. *The correlation functions of the KdV curve read:*

$$\omega_{g,n}(z_1,\dots,z_n) = \left(-\frac{\beta}{2\alpha h_1}\right)^{2g+n-2}\beta^{g+n-1}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!} \tag{78}$$
$$\sum_{\vec{\alpha}\in\mathbb{N}^{*m}}\prod_{k=1}^{m}(2\alpha_k+1)!!\frac{h_{2\alpha_k+1}}{h_1}\prod_{i=1}^{n}\frac{(2d_i+1)!!\,dz_i}{z_i^{2d_i+2}}\left\langle\prod_{j=1}^{n}\tau_{d_j}\prod_{k=1}^{m}\tau_{\alpha_k+1}\right\rangle_{g,n+m}.$$

PROOF. Once again the proof can be found in the literature [**23**, **24**, **28**]. However, let us study a graphical interpretation of this result when considering an arbitrary convention for the topological recursion. For $f(z)$ an analytic function around $z \to 0$ and $\{T_k\}_{k\in\mathbb{Z}}$ a set of parameters, one can compute

$$\operatorname*{Res}_{Z_1\to 0}\operatorname*{Res}_{Z_2\to 0} K(Z_1,z)\left\{\left(\sum_{k\geq 1}T_kZ_1^k\right)dZ_1\,K(Z_2,-Z_1)f(Z_2)\,[dZ_2]^2 - \left(\sum_{k\geq 1}T_k(-Z_1)^k\right)dZ_1\,K(Z_2,Z_1)f(Z_2)\,[dZ_2]^2\right\} \tag{79}$$

where the recursion kernel is the one of the Airy curve, i.e. the one for which $h_k = 0$ for $k \geq 2$:

$$K(z,z_0) = \frac{\beta}{2\alpha h_1}\frac{dz_0}{2z\,dz}\frac{1}{(z_0^2-z^2)}. \tag{80}$$

One can move the integration contours to get

$$\operatorname*{Res}_{Z_1\to 0}\operatorname*{Res}_{Z_2\to 0} = \operatorname*{Res}_{Z_2\to 0}\operatorname*{Res}_{Z_1\to 0} + \operatorname*{Res}_{Z_2\to 0}\operatorname*{Res}_{Z_1\to Z_2} + \operatorname*{Res}_{Z_2\to 0}\operatorname*{Res}_{Z_1\to -Z_2}. \tag{81}$$

The first term of the right hand side vanishes since the integrand does not have any pole at $Z_1 \to 0$. Let us now compute one of the other two terms:

$$\operatorname*{Res}_{Z_2\to 0}\operatorname*{Res}_{Z_1\to Z_2} K(Z_1,z)\left(\sum_{k\geq 1} T_k Z_1^k dZ_1\right) K(Z_2,-Z_1) f(Z_2)\,[dZ_2]^2 \tag{82}$$

$$= -\operatorname*{Res}_{Z_2\to 0}\frac{\beta}{2\alpha h_1}\frac{dZ_2}{2Z_2} f(Z_2)\operatorname*{Res}_{Z_1\to Z_2}\frac{dz}{2Z_1}\frac{1}{(z^2-Z_1^2)}\frac{\beta}{2\alpha h_1}\frac{1}{(Z_1^2-Z_2^2)}\left(\sum_{k\geq 1} T_k Z_1^k dZ_1\right)$$

$$= -\operatorname*{Res}_{Z_2\to 0}\frac{\beta}{2\alpha h_1}\frac{dZ_2 dz}{2Z_2} f(Z_2)\frac{1}{(z^2-Z_2^2)}\frac{\beta}{2\alpha h_1}\sum_{k\geq 1}\frac{T_k}{4} Z_2^{k-2}.$$

In the same way,

$$\operatorname*{Res}_{Z_2\to 0}\operatorname*{Res}_{Z_1\to -Z_2} K(Z_1,z)\left(\sum_{k\geq 1} T_k Z_1^k dZ_1\right) K(Z_2,-Z_1) f(Z_2)\,[dZ_2]^2 \tag{83}$$

$$= -\operatorname*{Res}_{Z_2\to 0}\frac{\beta}{2\alpha h_1}\frac{dZ_2 dz}{2Z_2} f(Z_2)\frac{1}{(z^2-Z_2^2)}\frac{\beta}{2\alpha h_1}\sum_{k\geq 1}\frac{T_k}{4}(-Z_2)^{k-2}.$$

The sum of these two terms reads

$$\operatorname*{Res}_{Z_2\to 0}\operatorname*{Res}_{Z_1\to \pm Z_2} K(Z_1,z)\left(\sum_{k\geq 1} T_k Z_1^k dZ_1\right) K(Z_2,-Z_1) f(Z_2)\,[dZ_2]^2 \tag{84}$$

$$= -\operatorname*{Res}_{Z_2\to 0}\frac{\beta}{2\alpha h_1}\frac{dZ_2 dz}{2Z_2} f(Z_2)\frac{1}{(z^2-Z_2^2)}\frac{\beta}{2\alpha h_1}\sum_{k\geq 1}\frac{T_{2k}}{2}(Z_2)^{2k-2}$$

and finally:

$$\operatorname*{Res}_{Z_1\to 0}\operatorname*{Res}_{Z_2\to 0} K(Z_1,z)\left\{\left(\sum_{k\geq 1} T_k Z_1^k\right) dZ_1\, K(Z_2,-Z_1) f(Z_2)\,[dZ_2]^2 \right. \tag{85}$$

$$\left. -\left(\sum_{k\geq 1} T_k (-Z_1)^k\right) dZ_1\, K(Z_2,Z_1) f(Z_2)\,[dZ_2]^2\right\} =$$

$$= \operatorname*{Res}_{Z_2\to 0}\frac{\beta}{2\alpha h_1}\frac{dZ_2 dz}{2Z_2} f(Z_2)\frac{1}{(z^2-Z_2^2)}\left(-\frac{\beta}{2\alpha h_1}\right)\sum_{k\geq 1} T_{2k}(Z_2)^{2k-2}.$$

On the other hand, plugging in the times h_k amounts to computing similar quantities:

$$\operatorname*{Res}_{z\to 0}\frac{\beta}{2\alpha h_1}\frac{dz_0}{2z\,dz}\frac{1}{(z_0^2-z^2)}\frac{1}{\left(1+\sum_{k=1}^{\infty}\frac{h_{2k+1}}{h_1}z^{2k}\right)} f(z)\,[dz]^2 \tag{86}$$

$$= \operatorname*{Res}_{z\to 0}\frac{\beta}{2\alpha h_1}\frac{dz_0}{2z\,dz}\frac{1}{(z_0^2-z^2)} f(z)\,[dz]^2\,(1-\sum_{k=1}^{\infty}\frac{h_{2k+1}}{h_1}z^{2k}+\left[\sum_{k=1}^{\infty}\frac{h_{2k+1}}{h_1}z^{2k}\right]^2+\dots)$$

The first term of this sum is the Airy recursion kernel. The second one is of the shape of the preceding one with $T_{2k+2} = \frac{2\alpha h_{2k+1}}{h_1}$ for $k \geq 1$ so that:

$$
\begin{aligned}
(87) \quad & -\operatorname*{Res}_{z\to 0} \frac{\beta}{2\alpha} \frac{dz_0}{2z\,dz} \frac{1}{(z_0^2 - z^2)} f(z)\,[dz]^2 \sum_{k=1}^{\infty} \frac{h_{2k+1}}{h_1} z^{2k} \\
&= \operatorname*{Res}_{Z_1\to 0} \operatorname*{Res}_{Z_2\to 0} K(Z_1, z_0) \Big\{ g(Z_1) dZ_1\, K(Z_2, -Z_1) f(Z_2)\,[dZ_2]^2 \\
&\qquad - g(-Z_1) dZ_1\, K(Z_2, Z_1) f(Z_2)\,[dZ_2]^2 \Big\}
\end{aligned}
$$

where

$$
g(z) := \sum_{k\geq 1} \frac{2\alpha h_{2k+1}}{\beta h_1} z^{2k+2}. \tag{88}
$$

This same procedure can be applied to the other terms of the sum. The k^{th} order term can be written as a sequence of $k+1$ residues computed with the Airy recursion kernel with $g(z)dz$ on one of the outgoing legs. This computation shows that introducing non-vanishing times amounts to introducing a non-vanishing $\omega_{0,1}(z) := g(z)dz$ in the topological recursion.

It is often useful to represent the topological recursion in a graphical form by representing the interaction kernel $K(z, z_0)$ by an edge oriented from z_0 towards a trivalent vertex labelled by z and the function $\omega_{0,2}(z_1, z_2)$ by a non-oriented edge (see [**27**] for more details about this set of graphs). In this form, $\omega_{g,n}(z_1, \ldots, z_n)$ is a sum over trivalent graphs of genus g with n leaves labelled by the arguments $z_1, \ldots, z_n$. The preceding computation shows that the correlation functions of the KdV curve can be obtained from the correlation functions of the Airy curve by introducing a set of new leaves, called dilation leaves, in the definition of the graphs used. A dilation leave decorated by a label k is weighted by

$$
(2d-1)!!\, \operatorname*{Res}_{z\to 0} g(z) \frac{dz}{z^{2d+1}} = (2d-1)!!\, \frac{2\alpha h_{2d-1}}{\beta}. \tag{89}
$$

Plugging this expression into the formula for the Airy correlation functions proves the result. □

3.3.4. *General case.* In this section we give a formula for the correlation function of the local topological recursion.

DEFINITION 3.14. Let $\Gamma_{g,n}$ be the subset of Γ (see Notation 3.6) consisting of graphs of genus g' such that $g' + \sum_{v\in V(\Gamma)} \mathfrak{g}(v) = g$ and with n ordinary leaves. Let us also introduce orderings on the ordinary leaves and denote by $\check{\Gamma}_{g,n}$ the set of all graphs from $\Gamma_{g,n}$ with all possible orderings on the ordinary leaves. For a given graph with a fixed ordering $\check{\gamma} \in \check{\Gamma}_{g,n}$ for an ordinary leaf of that graph $l \in L^*(\check{\gamma})$ we denote by $\mathfrak{m}(l)$ the index of this particular leaf (then $\mathfrak{m}(l)$ is an integer from 1 to n such that different leaves have different values $\mathfrak{m}(l)$ assigned to them).

THEOREM 3.15. *The correlation functions can be written as a sum over decorated graphs whose vertices are weighted by intersection of ψ-classes on $\overline{\mathcal{M}}_{g,n}$, edges by the jumps, ordinary leaves by primitives of B and dilaton leaves by the times.*

For $2-2g-n<0$, one has

$$\omega_{g,n}(\vec{z}) = \frac{1}{n!} \sum_{\check{\gamma}\in\check{\Gamma}_{g,n}} \prod_{v\in V(\check{\gamma})} \left(-2h_1^{\mathfrak{i}(v)}\right)^{2-2\mathfrak{g}(v)-\mathrm{val}(v)} \left\langle \prod_{h\in H(v)} \tau_{\mathfrak{k}(h)} \right\rangle_{\mathfrak{g}(v),\mathrm{val}(v)} \prod_{e\in E(\check{\gamma})} \check{B}^{\mathfrak{i}(\mathfrak{v}_1(e)),\mathfrak{i}(\mathfrak{v}_2(e))}_{\mathfrak{k}(\mathfrak{h}_1(e)),\mathfrak{k}(\mathfrak{h}_2(e))} \prod_{l\in L^*(\check{\gamma})} \sum_{j=1}^{N} \mathrm{d}\xi^{\mathfrak{i}(\mathfrak{v}(l))}_{\mathfrak{k}(l)}(z_{\mathfrak{m}(l)},j) \prod_{\lambda\in L^\bullet(\check{\gamma})} \check{h}^{\mathfrak{i}(\mathfrak{v}(\lambda))}_{\mathfrak{k}(\lambda)} \tag{90}$$

with

$$\check{h}^i_k := 2(2k-1)!!h^i_{2k-1}, \tag{91}$$

$$d\xi^i_d(z_\alpha,j) := \operatorname*{Res}_{z\to 0} \frac{(2d+1)!!dz}{z^{2d+2}} \int^z B^{i,j}(z,z_\alpha), \tag{92}$$

$$\check{B}^{i,j}_{d_1,d_2} := B^{i,j}_{2d_1,2d_2}\ (2d_1-1)!!\ (2d_2-1)!! \tag{93}$$

and

$$\left\langle \prod_{i=1}^{n} \tau_{k_i} \right\rangle_{g,n} := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1}\psi_2^{k_2}\dots\psi_n^{k_n}. \tag{94}$$

Proof. The proof is very similar to the one presented in [**25**, **42**]. However, we prefer to present a completely graphical proof so that the link with the next sections becomes clear.

We follow the proof of [**25**]. From the definition, one can write the correlation functions as a sum over graphs with oriented and non-oriented arrows linking trivalent vertices resulting in the following expression:

$$\omega^{\vec{i}}_{g,n}(\vec{z}) = \sum_{G\in\widehat{G}_{g,n}} \omega(G) \tag{95}$$

with $\widehat{G}_{g,n}$ the set of genus g trivalent graphs with one root and $n-1$ leaves labelled by the arguments z_i and a skeleton tree of oriented edges pointing from the root towards the leaves weighted by

$$\omega(G) = \vec{\prod_{v\in V(G)}} \operatorname*{Res}_{Z_v\to 0} \prod_{e\in E_{\text{oriented}}(G)} K^{i(v_1(e)),i(v_2(e))}(Z_{v_1(e)},Z_{v_2(e)}) \prod_{e\in E_{\text{unoriented}}(G)} B^{i(v_1(e)),i(v_2(e))}(Z_{v_1(e)},Z_{v_2(e)})$$

where each leaf is considered as a one-valent vertex v and one denotes Z_v the variable z_i associated to this leaf in the correlation function, $E_{\text{oriented}}(G)$ is the set of oriented leaves of G and $E_{\text{unoriented}}(G)$ is the set of unoriented leaves of G (see [**27**] for further details). The product of residues $\vec{\prod}_{v\in V(G)} \operatorname{Res}_{Z_v\to 0}$ is oriented following the arrows, i.e. one first computes the residue corresponding to the end of an arrow before the one associated to its root.

It is useful to remark, that, for any edge, oriented or not, one has two types of contributions. Indeed, the functions $B^{i,j}(z,z')$ have a singular part

$$B^{i,j}_{\mathrm{KdV}}(z,z') := \delta_{i,j}\frac{dz\otimes dz'}{(z-z')^2} \tag{97}$$

and a regular part

$$B^{i,j}_{\mathrm{reg}}(z,z') := \sum_{k,l=0}^{\infty} B^{i,j}_{k,l} z^k z'^l dz\otimes dz' \tag{98}$$

when $z\to z'$:

$$B^{i,j}(z,z') = B^{i,j}_{\mathrm{KdV}}(z,z') + B^{i,j}_{\mathrm{reg}}(z,z'). \tag{99}$$

In the same way, one has

$$K^{i,j}(z,z') = K^{i,j}_{\mathrm{KdV}}(z,z') + K^{i,j}_{\mathrm{reg}}(z,z'). \tag{100}$$

One can translate this by representing $B^{i,j}_{\mathrm{KdV}}(z,z')$ (resp. $B^{i,j}_{\mathrm{reg}}(z,z')$) by dashed (resp. dotted) unoriented edges and $K^{i,j}_{\mathrm{KdV}}(z,z')$ (resp. $K^{i,j}_{\mathrm{reg}}(z,z')$) by dashed (resp. dotted) oriented edges form z to z'. The preceding sum is thus transformed into a sum over graphs where the edges are dotted or dashed and weighted accordingly.

The dashed edges can be expressed in a slightly different way. Indeed, one has

$$B^{i,j}_{\mathrm{reg}}(z,z') = \mathop{\mathrm{Res}}_{z_1\to z}\mathop{\mathrm{Res}}_{z_2\to z'} B^{i,i}_{\mathrm{KdV}}(z,z_1)\left[\int^{z_1}\int^{z_2} B^{i,j}_{\mathrm{reg}}(z_1,z_2)\right] B^{j,j}_{\mathrm{KdV}}(z_2,z') \tag{101}$$

and

$$K^{i,j}_{\mathrm{reg}}(z,z') = \mathop{\mathrm{Res}}_{z_1\to z}\mathop{\mathrm{Res}}_{z_2\to \pm z'} B^{i,i}_{\mathrm{KdV}}(z,z_1)\left[\int^{z_1}\int^{z_2} B^{i,j}_{\mathrm{reg}}(z_1,z_2)\right] K^{j,j}_{\mathrm{KdV}}(z_2,z') \tag{102}$$

by a simple application of the Cauchy formula.

Remember that such an edge comes with integration of its boundary variables, thus, one typically has to compute

$$\mathop{\mathrm{Res}}_{z\to 0}\mathop{\mathrm{Res}}_{z'\to 0} g(z)K^{i,j}_{\mathrm{reg}}(z,z')f(z') \tag{103}$$

which reads

$$\mathop{\mathrm{Res}}_{z\to 0}\mathop{\mathrm{Res}}_{z_1\to z}\mathop{\mathrm{Res}}_{z'\to 0}\mathop{\mathrm{Res}}_{z_2\to \pm z'} g(z)B^{i,i}_{\mathrm{KdV}}(z,z_1)\left[\int^{z_1}\int^{z_2} B^{i,j}_{\mathrm{reg}}(z_1,z_2)\right] K^{j,j}_{\mathrm{KdV}}(z_2,z')f(z'). \tag{104}$$

One can move the integration contours around 0 thanks to:

$$\mathop{\mathrm{Res}}_{z\to 0}\mathop{\mathrm{Res}}_{z_1\to z} = \mathop{\mathrm{Res}}_{z_1\to 0}\mathop{\mathrm{Res}}_{z\to 0} - \mathop{\mathrm{Res}}_{z\to 0}\mathop{\mathrm{Res}}_{z_1\to 0} \tag{105}$$

and

$$\mathop{\mathrm{Res}}_{z'\to 0}\mathop{\mathrm{Res}}_{z_2\to \pm z'} = \mathop{\mathrm{Res}}_{z_2\to 0}\mathop{\mathrm{Res}}_{z'\to 0} - \mathop{\mathrm{Res}}_{z'\to 0}\mathop{\mathrm{Res}}_{z_2\to 0}. \tag{106}$$

Since, the integrand does not have any pole as $z_1\to 0$ nor $z_2\to 0$, this shows that 103 is equal to

$$\mathop{\mathrm{Res}}_{z_1\to 0}\mathop{\mathrm{Res}}_{z\to 0}\mathop{\mathrm{Res}}_{z_2\to 0} g(z)\, B^{i,i}_{\mathrm{KdV}}(z,z_1)\left[\int^{z_1}\int^{z_2} B^{i,j}_{\mathrm{reg}}(z_1,z_2)\right]\mathop{\mathrm{Res}}_{z'\to 0} K^{j,j}_{\mathrm{KdV}}(z_2,z')\, f(z'). \tag{107}$$

In the same way, one gets that

$$\operatorname*{Res}_{z\to 0}\operatorname*{Res}_{z'\to 0} g(z)B^{i,j}_{\mathrm{reg}}(z,z')f(z') \tag{108}$$

is equal to

$$\operatorname*{Res}_{z_1\to 0}\operatorname*{Res}_{z\to 0}\operatorname*{Res}_{z_2\to 0} g(z)\,B^{i,i}_{\mathrm{KdV}}(z,z_1)\left[\int^{z_1}\int^{z_2} B^{i,j}_{\mathrm{reg}}(z_1,z_2)\right]\operatorname*{Res}_{z'\to 0}\; B^{j,j}_{\mathrm{KdV}}(z_2,z')\,f(z'). \tag{109}$$

One can finally proceed in a similar way for re-expressing the weights of the root and the leaves by writing[1]

$$\operatorname*{Res}_{z'\to 0} K^{i,j}(z,z')f(z') = \operatorname*{Res}_{z_2\to 0}\left[\int^{z_2} B^{i,j}(z,z_2)\right]\operatorname*{Res}_{z'\to 0}\; K^{j,j}_{\mathrm{KdV}}(z_2,z')\,f(z') \tag{110}$$

and

$$\operatorname*{Res}_{z\to 0} g(z)B^{i,j}(z,z') = \operatorname*{Res}_{z_1\to 0}\operatorname*{Res}_{z\to 0} g(z)\,B^{i,i}_{\mathrm{KdV}}(z,z_1)\left[\int^{z_1} B^{i,j}(z_1,z')\right]. \tag{111}$$

As a result, by applying this transformation to each dotted line, any graph is composed of a set of dotted subgraphs whose vertices have the same label separated by dashed lines. Since each subgraph with label i also includes a root and leaves, it is a contribution to the correlation functions obtained for the case $N=1$, times h^i_k and vanishing jumps $B^{i,i}_{k,l}=0$. In the sum over graphs, one can thus replace every sum over such sub-graphs by vertices of corresponding genus weighted by the correlation function for $N=1$, which reads

$$\omega^{\vec{i}}_{g,n}(\vec{z}) = \sum_{\gamma\in\Gamma_{g,n}} \Omega(\gamma) \tag{112}$$

where

$$\begin{aligned}\Omega(\gamma) \quad = \quad & \prod_{v\in V(\gamma)}\prod_{h\in H(v)} \operatorname*{Res}_{Z_h\to 0}\omega^{\mathrm{KdV},i(v)}_{g(v),\mathrm{val}(v)}\left(\{Z_h\}_{h\in H(v)}\right)\\ & \prod_{e\in E(\gamma)}\int^{Z_{h_1(e)}}\int^{Z_{h_2(e)}} B^{i(v_1(h)),i(v_2(h))}_{\mathrm{reg}}(Z_{h_1(e)},Z_{h_2(e)})\\ & \prod_{h\in L^*(\gamma)}\int^{Z_h} B^{i,j}(Z_h,z_h)\end{aligned} \tag{113}$$

where $\omega^{\mathrm{KdV},i}_{g,n}(z_1,\dots,z_n)$ is the genus g, n-pointed correlation function obtained from the topological recursion in the case $N=1$ and the initial data:

$$\begin{cases} x(z) = z^2 + a_i\\ y(z) = \displaystyle\sum_{k=1}^{\infty} h^i_k z^k\\ B(z,z') = B_{\mathrm{KdV}}(z,z') = \dfrac{dz\otimes dz'}{(z-z')^2}\end{cases}. \tag{114}$$

[1]Remark that, for the roots and leaves, in opposition to the inner edges, the functions are the full ones, not just the regular part.

As explained in the preceding section, it can be expressed in terms of intersection numbers:

$$\omega_{g,n}^{\mathrm{KdV},i}(z_1,\dots,z_n) = \left(-2h_1^i\right)^{2-2g-n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{\vec{\alpha}\in\mathbb{N}^{*m}} \prod_{k=1}^{m} (2\alpha_k+1)!! \frac{h^i_{2\alpha_k+1}}{h_1^i} \prod_{i=1}^{n} \frac{(2d_i+1)!!\, dz_i}{z_i^{2d_i+2}} \left\langle \prod_{j=1}^{n} \tau_{d_j} \prod_{k=1}^{m} \tau_{\alpha_k+1} \right\rangle_{g,n+m} \tag{115}$$

which can be made more symmetric under the exchange of the ordinary and dilation leaves by writing

$$\omega_{g,n}^{\mathrm{KdV},i}(z_1,\dots,z_n) = \sum_{m=0}^{\infty} (-2h_1)^{2-2g-n-m} \frac{1}{m!} \sum_{\vec{\alpha}\in\mathbb{N}^{*m}} \prod_{k=1}^{m} (2\alpha_k-1)!! 2h^i_{2\alpha_k-1} \prod_{i=1}^{n} \frac{(2d_i+1)!!\, dz_i}{z_i^{2d_i+2}} \left\langle \prod_{j=1}^{n} \tau_{d_j} \prod_{k=1}^{m} \tau_{\alpha_k} \right\rangle_{g,n+m} \tag{116}$$

Absorbing the factors of the form

$$\frac{(2d+1)!!\mathrm{d}z}{z^{2d+2}} \tag{117}$$

into the corresponding half-edge contribution, the weight of an inner edge becomes

$$\operatorname*{Res}_{z_1\to 0} \operatorname*{Res}_{z_2\to 0} \int^{z_1}\int^{z_2} B^{i,j}_{\mathrm{reg}}(z_1,z_2)\, \frac{(2d_1+1)!!\mathrm{d}z_1}{z_1^{2d_1+2}}\, \frac{(2d_2+1)!!\mathrm{d}z_1}{z_2^{2d_2+2}} \tag{118}$$

which is equal to

$$\check{B}^{i,j}_{d_1,d_2} := B^{i,j}_{2d_1,2d_2}\, (2d_1-1)!!\, (2d_2-1)!! \tag{119}$$

while the weight of the ordinary leaves becomes

$$\mathrm{d}\xi^i_d(z_\alpha,j) := \operatorname*{Res}_{z\to 0} \frac{(2d+1)!!\mathrm{d}z}{z^{2d+2}} \int^z B^{i,j}(z,z_\alpha), \tag{120}$$

where one considers both the singular and non-singular part of $B^{i,j}(z,z_\alpha)$. Collecting these contributions together proves the theorem. ☐

3.3.5. *Change of scales.* An important property of the correlation functions built in this way is their homogeneity property which reads

$$\forall\lambda\in\mathbb{C}\,,\ \omega_{g,n}(\vec{z}_N|x,\lambda y,B) = \lambda^{2-2g-n}\omega_{g,n}(\vec{z}_N|x,y,B) \tag{121}$$

One can thus get an additional factor λ^i by replacing $h^i_k \to \lambda^i h^i_k$ resulting in a rescaling of the weight of the vertices by $\left(\lambda^{\mathrm{i}(v)}\right)^{2-2g(v)-\mathrm{val}(v)}$.

3.3.6. *Weights, Laplace transform and recursive definition.* It is interesting to note that the weights of the edges are the coefficient of the Laplace transform of B:

$$\check{B}^{i,j}(u,v) := \sum_{(k,l)\in\mathbb{N}^2} \check{B}^{i,j}_{k,l} u^{-k} v^{-l} \tag{122}$$

is equal to

$$\check{B}^{i,j}(u,v) = \delta_{i,j}\frac{uv}{u+v}+ \frac{\sqrt{uv}e^{ua_i+va_j}}{2\pi}\int_{x(z)-a_i\in\mathbb{R}^+}\int_{x(z')-a_j\in\mathbb{R}^+} B^{i,j}(z,z')e^{-ux(z)-vx(z')}. \tag{123}$$

In [**25**], it was proved that, if dx is a meromorphic form defined on a Riemann surface, $\check{B}^{i,j}(u,v)$ can be factorized and expressed in terms of some basic functions. Here, we will consider the converse and build $B^{i,j}_{k,l}$ by induction in such a way that there exist a set of functions $\{f_{i,j}(u)\}_{i,j=1}^N$ such that

$$\check{B}^{i,j}(u,v) = \frac{uv}{u+v}\left(\delta_{i,j} - \sum_{k=1}^N f_{i,k}(u)f_{k,j}(v)\right). \tag{124}$$

Let us define the coefficients $B^{i,j}_{k,l}$ recursively in terms of the initial data $B^{i,j}_{k,0}$ by imposing that

$$\xi^i_{d+1}(z,j) := -2\frac{d\xi^i_d(z,j)}{dx^{[j]}(z)} - \sum_{k=1}^{\mathfrak{b}} \check{B}^{i,k}_{d,0}\,\xi^k_0(z,j), \tag{125}$$

or, in terms of the Laplace transform

$$\begin{aligned} f^i_d(u,j) &:= \frac{\sqrt{u}}{2\sqrt{\pi}}\int_{x(z)-a_j\in\mathbb{R}^+} e^{-u(x(z)-a_j)}dx^j(z)\,\xi^i_d(z) \\ &= \delta_{i,j}(-1)^d u^d - \sum_{d'} \check{B}^{i,j}_{d,d'}u^{-d'-1}, \end{aligned} \tag{126}$$

it reads

$$f^i_{d+1}(u,j) := -2uf^i_d(u,j) - \sum_{k=1}^N \check{B}^{i,k}_{d,0}\,f^k_0(z,j). \tag{127}$$

With this definition, one has

$$\check{B}^{i,j}(u,v) = \frac{uv}{u+v}\left(\delta_{i,j} - \sum_{k=1}^{\mathfrak{b}} f^k_0(u,i)f^k_0(v,j)\right). \tag{128}$$

3.4. Identification of the two theories. In this section we show how to find a local spectral curve corresponding to any semi-simple conformal Frobenius manifold.

Suppose some local spectral curve is given. For any $i\in\{1,\dots,N\}$ and $k\in\mathbb{Z}_{\geq 0}$ define

$$W^i_k(z) := \sum_{j=1}^N d\left(\left(-\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right)^k \xi^i_0(z,j)\right).$$

THEOREM 3.16. *Let R be some series of operators on an N-dimensional vector space V as in Section 3.1. Let $Z = \hat{R}\hat{\Delta}\mathcal{T}$, where $\mathcal{T}$ is a product of N KdV τ-functions, be the partition function of the corresponding semi-simple cohomological field theory.*

Define a local spectral curve by the following data

$$\check{B}^{i,j}_{p,q} := [z^p w^q] \frac{\delta^{ij} - \sum_{s=1}^N R^i_s(-z) R(-w)^j_s}{z+w} \tag{129}$$

and

$$\check{h}^i_k := [z^{k-1}] \left(-R(-z)\right)^i_{\mathbf{1}} \tag{130}$$

$$h^i_1 := -\frac{1}{2\sqrt{\Delta^i}}. \tag{131}$$

Let $\omega_{g,n}$ be the genus g, n-pointed topological recursion invariant of this spectral curve and denote by

$$\Omega(\{v^{d,i}\}) = \left(\sum_{g,d} \omega_{g,d}(z_1, \ldots, z_d) \Big|_{W^i_d(z_m) = v^{d,i}} \hbar^{g-1} \right)$$

their sum after a change of variables $W^i_k(z_m) \leftrightarrow v^{d,i}$ for all m. Then the partition function of the cohomological field theory and the topological recursion invariants agree in the following sense:

$$Z(\{v^{d,i}\}) = \exp\left(\Omega(\{v^{d,i}\})\right). \tag{132}$$

PROOF. In Sections 3.1 and 3.3 we have given representations of Z and $\omega_{g,n}$ as sums over the set Γ (in fact, in the case of $\omega_{g,n}$ this set is $\check{\Gamma}$ rather than Γ, but after changing the variables $W^i_k(z_m) \leftrightarrow v^{d,i}$ we can take the sum over orderings and arrive at the sum over Γ acquiring an additional factor of $n!$, which cancels with the corresponding factor in (90)). We prove the theorem by showing that the contribution of each individual graph to Z is equal to the contribution to Ω.

Let $\gamma \in \Gamma$ be some graph. Note that on both sides we assign the same weight to the vertices of γ, namely to a vertex labelled (g,i) with n half-edges attached to it labelled $d_1, \ldots, d_n$ we associate

$$(-2h^i_1)^{2-2g-n} \left\langle \tau_{d_1} \cdots \tau_{d_n} \right\rangle_{g,n}. \tag{133}$$

Furthermore, by equation (129), any edge in γ contributes the same to Z and Ω.

Let l be an ordinary leaf of γ labelled by k attached to a vertex labelled by (g,i). We use induction on k to show that the contribution to Z is the same as the contribution to Ω.

The contribution of l to Z is given by

$$\mathcal{L}^i_k(l) = [z^k] \left(\sum_d (R(-z))^i_j v^{d,j} z^d \right) = \sum_{d=0}^k (-1)^{k-d} (R_{k-d})^i_j v^{d,j}. \tag{134}$$

When $k = 0$, the contribution of l to Ω is given by

$$\sum_j d\xi^i_0(z_j, j) = W^i_0. \tag{135}$$

Since $(R_0)^i_j = \delta^i_j$, the contributions to Z and Ω agree when $k = 0$.

Now suppose that they agree for some $k \in \mathbb{Z}_{\geq 0}$. That is, suppose that

$$\sum_j d\xi_k^i(z^j, j) = \sum_{l=0}^{k} (-1)^{k-l} (R_{k-l})_s^i W_l^s. \tag{136}$$

Then, using Equation (125), the contribution of the leaf to Ω for the index $k+1$ is given by

$$\begin{aligned}\sum_j d\xi_{k+1}^i(z^j, j) &= \sum_j d\left(-2\frac{\partial \xi_k^i(z^j, j)}{\partial x^j} - \sum_{t=1}^{N} \check{B}_{k,0}^{i,t} \xi_0^t(z^j, j)\right) \\ &= \sum_j d\left(-\frac{1}{z^j}\frac{\partial}{\partial z^j}\xi_k^i(z^j, j) - \sum_{t=1}^{n} -(-1)^{k+1}(R_{k+1})_t^i \xi_0^t(z^j, j)\right) \\ &= \sum_{l=0}^{k} (-1)^l (R_l)_t^i W_{k+1-l}^t + (-1)^{k+1}(R_{k+1})_t^i W_0^t = \sum_{l=0}^{k+1} (-1)^l (R_l)_t^i W_{k+1-l}^t,\end{aligned} \tag{137}$$

where we used equation (129) to write

$$\check{B}_{k,0}^{i,t} = -(-1)^{k+1}(R_{k+1})_t^i. \tag{138}$$

This completes the induction, and since it is clear that the dilaton leaves contribute the same in both cases, it also completes the proof of the theorem. □

REMARK 3.17. The theorem above deals with the potential of a cohomological field theory written in terms of formal variables $v^{d,i}$ corresponding to normalized canonical basis. In order to pass to flat coordinates one can change the variables in the following way:

$$v^{d,i} = \Psi_\mu^i t^{d,\mu}. \tag{139}$$

On the spectral curve side it will correspond to changing the variables W_k^i in the following way:

$$W_k^i = \Psi_\mu^i V_k^\mu. \tag{140}$$

Thus, the theorem holds in the same form for the potential of cohomological field theory written in terms of formal variables $t^{d,\mu}$, only one should identify $t^{d,\mu}$ with V_d^μ.

REMARK 3.18. Above we established the correspondence between cohomological field theories and symplectic invariants of spectral curves. However, as noted in Remark 3.5, in the case of Gromov-Witten theories we cannot disregard quadratic terms. So, in the formula for the total descendent potential an additional operator $\hat{S}$ appears. In some cases, again see Remark 3.5, it performs only a linear change of formal variables $t^{d,\mu}$ on which the potential depends. Thus, to establish the correspondence in this case, one has to change the variables W_k^i in precisely the same way, and then identify the resulting variables with $t^{d,\mu}$, similar to the case of previous remark. Occasionally, the changes of variables preformed by $\hat{\Psi}$ and $\hat{S}^{-1}$ can be a re-expansion of $\omega_{g,n}$ in a new coordinate on the spectral curve. We explain this procedure in detail for the case of $\mathbb{C}\mathrm{P}^1$ below in section 3.5.

REMARK 3.19. The system of equations obtained via a Laplace transform from the equations of Givental for the R-matrix (that is, the so-called equations of deformed flat connection) is studied in detail in [**13**, Section 5]. This gives, in particular, a recipe to reconstruct the two-point function directly from the Frobenius structure bypassing the reconstruction of the R-matrix. This also explains why we call the critical values $a_1, \dots, a_N$ of x the canonical coordinates.

3.5. The Norbury-Scott conjecture. In this section we recall and prove the Norbury-Scott conjecture on the stationary sector of the Gromov-Witten theory of $\mathbb{C}\mathrm{P}^1$.

3.5.1. *Gromov-Witten theory of* $\mathbb{C}\mathrm{P}^1$. The Gromov-Witten theory of $\mathbb{C}\mathrm{P}^1$ is discussed from the geometric point of view in many sources, see e. g. [**53**]. Givental proved in [**36**] that his formula for the formal Gromov-Witten potential coincides with the geometric Gromov-Witten potential of $\mathbb{C}\mathrm{P}^1$, so we discuss it here only from the Givental point of view, ignoring the geometric background. The same computations one can find in [**58, 59**].

The underlying structure of Frobenius manifold is determined by the following solution of the WDVV equation

$$\frac{1}{2}(t^1)^2 t^2 + e^{t^2}, \tag{141}$$

and the scalar product given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{142}$$

All ingredients of the Givental formula depend on a particular choice of the point on the Frobenius manifold, and in this case we choose the point $(0,0)$ in the coordinates (t_1, t_2).

We perform a direct computation following the recipe of Givental in [**35**], see also Section 3.1.2. As a possible choice of the canonical coordinates, we use

$$u^1 = t^1 + 2\exp(t^2/2); \tag{143}$$

$$u^2 = t^1 - 2\exp(t^2/2). \tag{144}$$

In particular, for $t^1 = t^2 = 0$ we have $u^1 = -u^2 = 2$. Then,

$$\Delta_1^{-1} = \left\langle \frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^1} \right\rangle = \frac{\exp(-t^2/2)}{2}; \tag{145}$$

$$\Delta_2^{-1} = \left\langle \frac{\partial}{\partial u^2}, \frac{\partial}{\partial u^2} \right\rangle = \frac{-\exp(-t^2/2)}{2}, \tag{146}$$

so we can choose the square roots as

$$\Delta_1^{-1/2} = \frac{\exp(-t^2/4)}{\sqrt{2}}; \tag{147}$$

$$\Delta_2^{-1/2} = \frac{-i\exp(-t^2/4)}{\sqrt{2}}, \tag{148}$$

and for this choice we have the following matrix of transition from the basis given by $(\partial/\partial t^1, \partial/\partial t^2)$ to the normalized canonical basis:

$$\Psi = \begin{pmatrix} \frac{\exp(-t^2/4)}{\sqrt{2}} & \frac{-i\exp(-t^2/4)}{\sqrt{2}} \\ \frac{\exp(t^2/4)}{\sqrt{2}} & \frac{i\exp(t^2/4)}{\sqrt{2}} \end{pmatrix}. \tag{149}$$

It is the matrix $\Psi = \Psi^i_\alpha$, where α labels the rows and corresponds to the flat basis, while i labels the columns and corresponds to the normalized canonical basis.

The recipe of reconstruction of the matrix R from [**35**] gives at the origin the matrix $R(\zeta) = \sum_{k=0}^{\infty} R_k \zeta^k$, where

$$R_k = \frac{(2k-1)!!(2k-3)!!}{2^{4k}k!} \cdot \begin{pmatrix} -1 & (-1)^{k+1}2ki \\ 2ki & (-1)^{k+1} \end{pmatrix} \tag{150}$$

The S matrix is given by the derivatives of the deformed flat coordinates, computed in [**15**, Example 3.7.9] At the origin we have:

$$\begin{aligned} S(\zeta^{-1}) =& \mathbf{I} + \zeta^{-1} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ &+ \sum_{k=1}^{\infty} \frac{\zeta^{-2k}}{(k!)^2} \begin{pmatrix} 1 - 2k\left(\frac{1}{1} + \cdots \frac{1}{k}\right) & 0 \\ 0 & 1 \end{pmatrix} \\ &+ \sum_{k=1}^{\infty} \frac{\zeta^{-2k-1}}{(k!)^2} \begin{pmatrix} 0 & -2\left(\frac{1}{1} + \cdots \frac{1}{k}\right) \\ \frac{1}{k+1} & 0 \end{pmatrix}. \end{aligned} \tag{151}$$

(Note once again that we are using the convention that the matrices are acting on vector rows, opposite to the standard one).

The unit vector at the origin in the normalized canonical basis is equal to

$$e = (1,0) \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} = \left(\frac{1}{\sqrt{2}}, \frac{-i}{\sqrt{2}}\right). \tag{152}$$

Therefore, the dilaton leaves (cf. Equation (59)) in the Givental formula for $\mathbb{C}\mathrm{P}^1$ at the origin are

$$(\mathcal{L}^\bullet)^1_{k+1} = \frac{1}{\sqrt{2}} \cdot \frac{(-1)^{k+1}\left((2k-1)!!\right)^2}{k!2^{4k}}; \tag{153}$$

$$(\mathcal{L}^\bullet)^2_{k+1} = \frac{i}{\sqrt{2}} \cdot \frac{\left((2k-1)!!\right)^2}{k!2^{4k}} \tag{154}$$

for $k \geq 0$.

PROPOSITION 3.20. *The Gromov-Witten potential of* $\mathbb{C}\mathrm{P}^1$,

$$Z_{\mathbb{C}\mathrm{P}^1}(\hbar, \{t^{\ell,1}, t^{\ell,2}\}_{\ell=0}^{\infty}), \tag{155}$$

is obtained from $\hat{R}\hat{\Delta}Z_{\mathrm{KdV}}^{\otimes 2}$ *(understood as a sum over graphs in the sense of Section* 3.2 *and written down in the normalized canonical basis, that is, in the variables* $v^{d,i}$, $d \geq 0$, $i = 1,2$*) via a linear change of variables given by*

$$\sum_{m \geq k}^{\infty} \left(t^{m,1}, t^{m,2}\right) S_k \zeta^{m-k} = \sum_{\ell=0}^{\infty} \left(v^{\ell,1}, v^{\ell,2}\right) \zeta^\ell \cdot \Psi^{-1}, \tag{156}$$

and a correction of the unstable terms (that is, (g,n)*-correlators with* $2g-2+n \leq 0$*).*

PROOF. In order to get the Gromov-Witten potential of $\mathbb{C}\mathrm{P}^1$ as given by the Givental formula, we have to apply the $\hat{\Psi}$- and $\hat{S}^{-1}$-action to the expression in terms of graphs discussed in Section 3.2 that corresponds to $\hat{R}\hat{\Delta}Z_{\mathrm{KdV}}^{\otimes 2}$. The $\hat{\Psi}$-action is just a linear change of variable by definition. The general S-action is discussed in [**32**, Section 4.2]. It is a combination of a shift of variables that vanishes in our case (indeed, $(1,0)S_1 = (0,0)$), the linear change of variables that we have in the statement of Proposition, and a correction of unstable terms that is not essential for us. □

3.5.2. *The Norbury-Scott conjecture.* Norbury and Scott [**52**] propose the following construction. They consider a spectral curve given by

$$\begin{cases} x = z + \frac{1}{z}; \\ y = \log z, \end{cases} \tag{157}$$

and the standard two-point function

$$B(z, z') = \frac{dz \otimes dz'}{(z - z')^2}. \tag{158}$$

Via topological recursion they obtain the n-forms $\omega_{g,n}$ that they consider in the global variable x, and they conjecture the following theorem (they prove it for $g = 0, 1$):

THEOREM 3.21. *For $2g - 2 + n > 0$, we have:*

$$\prod_{j=1}^{n} \left(- \operatorname*{Res}_{x_j=\infty} \frac{1}{(a_j+1)!} x_1^{a_j+1} \right) \omega_{g,n}(x_1, \dots, x_n) = \langle \prod_{j=1}^{n} \tau_{2,a_j} \rangle_g, \tag{159}$$

where $\langle \prod_{j=1}^{n} \tau_{2,a_j} \rangle_g$ is the corresponding correlator in $Z_{\mathbb{CP}^1}$, that is, the coefficient of $\hbar^{g-1} \prod_{j=1}^{n} t_{2,a_j} / |Aut(a_1, \dots, a_n)|$ in $\log Z_{\mathbb{CP}^1}$.

In the rest of this section we prove this theorem, identifying all ingredients of the topological recursion with the corresponding parts of the Givental formula.

3.5.3. *Proof of the Norbury-Scott conjecture.*

3.5.4. *Local coordinates near the branch points.* We denote the local coordinates by $z_1 = \sqrt{x-2}$ and $z_2 = \sqrt{x+2}$. Then we have:

$$x = z_1^2 + 2 \text{ near } x = 2,\ z = 1,\ z_1 = 0; \tag{160}$$

$$x = z_2^2 - 2 \text{ near } x = -2,\ z = -1,\ z_2 = 0. \tag{161}$$

Therefore,

$$z = 1 + \frac{z_1^2}{2} \pm z_1 \sqrt{1 + \frac{z_1^2}{4}}; \tag{162}$$

$$z = -1 + \frac{z_2^2}{2} \pm i z_2 \sqrt{1 - \frac{z_2^2}{4}}. \tag{163}$$

In both cases we choose $+$ for $\pm$.

3.5.5. *Expansion of y.* Recall that $y = \log z$. The direct computation shows:

$$y = \int \frac{dz_1}{\sqrt{1 + \frac{z_1^2}{4}}}; \tag{164}$$

$$y = \int \frac{-i\, dz_2}{\sqrt{1 - \frac{z_2^2}{4}}}; \tag{165}$$

Note that

$$\frac{1}{\sqrt{1+\frac{z_1^2}{4}}} = 1 + \sum_{k=1}^{\infty} z_1^{2k} \cdot \frac{(-1)^k(2k-1)!!}{k!2^{3k}}; \tag{166}$$

$$\frac{-i}{\sqrt{1-\frac{z_2^2}{4}}} = -i + \sum_{k=1}^{\infty} z_2^{2k} \cdot \frac{(-i)\cdot(2k-1)!!}{k!2^{3k}}. \tag{167}$$

Therefore

$$y = z_1 + \sum_{k=1}^{\infty} z_1^{2k+1} \cdot \frac{(-1)^k(2k-1)!!}{k!2^{3k}(2k+1)}; \tag{168}$$

$$y = -iz_2 + \sum_{k=1}^{\infty} z_2^{2k+1} \cdot \frac{(-i)\cdot(2k-1)!!}{k!2^{3k}(2k+1)}. \tag{169}$$

Thus the coefficients $\check{h}^i_{k+1}$, $k \geq 0$, are given by the following formulas:

$$\check{h}^1_{k+1} = 2 \cdot \frac{(-1)^k\left((2k-1)!!\right)^2}{k!2^{3k}}; \tag{170}$$

$$\check{h}^2_{k+1} = 2 \cdot \frac{(-i)\cdot\left((2k-1)!!\right)^2}{k!2^{3k}}. \tag{171}$$

3.5.6. *Matrix $f_{i,j}(w)$.* We use the following definition of the matrix $f_{ij}(w)$ (cf. Equation (124)):

$$f_{ij}(w) = \delta_{ij} - w\check{B}^{[ij]}(0, w^{-1}), \tag{172}$$

where $w = v^{-1}$. We use $\tilde{B}^{ij}_{0,l} = (B^{ij}_{\mathrm{reg}})_{0,2l}(2l-1)!!$, and the following expressions:

$$B^{11}_{\mathrm{reg}}(0, z_1) = \left[\frac{dz(z_1')\otimes dz(z_1)}{(z(z_1')-z(z_1))^2} - \frac{dz_1'\otimes dz_1}{(z_1'-z_1)^2}\right]_{z_1'=0} \tag{173}$$

$$B^{12}_{\mathrm{reg}}(0, z_2) = \left[\frac{dz(z_1')\otimes dz(z_2)}{(z(z_1')-z(z_2))^2}\right]_{z_1'=0} \tag{174}$$

$$B^{21}_{\mathrm{reg}}(0, z_1) = \left[\frac{dz(z_2')\otimes dz(z_1)}{(z(z_2')-z(z_1))^2}\right]_{z_2'=0} \tag{175}$$

$$B^{22}_{\mathrm{reg}}(0, z_2) = \left[\frac{dz(z_2')\otimes dz(z_2)}{(z(z_2')-z(z_2))^2} - \frac{dz_2'\otimes dz_2}{(z_2'-z_2)^2}\right]_{z_2'=0} \tag{176}$$

Therefore,

$$B^{11}_{\mathrm{reg}}(0,z_1) = \frac{1}{z_1^2}\left(\frac{1}{\sqrt{1+\frac{z_1^2}{4}}}-1\right) \tag{177}$$

$$B^{12}_{\mathrm{reg}}(0,z_2) = \frac{i}{4(1-\frac{z_2^2}{4})^{3/2}} \tag{178}$$

$$B^{21}_{\mathrm{reg}}(0,z_1) = \frac{i}{4(1+\frac{z_1^2}{4})^{3/2}} \tag{179}$$

$$B^{22}_{\mathrm{reg}}(0,z_2) = \frac{1}{z_2^2}\left(\frac{1}{\sqrt{1-\frac{z_2^2}{4}}}-1\right) \tag{180}$$

So, we have the following expansions:

$$B^{11}_{\mathrm{reg}}(0,z_1) = \sum_{k=0}^{\infty} z_1^{2k}\cdot\frac{(-1)^{k+1}(2k+1)!!}{(k+1)!2^{3(k+1)}} \tag{181}$$

$$B^{12}_{\mathrm{reg}}(0,z_2) = \sum_{k=0}^{\infty} z_2^{2k}\cdot\frac{i(2k+1)!!}{(k)!2^{3k+2}} \tag{182}$$

$$B^{21}_{\mathrm{reg}}(0,z_1) = \sum_{k=0}^{\infty} z_2^{2k}\cdot\frac{i(-1)^{k}(2k+1)!!}{(k)!2^{3k+2}} \tag{183}$$

$$B^{22}_{\mathrm{reg}}(0,z_2) = \sum_{k=0}^{\infty} z_2^{2k}\cdot\frac{(2k+1)!!}{(k+1)!2^{3(k+1)}}. \tag{184}$$

The formulas for $f_{ij}(w)$ are then

$$f_{11}(w) = 1+\sum_{k=1}^{\infty} w^k\cdot\frac{(-1)^{k+1}(2k-1)!!(2k-3)!!}{k!2^{3k}} \tag{185}$$

$$f_{12}(w) = \sum_{k=1}^{\infty} w^k\cdot\frac{-i(2k-1)!!(2k-3)!!}{(k-1)!2^{3k-1}} \tag{186}$$

$$f_{21}(w) = \sum_{k=1}^{\infty} w^k\cdot\frac{(-1)^{k}i(2k-1)!!(2k-3)!!}{(k-1)!2^{3k-1}} \tag{187}$$

$$f_{22}(w) = 1+\sum_{k=1}^{\infty} w^k\cdot\frac{-(2k-1)!!(2k-3)!!}{k!2^{3k}} \tag{188}$$

This coincides with the formula for the $\sum_{k=0}^{\infty} R_k 2^k(-w)^k$ at the point $(0,0)$.

3.5.7. *Comparison of the coefficient of (g,n,m)-vertex.* In this section we consider a vertex of genus g with n attached half-edges or ordinary leaves, and m dilaton leaves, with an associated intersection number $\langle\prod_{i=1}^n \tau_{d_i}\prod_{i=1}^m \tau_{a_i+1}\rangle_{g,n+m}$. There are vertices of type 1 and type 2, depending on the canonical coordinate that we associate to the vertex. We compare the coefficients that we associate to these vertices in the Givental case, using the data from Section 3.5.1 in Formula (62), and in the case of local topological recursion, using the data from Sections 3.5.4-3.5.6 in Formula (90).

The coefficients that we have in Formula (90) (at the vertex of the type 1 and 2 resp.):

$$(189)\qquad (-2)^{2-2g-n-m} \quad \text{and} \quad (2i)^{2-2g-n-m}.$$

Let us compute how these coefficients change if we take into account all the differences between R-matrix and the dilaton leaves. First, observe that the extra factor of 2^k in R_k and, in addition, an extra factor of $\sqrt{2}$ that we have to put by hand on each ordinary leave give us together the extra factors of

$$(190)\qquad 2^{\sum_{i=1}^n d_i} 2^{n/2} \quad \text{and} \quad 2^{\sum_{i=1}^n d_i} 2^{n/2}.$$

Then the quotient of the contributions of the dilaton leaves gives us the extra factors of

$$(191)\qquad 2^{\sum_{i=1}^m (a_i+1)} 2^{m/2}(-1)^m \quad \text{and} \quad 2^{\sum_{i=1}^m (a_i+1)} 2^{m/2}(-1)^m.$$

Let us assign by hand an extra factor of $(-1)^{2g-2+n}$ to each (g,n,m)-vertex. This way we get the following coefficients:

$$(192)\qquad 2^{g-1+n/2+m/2} \quad \text{and} \quad 2^{g-1+n/2+m/2} i^{2g-2+n+m}.$$

These coefficients are precisely

$$(193)\qquad \left((\Delta_1)^{1/2}\right)^{2g-2+n+m} \quad \text{and} \quad \left((\Delta_2)^{1/2}\right)^{2g-2+n+m}.$$

Therefore, the coefficient of $\prod_{k=1}^{\hat{n}} W_{d_k}^{i_k}$ in a graph of global genus $\hat{g}$ with $\hat{n}$ marked leaves in the Formula (90) for the set up of Norbury-Scott, multiplied by

$$(194)\qquad 2^{\hat{n}/2}(-1)^{2\hat{g}-2+\hat{n}} = (-\sqrt{2})^{\hat{n}},$$

is equal to the coefficient of $\prod_{k=1}^{\hat{n}} t^{d_k,i_k}$ in the same graph in Formula (62). This extra factor will be taken into account via rescaling of the variables by $-\sqrt{2}$.

3.5.8. *The Ψ-action.* Let us apply the Ψ-operator to the leaves. After comparing the R-action with graph expansion given formulas (62) and (90), and taking into account the extra factor of $-\sqrt{2}$, we have the following identification of the marking on the leaves:

$$(195)\qquad \sum_{a-b=c} (t^{a,1}, t^{a,2}) S_b = \left(W_c^1, W_c^2\right) \Psi^{-1}/(-\sqrt{2}).$$

Here

$$(196)\qquad W_0^1 = \left.\frac{dz}{(1-z)^2}\right|_{z=z(z_1)} + \left.\frac{dz}{(1-z)^2}\right|_{z=z(z_2)}$$

$$(197)\qquad W_0^2 = \left.\frac{i\,dz}{(1+z)^2}\right|_{z=z(z_1)} + \left.\frac{i\,dz}{(1+z)^2}\right|_{z=z(z_2)},$$

and

$$(198)\qquad W_c^i = d\left(\left(-2\frac{d}{dx}\right)^c \int W_0^i\right),$$

so we can work in the global coordinate z rather than in the local coordinates z_1, z_2.

Since

$$(199)\qquad \Psi^{-1}/(-\sqrt{2}) = \begin{pmatrix} \frac{-1}{2} & \frac{-1}{2} \\ \frac{-i}{2} & \frac{i}{2} \end{pmatrix},$$

we have:

$$\sum_{a-b=c}(t^{a,1},t^{a,2})S_b = \left(U_c^1, U_c^2\right), \tag{200}$$

where

$$U_0^1 = \frac{1}{2}\left(-\frac{dz}{(1-z)^2}+\frac{dz}{(1+z)^2}\right) \tag{201}$$

$$U_0^2 = \frac{-1}{2}\left(\frac{dz}{(1-z)^2}+\frac{dz}{(1+z)^2}\right) \tag{202}$$

and

$$U_c^i = d\left(\left(-2\frac{d}{dx}\right)^c \int U_0^i\right), \qquad i=1,2; c=0,1,2,\dots. \tag{203}$$

3.5.9. *The S-action.* The S-action is just a linear change of variables prescribed by Equation (200). This means that we replace each U_c^i with a linear combination of times $t^{a,j}$, $a \geq c$, where the coefficient of $t^{a,2}$ (this is the series of variables corresponding to the stationary sector) is equal to

$$\begin{cases} 0, & \text{if } a-c \text{ is even;} \\ \frac{1}{(k+1)\cdot(k!)^2}, & \text{if } a-c=2k+1. \end{cases} \tag{204}$$

for $i=1$, and

$$\begin{cases} \frac{1}{(k!)^2}, & \text{if } a-c=2k; \\ 0, & \text{if } a-c \text{ is odd.} \end{cases} \tag{205}$$

for $i=2$.

Norbury and Scott make the same kind of a linear change of variables, with the coefficient of $t^{a,2}$ in U_c^j, $j=1,2$, given by

$$-\operatorname*{Res}_{x=\infty}\frac{1}{(a+1)!}x^{a+1}U_c^j = \frac{1}{(a+1)!}\operatorname*{Res}_{z=0}\left(z+\frac{1}{z}\right)^{a+1}U_c^j. \tag{206}$$

In order to complete the proof of Theorem 3.21, we have to check two things: (1) that the Norbury-Scott formula for the contribution depends only on the difference $a-c$; (2) that for $c=0$ Equation (206) gives exactly the same coefficients as we have in Equations (204) and (205).

The first thing follows directly from the formula. Indeed,

$$\begin{aligned} -\oint \frac{x^{a+1}}{(a+1)!}d\left(\left(-2\frac{d}{dx}\right)^c\int U_0^j\right) &= \oint \frac{x^a}{(a)!}\left(\left(-2\frac{d}{dx}\right)^c\int U_0^j\right)dx \\ &= 2^c\oint\frac{x^{a-c}}{(a-c)!}\left(\int U_0^j\right)dx \\ &= -2^c\oint\frac{x^{a+1-c}}{(a+1-c)!}d\left(\int U_0^j\right). \end{aligned} \tag{207}$$

In particular, we see that the coefficient is equal to 0 if $a<c$.

Then, a direct computation shows that

$$\begin{aligned}
(208)\quad & \frac{1}{(a+1)!}\operatorname*{Res}_{z=0}\left(z+\frac{1}{z}\right)^{a+1} U_0^1 \\
&= \frac{1}{(a+1)!}\operatorname*{Res}_{z=0}\left(z+\frac{1}{z}\right)^{a+1} \frac{1}{2}\left(-\frac{dz}{(1-z)^2}+\frac{dz}{(1+z)^2}\right) \\
&= \frac{1}{(a+1)!}\operatorname*{Res}_{z=0}\left(z+\frac{1}{z}\right)^{a+1} \frac{-2z\,dz}{(1-z^2)^2} \\
&= \begin{cases} 0, & \text{if } a \text{ is even;} \\ \frac{-2}{(2k+2)!}\left(\binom{2k+2}{0}\cdot(k+1)+\binom{2k+2}{1}\cdot k+\cdots\binom{2k+2}{k}\cdot 1\right) & \text{if } a=2k+1. \end{cases} \\
&= \begin{cases} 0, & \text{if } a \text{ is even;} \\ \frac{-1}{(k+1)(k!)^2} & \text{if } a=2k+1. \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
(209)\quad & \frac{1}{(a+1)!}\operatorname*{Res}_{z=0}\left(z+\frac{1}{z}\right)^{a+1} U_0^2 \\
&= \frac{1}{(a+1)!}\operatorname*{Res}_{z=0}\left(z+\frac{1}{z}\right)^{a+1} \frac{-1}{2}\left(\frac{dz}{(1-z)^2}+\frac{dz}{(1+z)^2}\right) \\
&= \frac{-1}{(a+1)!}\operatorname*{Res}_{z=0}\left(z+\frac{1}{z}\right)^{a+2} \frac{z\,dz}{(1-z^2)^2} \\
&= \begin{cases} \frac{-1}{(2k+1)!}\left(\binom{2k+2}{0}\cdot(k+1)+\binom{2k+2}{1}\cdot k+\cdots\binom{2k+2}{k}\cdot 1\right) & \text{if } a=2k; \\ 0, & \text{if } a \text{ is odd} \end{cases} \\
&= \begin{cases} \frac{-1}{(k!)^2} & \text{if } a=2k; \\ 0, & \text{if } a \text{ is odd.} \end{cases}
\end{aligned}$$

We see that there is an extra factor of (-1) in all coefficients. This means that the (g,n)-correlation functions of Norbury-Scott differ from the stationary Gromov-Witten invariants of $\mathbb{C}\mathrm{P}^1$ by the factor of $(-1)^n$. But this factor is exactly the difference we must have because Norbury and Scott using a different convention on the sign in the topological recursion, cf. Remark 3.12. This completes the proof of Theorem 3.21.

4. Dubrovin's superpotential as a global spectral curve

In this section we describe a way of constructing global spectral curves for cohomological field theories when certain conditions are satisfied.

4.1. Spectral curve topological recursion via CohFTs. Here we reformulate the results of Section 3.4 in a different way which would suit us better for what follows.

Namely, we now consider a global spectral curve. Note that it gives rise to a local spectral curve when one takes the germs of all the functions at the branching points of x. Suppose that this local spectral curve corresponds to a cohomological field theory according to Section 3.4. Then let us formulate what that would mean in terms of the global spectral curve information.

We choose the local coordinates w_i in the domains U_i such that $x|_{U_i} = -w_i^2/2 + x(c_i)$, $i = 1, \dots, n$. The identification with the data of a CohFT then goes as follows:

$$\Delta_i^{-\frac{1}{2}} = \frac{dy}{dw_i}(0); \tag{210}$$

$$R^{-1}(\zeta^{-1})_i^j = -\frac{1}{\sqrt{2\pi\zeta}} \int_{-\infty}^{\infty} \frac{B(w_i, w_j)}{dw_i}\bigg|_{w_i=0} \cdot e^{(x(w_j)-x(c_j))\zeta}; \tag{211}$$

$$\sum_{k=1}^{n} (R^{-1}(\zeta^{-1}))_k^i \Delta_k^{-\frac{1}{2}} = \frac{\sqrt{\zeta}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy(w_i) \cdot e^{(x(w_i)-x(c_i))\zeta}. \tag{212}$$

Note that Equation (210) is in fact a consequence of Equation (212).

There is an extra condition on the bi-differential B that can be formulated as a requirement on decomposition of its Laplace transform as

$$\begin{aligned} &\frac{\sqrt{\zeta_1\zeta_2}}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B(w_i, w_j) e^{(x(w_i)-x(c_i))\zeta_1 + (x(w_j)-x(c_j))\zeta_2} \\ &= \frac{\sum_{k=1}^{n} R^{-1}(\zeta_1^{-1})_k^i R^{-1}(\zeta_2^{-1})_k^j}{\zeta_1^{-1} + \zeta_2^{-1}}. \end{aligned} \tag{213}$$

This assumption is always satisfied if the curve is compact and the differential dx is meromorphic. This uses a general finite decomposition for $B(p, q)$ proven by Eynard in Appendix B of [**25**] together with (211).

This data (the constants $\Delta_i^{-\frac{1}{2}}$ and the matrix $R^{-1}(\zeta^{-1})_i^j$) determine for us a semi-simple CohFT $\{\alpha_{g,k}\}$ with an n-dimensional space of primary fields $V := \langle e_1, \dots, e_n \rangle$. The differentials $\omega_{g,k}$ can be written in terms of the auxiliary functions

$$\xi^i(z) := \int^z \frac{B(w_i, \bullet)}{dw_i}\bigg|_{w_i=0} \tag{214}$$

as

$$\omega_{g,k} = \sum_{\substack{i_1,\dots,i_k \\ d_1,\dots,d_k}} \int_{\overline{\mathcal{M}}_{g,k}} \alpha_{g,k}(e_{i_1}, \dots, e_{i_k}) \prod_{j=1}^{k} \psi_j^{d_j} d\left(\left(\frac{d}{dx}\right)^{d_j} \xi^{i_j}\right). \tag{215}$$

(These kind of formulas are typically of ELSV-type, see [**21**] for explanation.) In terms of the underlying Frobenius manifold structure, the basis $e_1, \dots, e_n$ corresponds to the normalized canonical basis.

4.2. Superpotential and function y. The goal of this Section is to prove that Dubrovin's superpotential provides us with a Riemann surface with two functions, $x := \lambda$ and $y := p$, such that the local expansion of y near the critical points of x reproduces the unit vector at the point $(u_1, \dots, u_n)$ of the underlying Frobenius manifold as well as the value of the matrix R^{-1} on the unit vector. These two local properties of y are precisely equivalent to the equations (210) and (212).

Consider Dubrovin's construction of a superpotential on the Riemann surface $\mathcal{D}$ described in Section 2.3. It is associated to a Frobenius manifold with given constants $\Delta_i^{-\frac{1}{2}}$ and the matrix $R^{-1}(\zeta^{-1})_i^j$ at the point with canonical coordinates $u_1, \dots, u_n$. Consider the points $c_i = p(u_i, u) \in \mathcal{D}$. These points are the critical points of the function $x := \lambda$.

THEOREM 4.1. *Given a semi-simple Frobenius manifold M, and Dubrovin's construction of a superpotential $\mathcal{D}$ for M, define spectral curve data by $\Sigma = \mathcal{D}$, $x := \lambda$, $y := p$ (with B yet to be defined). Then equations* (210) *and* (212) *are satisfied for the constants $\Delta_i^{-\frac{1}{2}}$ and the matrix $R^{-1}(\zeta^{-1})_i^j$ associated to M.*

PROOF. Let us prove the first statement, namely, Equation (210) (though it is a corollary of Equation (212), it is convenient to check it directly). Indeed, Equation (17) states that near the points c_i the function p looks like

$$p = c_i + \Psi_{i,\mathbb{1}}(u)\sqrt{2(u_j - \lambda)} + O(u_j - \lambda).$$

Therefore, the derivative of p with respect to the local coordinate $w_i = \sqrt{2(u_i - \lambda)}$ at the point c_i is equal to $\Psi_{i,\mathbb{1}}(u) = \Delta_i^{-\frac{1}{2}}$.

Now we prove Equation (212). We can assume that the contour of integration on the right hand side in Equation (212) is the image of L_i under the map p. Then,

$$\frac{\sqrt{\zeta}}{\sqrt{2\pi}} \int_{p(L_i)} dp \cdot e^{(\lambda - u_i)\zeta} = \frac{\sqrt{\zeta}}{\sqrt{2\pi}} \int_{p(L_i)} \frac{dp}{d\lambda} \cdot e^{(\lambda - u_i)\zeta} d\lambda. \tag{216}$$

Here we treat dp and $d\lambda$ as 1-forms defined on the surface $\mathcal{D}$.

Observe that from equation (8) we have

$$\frac{d\phi^T}{d\lambda} = \phi^T \left(\frac{1}{2} - V\right)(U - \lambda)^{-1}. \tag{217}$$

Therefore, using definition (16), we get

$$\begin{aligned}\frac{dp}{d\lambda} &= \frac{d}{d\lambda}\frac{\sqrt{2}}{1-d}\phi^T(U-\lambda)\Psi\mathbb{1} = \frac{\sqrt{2}}{1-d}\phi^T\left(\frac{1}{2} - V\right)\Psi\mathbb{1} - \frac{\sqrt{2}}{1-d}\phi^T\Psi\mathbb{1} \\ &= \frac{\sqrt{2}}{1-d}\phi^T\Psi\Psi^{-1}\left(-\frac{1}{2} - V\right)\Psi\mathbb{1} = \frac{\sqrt{2}}{1-d}\phi^T\Psi\left(-\frac{1}{2} - \mu\right)\mathbb{1} \\ &= \frac{\sqrt{2}}{1-d}\phi^T\Psi\left(-\frac{1}{2} + \frac{d}{2}\right)\mathbb{1} = -\frac{1}{\sqrt{2}}\phi^T\Psi\mathbb{1}.\end{aligned} \tag{218}$$

(In this computation we used the fact that $\mu\mathbb{1} = (-d/2)\mathbb{1}$.)

Equations (14) and (15) imply that on the contour $p(L_i)$ the vector ϕ is equal to $\phi^{(i)} + E_i$, where E_i is some holomorphic function of $(u_i - \lambda)$. Recall also that $(\Psi\mathbb{1})_k = \Delta_k^{-\frac{1}{2}}$. Therefore,

$$\frac{\sqrt{\zeta}}{\sqrt{2\pi}} \int_{p(L_i)} \frac{dp}{d\lambda} \cdot e^{(\lambda - u_i)\zeta} d\lambda = \sum_{k=1}^{n} \Delta_k^{-\frac{1}{2}} \cdot \frac{-\sqrt{\zeta}}{2\sqrt{\pi}} \int_{p(L_i)} \phi_k^{(i)} \cdot e^{(\lambda - u_i)\zeta} d\lambda. \tag{219}$$

Dubrovin shows in [**13**, Proof of Lemma 5.4] that the second factor in this expression is $(R^{-1}(\zeta^{-1}))_k^i$. Thus the right hand side of Equation (219) coincides with the left hand side of Equation (212). This completes the proof of the Theorem. □

REMARK 4.2. Note that we have not used the specific formula for ϕ in the proof. We used only Equation (8) and the fact that the local expansion of ϕ for $\lambda \to u_i$ coincides with the local expansion of $\phi^{(i)}$ up to some holomorphic non-branching term. Thus, if we have a solution for (8) satisfying this property, we can use it directly in the formula for the superpotential (16), bypassing the requirement for G^{ij} to be non-degenerate. This will be important below in certain applications.

REMARK 4.3. *Flat identity.* Topological recursion satisfies the string equation.

$$\sum_{i=1}^{n} \operatorname*{res}_{p=c_i} y(p)\omega_{g,k+1}(p,p_1,...,p_k) = -\sum_{j=1}^{k} d_{p_j}\partial z_j \left(\frac{\omega_{g,k}(p_1,...,p_k)}{dx(p_j)}\right) \tag{220}$$

where the sum is over the zeros $dx(c_i) = 0$ and d_{p_j} is exterior derivative in the variables p_j. The operator $\omega \mapsto \sum_i \mathrm{Res}_{p=c_i} y(p)\omega(p)$ acts on differentials ω. It is non-zero (and evaluates to 1) on the auxiliary differential $\sum_j a_j d\xi^j$ corresponding to the flat identity and annihilates all others. In particular

$$\sum_i \operatorname*{res}_{p=c_i} d\left(\left(\frac{d}{dx}\right)^{d_j}\xi^{i_j}\right) = 0, \quad d_j > 0.$$

This corresponds to insertion/removal of the identity vector in ancestor invariants.

4.3. Compatibility between B and y. In this section we discuss a necessary condition on a spectral curve to be able to apply the inverse construction of Section 3, i.e. so that a CohFT can be reconstructed from this spectral curve.

More precisely, for a given data of a spectral curve (Σ, x, y, B) (maybe, local) Equations (211) and (212), (210) imply some relation for x, y, and B, and we want to state this relation in a direct geometric way rather than in terms of the Laplace transform.

The compatibility condition below is equivalent to differentiation of the potential of a CohFT by t_1 producing the string equation. In the language of [**27**], $\delta(ydx) = \int (dy/dx)(p')\,B(p,p') = d(dy/dx)$ gives rise to variations of $\omega_{g,k}$ corresponding to the string equation (220).

Recall that x defines a local involution σ_i near each zero c_i of dx, $i = 1,\dots,n$.

THEOREM 4.4. *A CohFT can be reconstructed from a spectral curve* (Σ, x, y, B) *via the inverse construction of Section* 3 *reformulated in Section* 4.1 *if and only if the 1-form on* Σ

$$\eta(z) = d\left(\frac{dy}{dx}(z)\right) + \sum_{i=1}^{n} \operatorname*{res}_{z'=c_i} \frac{dy}{dx}(z')B(z,z'). \tag{221}$$

is invariant under each local involution σ_i, $i = 1,\dots,n$.

PROOF. The construction of Section 3 requires equations (210), (211), and (212) to hold. We will prove that the 1-form (221) is invariant under each local involution σ_i, $i = 1,\dots,n$ if and only if equations (210), (211), and (212) are compatible (as equations for the unknown variables R^{-1} and $\Delta_i^{-\frac{1}{2}}$, $i = 1,\dots,n$).

Recall that $x = x(c_i) - w_i^2/2$ in a neighborhood of c_i. Note that

$$\begin{aligned} \operatorname*{res}_{w_i=0} \frac{dy}{dx}(w_i)B(z,w_i) &= \operatorname*{res}_{w_i=0} \frac{dy}{dw_i}(w_i)\cdot\frac{dw_i}{dx}\cdot B(z,w_i) \\ &= -\operatorname*{res}_{w_i=0} \frac{dy}{dw_i}(w_i)\cdot\frac{dw_i}{w_i}\cdot\frac{B(z,w_i)}{dw_i} = \frac{dy}{dw_i}(0)\cdot\left.\frac{B(z,w_i)}{dw_i}\right|_{w_i=0}. \end{aligned} \tag{222}$$

An equivalent way to say that η is σ_i-invariant is to say that the following Laplace transform of η is equal to zero:

$$\int_{-\infty}^{\infty} \eta(w_i)e^{(x(w_i)-x(c_i))\zeta} = 0. \tag{223}$$

On the other hand,

(224)
$$\int_{-\infty}^{\infty} \eta(w_i)e^{(x(w_i)-x(c_i))\zeta} = -\zeta \int_{-\infty}^{\infty} \frac{dy}{dx}(w_i)e^{(x(w_i)-x(c_i))\zeta}dx \\ - \sum_{j=1}^{n} \frac{dy}{dw_j}(0) \int_{-\infty}^{\infty} \left.\frac{B(w_i,w_j)}{dw_j}\right|_{w_j=0} e^{(x(w_i)-x(c_i))\zeta}.$$

Thus, Equation (223) is satisfied if and only if

$$\frac{\sqrt{\zeta}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy(w_i)e^{(x(w_i)-x(c_i))\zeta} \\ = \sum_{j=1}^{n} \frac{dy}{dw_j}(0) \cdot \frac{-1}{\sqrt{2\pi\zeta}} \int_{-\infty}^{\infty} \left.\frac{B(w_i,w_j)}{dw_j}\right|_{w_j=0} e^{(x(w_i)-x(c_i))\zeta}, \tag{225}$$

which is precisely the compatibility condition for Equations (210), (211), and (212). □

We can state (221) in simpler terms when the spectral curve is connected.

Corollary 4.5. *For a connected spectral curve, equations* (210), (211), *and* (212) *are compatible if and only if the 1-form defined in* (221) *is a pull-back of a 1-form downstairs, i.e.* $\eta(z) = x^*\omega$.

Proof. If $\eta(z) = x^*\omega$ for ω a differential downstairs then it is invariant under local involutions hence Theorem 4.4 applies. On a comnnected spectral curve Σ the converse is also true. This follows from the more general fact that any $\eta(z)$ which is invariant under local involutions defined around simple ramification points of $x : \Sigma \to \mathbb{C}$ is the pull-back of a differential downstairs. Take any regular (w.r.t. function x) point $p \in \Sigma$ and a path γ from p to a zero b of dx. Then $x(\gamma)$ is covered by a path $\tilde{\gamma} \subset \Sigma$ that contains p and p' where $x(p) = x(p')$. The local involution defined by x in a neighbourhood of b can be analytically continued along $\tilde{\gamma}$. Since $\eta(z)$ is invariant under the local involution at b, it is invariant under the continued involution above a neighbourhood of $x(\gamma)$. So $\eta(z)$ agrees (via identification of cotangent bundles using x) around p and p'. Connectedness of Σ guarantees that the monodromy of the cover defined by x is transitive and generated by local involutions. Hence we can find paths γ_i that can be used to show that $\eta(z)$ agrees around p and any point in the fibre over $x(p)$. Hence $\eta(z) = x^*\omega$ locally and this pieces together to give the global result. The result isn't true on disconnected curves, in particular local curves, because monodromy is not transitive. □

Let us show how this compatibility test can be used.

Proposition 4.6. *The differential* $\eta \equiv 0$*, hence Equation* (223) *is satisfied, when* Σ *is a global curve equipped with a canonical bidifferential* B *normalized so that* $\int_{p' \in \alpha_i} B(p,p') = 0$ *for a choice of A-cycles* α_i*, and one of the following holds:*

1. Σ *is rational with global coordinate* z *chosen so that* $x(z = \infty) = \infty$*;*
2. dy *is a meromorphic differential such that* $\frac{dy}{dx}$ *has poles only at the zeros of* dx*, for example* dy *is a holomorphic differential.*

Note that in case (2) above, we take y to be the (multiply-defined) primitive of a differential which is sufficient for the purposes of topological recursion—see Remark (2.6).

PROOF. Recall the property that for any function f on Σ, $\mathrm{Res}_{p'=p} f(p')B(p,p') = df(p)$ (independent of the choice of A-cycles along which B is normalized). For example, in the rational case $B = \frac{dzdz'}{(z-z')^2}$ and this property is the Cauchy integral formula. Since $\frac{dy}{dx}$ has poles only at the zeros of dx

$$\sum_{i=1}^{n} \operatorname*{res}_{p'=c_i} \frac{dy}{dx}(p')B(p,p') = -\operatorname*{res}_{p'=p} \frac{dy}{dx}(p')B(p,p') = -d\left(\frac{dy}{dx}(p)\right)$$

hence $\eta \equiv 0$. □

EXAMPLE 4.7. Consider $x = z+1/z$, $y = p(z)$ a polynomial. Then $\frac{dy}{dx} = \frac{z^2 p'(z)}{z^2-1}$ has poles at $z = \pm 1$ and possibly $z = \infty$. Hence $\eta(z) = dq(z)$ where $q(z)$ is a polynomial given by the principal part of dy/dx at $z = \infty$. A non-trivial polynomial has poles only at $z = \infty$ so if $\eta \neq 0$ it cannot be the pull-back of a differential form downstairs since it would necessarily require poles at $x^{-1}(\infty) = \{0, \infty\}$. Hence this fails the compatibility test, unless $\eta(z) \equiv 0$ i.e. $\deg p(z) \leq 1$. If $\deg p(z) = 1$ then Equation (223) is satisfied.

EXAMPLE 4.8. Consider $x = z + 1/z$, $y = \ln z$. Then $\frac{dy}{dx} = \frac{z}{z^2-1}$ has poles only at $z = \pm 1$ so Equation (223) is satisfied.

EXAMPLE 4.9. Since the compatibility test is a linear condition in y, $x = z+1/z$, $y = \ln z + cz$ also satisfies the compatibility test and leads to a CohFT with a flat unit. This was also observed in [**33**].

4.4. Superpotential as a global spectral curve in genus 0 case. In this Section we discuss a special case of Dubrovin's superpotential defined in Section 2.3 and show that it indeed gives a proper spectral curve for the correponding cohomological field theory.

More precisely, we start with a homogeneous cohomological field theory. Its genus zero part without descendants defines a Frobenius manifold that we assume to be semi-simple. Consider Dubrovin's construction in Section 2.3. Assume that this construction goes through in such a way that

(1) The form $d_\lambda p$ has no zeros in $\mathbb{C} \setminus \cup_{i=1}^n L_i$;
(2) $\lambda(p = \infty) = \infty$;
(3) The resulting curve $\mathcal{D}$ is a compact curve of genus 0 and p is a global coordinate on it;
(4) There is exactly one critical point in each singular fiber of function λ.

THEOREM 4.10. *Under the conditions* (1)-(4) *above, the correlators of the CohFT are related by Equation* (215) *to the correlator differentials obtained through spectral curve topological recursion on a curve* $\mathcal{D}$ *with* $x = \lambda$, $y = p$ *and* $B(p_1, p_2) = dp_1 dp_2/(p_1 - p_2)^2$.

In other words, in this case the ancestor potential of CohFT is reproduced by global topological recursion related to Dubrovin's superpotential. Note that this identification happens over an open ball in the underlying Frobenius manifold.

PROOF. First of all, note that since p is a global coordinate and $\lambda(p=\infty)=\infty$, this spectral curve satisfies the compatibility condition of Theorem 4.4, which means that one can reconstruct a CohFT such that Equation (215) is satisfied. We only need to prove that this CohFT is the same as the original one.

Theorem 4.1 implies that we have the right function y, so, in particular, the functions $\Delta_i^{-\frac{1}{2}}(u)$ are correctly reproduced on an open ball in the space of parameters $u_1,\dots,u_n$. Note that these functions determine completely the structure of Frobenius multiplication, so we can conclude that the CohFT reconstructed from the spectral curve data coincides with the original one in genus zero.

Higher genera correlators of a semi-simple CohFT are determined uniquely by genus 0 data in homogeneous cases [**60**]. Therefore, it is sufficient to prove that the CohFT reconstructed from the spectral curve data is homogeneous. We do this by proving the Euler equation for the corresponding R-matrix. Namely, a CohFT with an R-matrix $R(\xi)$ is homogeneous if and only if the R-matrix satisfies the Euler equation [**36**]:

$$\left(\xi\frac{d}{d\xi}+\sum_{i=1}^n u_i\frac{\partial}{\partial u_i}\right)R(\xi,u)=0 \tag{226}$$

(or, equivalently, we can consider the same equation for $R^{-1}(\xi,u)=R(-\xi,u)^T$). Using Equation (213), the Euler equation for the R-matrix can be rewritten as

$$\left(1+\xi_1\frac{\partial}{\partial\xi_1}+\xi_2\frac{\partial}{\partial\xi_2}+\sum_{i=1}^n u_i\frac{\partial}{\partial u_i}\right)\check{B}=0 \tag{227}$$

for $\check{B}=\check{B}^{ij}(\xi_1,\xi_2)$ given by

$$\frac{e^{-\frac{u_i}{\xi_1}-\frac{u_j}{\xi_2}}}{2\pi\sqrt{\xi_1\xi_2}}\int_{p(L_i)}\int_{p(L_j)}B\cdot e^{\frac{\lambda_1}{\xi_1}+\frac{\lambda_2}{\xi_2}}. \tag{228}$$

Recall that we consider the case when $d_\lambda p$ does not have zeros in $\mathbb{C}\setminus\cup_{i=1}^n L_i$, and the Riemann surface $\mathcal{D}$ that we get through Dubrovin's construction has genus 0. The Bergman kernel $B(p_1,p_2)$ has the form $dp_1dp_2/(p_1-p_2)^2$.

PROPOSITION 4.11. *Under these conditions Equation* (227) *is satisfied.*

We prove this proposition below. It implies that the R-matrix associated to the Bergman kernel in this case satsifies the Euler equation, and, therefore, the corresponding CohFT is homogeneous. This proposition completes the proof of Theorem 4.10. □

For the proof of Proposition 4.11 we need the following technical lemma:

LEMMA 4.12. *We have:*

$$\left(\lambda\frac{d}{d\lambda}+\sum_{i=1}^n u_i\frac{\partial}{\partial u_i}\right)p(\lambda,u)=\frac{1-d}{2}p(\lambda,u). \tag{229}$$

PROOF. Recall Equation (218):

$$d_\lambda p(\lambda,u)=-\frac{1}{\sqrt{2}}\phi^T d\lambda\Psi\mathbb{1}. \tag{230}$$

In the same way we prove that

$$d_u p(\lambda, u) = \frac{1}{\sqrt{2}} \phi^T dU \Psi \mathbb{1} \tag{231}$$

(this is [**13**, equation (5.66)]; note that there is a misprint in this equation in [**13**]). Combining these equations, we get

$$\left(\lambda \frac{d}{d\lambda} + \sum_{i=1}^{n} u_i \frac{\partial}{\partial u_i} \right) p(\lambda, u) = \frac{1}{\sqrt{2}} \phi^T (U - \lambda) \Psi \mathbb{1} = \frac{1-d}{2} p(\lambda, u). \tag{232}$$

□

Proof of Proposition 4.11. We have:

$$\begin{aligned} &\left(1 + \xi_1 \frac{\partial}{\partial \xi_1} + \xi_2 \frac{\partial}{\partial \xi_2} + \sum_{i=1}^{n} u_i \frac{\partial}{\partial u_i} \right) \check{B} \\ &= \frac{e^{-\frac{u_i}{\xi_1} - \frac{u_j}{\xi_2}}}{2\pi\sqrt{\xi_1 \xi_2}} \iint \frac{d\lambda_1 d\lambda_2}{(p(\lambda_1) - p(\lambda_2))^2} \frac{dp}{d\lambda}(\lambda_1) \frac{dp}{d\lambda}(\lambda_2) e^{\frac{\lambda_1}{\xi_1} + \frac{\lambda_2}{\xi_2}} X, \end{aligned} \tag{233}$$

where

$$\begin{aligned} X = &- \frac{\lambda_1}{\xi_1} - \frac{\lambda_2}{\xi_2} - 2 \cdot \frac{\left(\sum_{i=1}^{n} u_i \frac{\partial}{\partial u_i} \right) (p(\lambda_1) - p(\lambda_2))}{p(\lambda_1) - p(\lambda_2)} \\ &+ \frac{\left(\sum_{i=1}^{n} u_i \frac{\partial}{\partial u_i} \right) \frac{dp}{d\lambda}(\lambda_1)}{\frac{dp}{d\lambda}(\lambda_1)} + \frac{\left(\sum_{i=1}^{n} u_i \frac{\partial}{\partial u_i} \right) \frac{dp}{d\lambda}(\lambda_2)}{\frac{dp}{d\lambda}(\lambda_2)}. \end{aligned}$$

Applying the integration by parts to the terms $-\lambda_1/\xi_1$ and $-\lambda_2/\xi_2$, we can rewrite the right hand side of Equation (233) as

$$\frac{e^{-\frac{u_i}{\xi_1} - \frac{u_j}{\xi_2}}}{2\pi\sqrt{\xi_1 \xi_2}} \iint \frac{d\lambda_1 d\lambda_2}{(p(\lambda_1) - p(\lambda_2))^2} \frac{dp}{d\lambda}(\lambda_1) \frac{dp}{d\lambda}(\lambda_2) e^{\frac{\lambda_1}{\xi_1} + \frac{\lambda_2}{\xi_2}} Y, \tag{234}$$

where

$$\begin{aligned} Y = 2 &+ \frac{\left(\lambda_1 \frac{d}{d\lambda_1} + \sum_{i=1}^{n} u_i \frac{\partial}{\partial u_i} \right) \frac{dp}{d\lambda}(\lambda_1)}{\frac{dp}{d\lambda}(\lambda_1)} + \frac{\left(\lambda_2 \frac{d}{d\lambda_2} + \sum_{i=1}^{n} u_i \frac{\partial}{\partial u_i} \right) \frac{dp}{d\lambda}(\lambda_2)}{\frac{dp}{d\lambda}(\lambda_2)} \\ &- 2 \cdot \frac{\left(\lambda_1 \frac{d}{d\lambda_1} + \sum_{i=1}^{n} u_i \frac{\partial}{\partial u_i} \right) p(\lambda_1) - \left(\lambda_2 \frac{d}{d\lambda_2} + \sum_{i=1}^{n} u_i \frac{\partial}{\partial u_i} \right) p(\lambda_2)}{p(\lambda_1) - p(\lambda_2)}. \end{aligned}$$

Using Equation (229), we rewrite Y as

$$\begin{aligned} Y = {} & 2 + \frac{\left(-1 + \frac{1-d}{2}\right)\frac{dp}{d\lambda}(\lambda_1)}{\frac{dp}{d\lambda}(\lambda_1)} + \frac{\left(-1 + \frac{1-d}{2}\right)\frac{dp}{d\lambda}(\lambda_2)}{\frac{dp}{d\lambda}(\lambda_2)} \\ & - 2 \cdot \frac{\frac{1-d}{2}p(\lambda_1) - \frac{1-d}{2}p(\lambda_2)}{p(\lambda_1) - p(\lambda_2)} \\ = {} & 2 + \left(-1 + \frac{1-d}{2}\right) + \left(-1 + \frac{1-d}{2}\right) - 2 \cdot \frac{1-d}{2} = 0, \end{aligned}$$

which proves the proposition. □

4.5. Superpotential as a global spectral curve for arbitrary genus. In this Section we extend the result of the previous section to the case of a compact global curve of arbitrary genus.

THEOREM 4.13. *Given a conformal Frobenius manifold, construct a superpotential $p(\lambda; u)$ which defines the Riemann surface $\mathcal{D}$ according to Dubrovin's construction of Section* 2.3. *Assume the following:*

- *$\mathcal{D}$ is a compact curve of genus g;*
- *there is exactly one critical point in each singular fiber of $\lambda : \mathcal{D} \to \mathbb{C}$.*

Fix a symplectic basis $(\mathcal{A}_i, \mathcal{B}_i)_{i=1}^g$ of $H_1(\mathcal{D}, \mathbb{Z})$ and define $B(p_1, p_2)$ as the only Bergman kernel on $\mathcal{D}$ normalized by

$$\forall i = 1, \ldots, g, \ \oint_{p_1 \in \mathcal{A}_i} B(p_1, p_2) = 0. \tag{235}$$

Further assume that:

- *the pair $(p, B(p_1, p_2))$ passes the compatibility test of Section* 4.3 *in any of its possible forms (given by Theorem* 4.4*, Corollary* 4.5*, or Proposition* 4.6*).*

Then the correlators of the CohFT associated to the Frobenius manifold are related by Equation (215) *to the correlator differentials obtained through spectral curve topological recursion on the Riemann surface $\mathcal{D}$ with $x = \lambda$, $y = p$ and $B(p_1, p_2)$.*

REMARK 4.14. This result extends Theorem 4.10 to an arbitrary compact curve. The new feature is that one needs to normalize the Bergman kernel on an arbitrary basis of cycles. In particular, for each basis, we recover a total ancestor potential for the same CohFT.

PROOF. The proof is very similar to the proof of the genus 0 case presented in the preceding section. However, it is important to remark that this proof only relies on Rauch's variational formula, i.e. it is valid for any compact curve presented as a ramified cover of the Riemann sphere with simple branch points. It does not require any knowledge about an auxiliary meromorphic form such as the super-potential.

Let us first show that the $(0,3)$ correlators are independent of choice of normalisation cycles for B. $\omega_{0,3}$ depends on these choices, but when decomposed into linear combinations of auxiliary differentials $d\xi^j = B/ds_j$ (for s_j defined by $x = (1/2)s_j^2 + a_j$) the coefficients are independent of A-cycles. By reconstruction, as in the proof of Theorem 4.10, this means that all correlators are the same. The formula

$$\begin{aligned}\omega_{0,3}(z_1,z_2,z_3) &= \sum_{i=1}^{n} \operatorname*{Res}_{p=c_i} B(p,z_1)B(p,z_2)B(p,z_3)/dx(p)dy(p) \\ &= \sum_{i=1}^{n} B(a_i,z_1)B(a_i,z_2)B(a_i,z_3)/x''(a_i)y'(a_i) \\ &= \sum_{i=1}^{n} \langle ....\rangle d\xi^i(z_1)d\xi^i(z_2)d\xi^i(z_3)\end{aligned}$$

shows the independence of the coefficients $\langle ...\rangle$ on the choice of B.

For the rest of the proof, the only part differing from the genus 0 case is the proof of the homogeneity of the CohFT, i.e. the fact that the R-matrix satisfies the Euler equation.

The first step consists in proving that there exist a R-matrix. This is due to a lemma of Eynard [**25**]:

LEMMA 4.15. *If $d\lambda$ is a meromorphic form on $\mathcal{D}$ and B the Bergman kernel normalized on a basis of $\mathcal{A}$-cycles as above, then the Laplace transform of the Bergman kernel satisfies Equation* (213) .

The Euler equation for the R-matrix is then equivalent to the following equation for the Laplace transform of B:

$$\left(1+\xi_1\frac{\partial}{\partial \xi_1}+\xi_2\frac{\partial}{\partial \xi_2}+\sum_{i=1}^{n} u_i\frac{\partial}{\partial u_i}\right)\check{B}=0 \tag{236}$$

for $\check{B}=\check{B}^{ij}(\xi_1,\xi_2)$ given by

$$\frac{e^{-\frac{u_i}{\xi_1}-\frac{u_j}{\xi_2}}}{2\pi\sqrt{\xi_1\xi_2}}\int_{p(L_i)}\int_{p(L_j)} B\cdot e^{\frac{\lambda_1}{\xi_1}+\frac{\lambda_2}{\xi_2}}. \tag{237}$$

By inverting the Laplace transform and integration by part, this is equivalent to

$$d_1\left(\frac{\lambda_1\, B(p_1,p_2)}{d\lambda_1}\right)+d_2\left(\frac{\lambda_2\, B(p_1,p_2)}{d\lambda_2}\right)+\sum_{i=1}^{n} u_i\frac{\partial}{\partial u_i}B(p_1,p_2). \tag{238}$$

In order to prove this equation, we remind Rauch's variational formula which expresses the variations of the Bergman kernel under deformation of the spectral curve. In particular

$$\frac{\partial B(p_1,p_2)}{\partial u_i}=\operatorname*{Res}_{r\to a_i}\frac{B(p_1,r)\,B(p_2,r)}{d\lambda(r)} \tag{239}$$

which implies that

$$\sum_{i=1}^{n} u_i\frac{\partial}{\partial u_i}B(p_1,p_2)=\sum_{i=1}^{n}\operatorname*{Res}_{r\to a_i}\frac{\lambda(r)B(p_1,r)\,B(p_2,r)}{d\lambda(r)}. \tag{240}$$

Moving the integration contours around the other poles of the integrands and reminding that the $\mathcal{A}$-periods of $B(p,r)$ are vanishing, this reads

(241)
$$\sum_{i=1}^{n} u_i \frac{\partial}{\partial u_i} B(p_1,p_2) = -\operatorname*{Res}_{r\to p_1,p_2} \frac{\lambda(r)B(p_1,r)\,B(p_2,r)}{d\lambda(r)} = -\left(\frac{d}{d\lambda_1}+\frac{d}{d\lambda_2}\right)B(p_1,p_2)$$

proving Equation 238. □

4.6. Global curves for A_n singularities. In this Section we apply the results of Sections 4.2, 4.3, and 4.4 in order to construct the spectral curve for the ancestor potential of A_n-singularities, $n=1,2,\dots$. The structure of this Frobenius manifolds is described in terms of Saito's theory on the space of polynomials

$$f(p,\tau) = p^{n+1} + \tau_1 p^{n-1} + \cdots \tau_n. \tag{242}$$

We refer to [**12**, **37**] for the detailed description of the structure of this Frobenius manifold. In particular, it is enough to say that $\lambda = f(p,\tau)$ is a superpotential of this Frobenius manifold.

The corresponding CohFT is well-studied. It was a subject of Witten's conjecture [**61**] proved in [**32**]. We refer to [**56**] for an exposition of this CohFT that includes an overview of its constructions; the CohFT whose correlators give the ancestor potential at the point τ of this Frobenius manifold is called there the shifted Witten class of A_n singularity.

Theorem 4.16. *The correlation differentials of the global spectral curve data $\Sigma := \mathbb{CP}^1$, $y := p$ (the global coordinate), $x := f(p,\tau)$, $B := dp_1dp_2/(p_1-p_2)^2$ are expressed via Equation* (215) *in terms of the shifted Witten class of A_n singularity.*

Proof. As we have already mentioned, the function $\lambda = f(p,\tau)$ is known to be a superpotential of the corresponding Frobenius manifold. We have to show that this superpotential can be obtained by Dubrovin's construction in Section 2.3. Then it is easy to see that all conditions of Theorem 4.10 are satisfied, which implies this theorem.

We construct solutions of Equation (8) in terms of the integrals over the vanishing cycles. Namely, consider the tangent bundle over the space of polynomials parametrized by $\tau \in \mathcal{T}$. It is identified with the space $\mathbb{C}[p]/(d_p f(p,\tau)/dp)$ by the map $v \mapsto (d_\tau f(p,\tau))(v)$ and equipped with a flat metric given by

$$(v_1,v_2) := \operatorname*{res}_{p=\infty} \frac{d_\tau f(v_1) d_\tau f(v_2)}{d_p f(p,\tau)/dp} dp. \tag{243}$$

For a cycle $\beta \in H_0(f^{-1}(\lambda),\mathbb{C})$ we denote by $I_\beta(\lambda,\tau)$ the section of the tangent bundle specified by the following formula:

$$(I_\beta(\lambda,\tau),v) := \int_\beta d_\tau f(v)\cdot \frac{dp}{d_p f}. \tag{244}$$

In normalized canonical coordinates $I_\beta(\lambda,\tau)$ is represented by the vector $\phi^\beta(\lambda,\tau)$ with components given by

$$\phi_i^\beta(\lambda,\tau) := \left(I_\beta(\lambda,\tau), \frac{\Delta_i^{\frac12}}{\sqrt{2}}\frac{\partial}{\partial u_i}\right) \tag{245}$$

is a solution of Equation (8) (see [**37**]). Let us discuss the singularities of this solution, depending on β.

Consider the λ-plane as the image of the map $\lambda = f(p,\tau)$. Let $u_1,\dots,u_n$ be the critical values of $f(p,\tau)$. We can alway choose a system of cuts L_i, $i=1,\dots,n$, from u_i to infinity such that the preimage $f^{-1}(\mathbb{C}\setminus\cup_{i=1}^n L_i)$ is a union of $n+1$ disks, $D_0, D_1,\dots,D_n$, glued along the boundary cuts in the following way:

- D_0 is glued to D_i along the boundary that is a double cover of L_i; in particular, their common boundary contains the critical preimage of u_i;
- All other lifts of the cut L_i are just cuts inside D_j, $j\neq 0,i$; the endpoints of these cuts are non-critical preimages of u_i.

For $n=1,2,3$ we give the corresponding pictures for a real orientable blow-up at infinity (that is, the boundary circle on the picture corresponds to the infinity point of the source sphere). The domain D_0 is shadowed there.

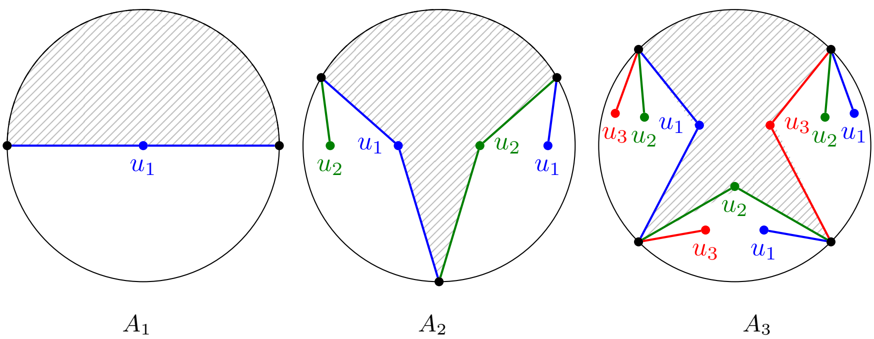

FIGURE 1. A_n-singularities, $1\le n\le 3$

Consider the vanishing cycles $\beta_i\in H_0(f^{-1}(\lambda),\mathbb{C})$ given by $\beta_i := p_0 - p_i$, where $\lambda = f(p_0,\tau) = f(p_i,\tau)$, and $p_0\in D_0$, $p_i\in D_i$. Then the system of solutions of Equation (8) given by $\phi^{(i)}(\lambda,\tau) := \phi^{\beta_i}(\lambda,\tau)$ satisfies the properties given by Equations (10)–(13). In particular, $G^{ij}=1/2$ for $i\neq j$ and $G^{ii}=1$. The inverse matrix is given by $G_{ii}=2n/(n+1)$ and $G_{ij}=-2/(n+1)$ for $i\neq j$. Therefore, Dubrovin's solution $\phi=\sum_{i,j=1}^n G_{ij}\phi^{(j)}$ is equal to ϕ^{β_0} for

$$\beta_0 = \sum_{i=1}^n \left(\frac{2n}{n+1} - (n-1)\frac{2}{n+1}\right)\beta_i = 2p_0 - \frac{2}{n+1}\sum_{i=0}^n p_i. \tag{246}$$

Recall that for the Frobenius structure A_n, $d=(n-1)/(n+1)$. Also we recall that for the Euler vector field $E=\sum_{i=1}^n u_i\frac{\partial}{\partial u_i}$ and the unit vector field $e=\sum_{i=1}^n\frac{\partial}{\partial u_i}$ so that we have:

$$Ef(p,\tau) = f(p,\tau) - \frac{p}{n+1}\frac{d_pf(p,\tau)}{dp}, \tag{247}$$

$$ef(p,\tau) = 1. \tag{248}$$

The formula $\phi^T(U-\lambda)\Psi\mathbb{1}$ can be written as $\big(I_{\beta_0}(\lambda,\tau),(E-\lambda e)/\sqrt{2}\big)$. Therefore,

$$\begin{aligned}\frac{\sqrt{2}}{1-d}\phi^T(U-\lambda)\Psi\mathbb{1} &= \frac{n+1}{2}\int_{\beta_0}(E-\lambda e)f(p,\tau)\cdot\frac{dp}{d_pf(p,\tau)}\\ &= \frac{n+1}{2}\int_{\beta_0}\left(f(p,\tau)-\frac{p}{n+1}\frac{d_pf(p,\tau)}{dp}-\lambda\right)\cdot\frac{dp}{d_pf(p,\tau)}.\end{aligned} \tag{249}$$

Since the cycle β_0 lies in $f^{-1}(\lambda)$, then $(f(p,\tau)-\lambda)|_{\beta_0}=0$. Therefore, the last integral can be rewritten as

$$\frac{n+1}{2}\int_{\beta_0}\frac{p}{n+1}=p_0(\lambda,\tau)-\frac{1}{n+1}\sum_{i=0}^{n}p_i(\lambda,\tau). \tag{250}$$

Since $\sum_{i=0}^{n}p_i(\lambda,\tau)=0$ (recall the form of the polynomial $f(p,\lambda)$), we conclude that the function $\frac{\sqrt{2}}{1-d}\phi^T(U-\lambda)\Psi\mathbb{1}$ is equal to the branch D_0 of $p=f^{-1}(\lambda,\tau)$.

So, $p(\mathbb{C}\backslash\cup_{i=1}^{n}L_i)=D_0$. It is obvious that $d_\lambda p$ has no zeros in $\mathbb{C}\backslash\cup_{i=1}^{n}L_i$, so $\hat{D}=D$, and one of the possible analytic continuation of the function $\lambda=f(p,\tau)|_{D_0}$ is its extension to the polynomial $f(p,\tau)$ defined on $\mathbb{CP}^1$. All condition of Theorem 4.10 are satisfied, so we apply it here to complete the proof. □

REMARK 4.17 (Relation to Milanov's spectral curve). The global spectral curve that we constructed differs from the one constructed by Milanov in [**50**]. Milanov gets a spectral curve with the same local behavior as $x=f(y,\tau)$ near the critical points, but, in our terms, he chooses a different analytic continuation of $\lambda|_D$. He constructs an analytic continuation using the action of the Weyl group , and obtains a curve where all preimages of the critical points in the x-plane are critical. In our terms, this can be achieved by gluing $n!$ copies of the curve $x=f(y,\tau)$ along the cuts connecting the non-critical preimages of the points u_i, $i=1,\dots,n$ such that each point belongs to exactly one cut. This makes all preimages of $u_1,\dots,u_n$ critical and will produce a curve of genus $1+\frac{n!}{2}(\frac{n^2}{2}-\frac{n}{2}-2)$ where the function x has degree $(n+1)!$, and it has $n!$ poles of degree $(n+1)$ each .

4.6.1. *Bouchard-Eynard recursion.* In this Section we discuss an application of Theorem 4.16. There is a more general formulation of topological recursion that works for functions x with higher order singular points [**6**]. Locally, a higher order singularity is given by $x=y^{n+1}$, $B=dy_1dy_2/(y_1-y_2)^2$. Bouchard and Eynard announced a theorem [**7**] that identifies the coefficients of the local expansion in y at $y=0$ of the correlation differentials of this spectral curve with the coefficients of the string solution of the $(r+1)$-Gelfand-Dickey hierarchy, also known as the total descendant potential of the A_r singularity. The proof of Bouchard and Eynard goes through analysis of matrix models. Here we give a new proof of their theorem, namely, we derive it directly from Theorem 4.16.

THEOREM 4.18. [**7**] *The Bouchard-Eynard recursion applied to $x=p^{n+1}$, $y=p$, $B=dp_1dp_2/(p_1-p_2)^2$ produces differentials $\omega_{g,k}$, whose expansions near infinity are given by*

$$\omega_{g,k}(p_1,\dots,p_k)=\sum_{\substack{0\le a_1,\dots,a_k\le n-1\\ d_1,\dots,d_k}}\langle\tau_{d_1a_1}\cdots\tau_{d_ka_k}\rangle_{g,k}\times \tag{251}$$
$$\prod_{j=1}^{k}\left(\frac{(a_j+1)(a_j+1+(n+1))\cdots(a_j+1+d_j(n+1))}{(-1)^{d_j}(n+1)^{d_j+1}}\frac{dp_j}{p_j^{(n+1)d_j+a_j+2}}\right),$$

where $\langle\tau_{d_1a_1}\cdots\tau_{d_ka_k}\rangle_{g,k}$ are the coefficients of the string solution of the $(n+1)$-Gelfand-Dickii hierarchy [**32, 37, 61**].

Note that we don't recall and don't use the definition of the Bouchard-Eynard recursion. The only property that we are using here is that it is compatible with the usual recursion on the curves with simple singularities and the limits [**6**]. In this case, we know that in the neighborhood of infinity the correlation differentials of the Bouchard-Eynard recursion are the limits for $\epsilon \to 0$ of the correlation differentials of the usual recursion applied to $x = y^{n+1} + \epsilon y$.

Let us now prove theorem 4.18.

PROOF. The flat coordinates $t_0 = t_{\mathbb{1}}, t_1, \ldots, t_{n-1}$ are given on the space of polynomials $f(p,\tau)$ defined in Equation (242) by the following formula:

$$f(p,\tau)^{\frac{1}{n+1}} = p + \frac{1}{n+1}\left(\frac{t_{n-1}}{p} + \frac{t_{n-2}}{p^2} + \cdots + \frac{t_0}{p^n}\right) + O\left(\frac{1}{p^{n+1}}\right). \tag{252}$$

Recall that the canonical coordinates are the critical values $u_1, \ldots, u_n$ of $f(p,\tau)$ and it is obvious that $\partial u_i / \partial t_{\mathbb{1}} = 1$. We denote by $c_1, \ldots, c_n$ the positions of the critical points of function $f(p,\tau)$; so $u_i = f(c_i, \tau)$, $i = 1, \ldots, n$.

We perform all computations only on a special curve in the space of polynomials, namely, $f(p,\tau) = p^{r+1} + \epsilon p$, and we are interested in all results only up $O(\epsilon)$ for $\epsilon \to 0$. In particular, we note that $t_a = O(\epsilon)$, $a = 0, \ldots, n-1$.

The full Jacobian of the change from the canonical to flat coordinates is then given by the following computation:

$$\begin{aligned}\frac{\partial u_i}{\partial t_a} = \frac{\partial f(c_i,\tau)}{\partial t_a} &= f(c_i,\tau)^{\frac{n}{n+1}}\left(\frac{1}{c_i^{n-a}} + O(\epsilon)\right) = c_i^a \frac{\partial u_i}{\partial t_0} + O(\epsilon) \\ &= c_i^a + O(\epsilon).\end{aligned} \tag{253}$$

The correlation differentials, written in terms of a CohFT considered in normalized canonical frame in Equation (215), can be rewritten in terms of the correlators of the ancestor potential of Givental [**37**] $\mathcal{A}_t(\{t_{d,\alpha}\})$ considered at the point t in flat coordinates $t_{d,\alpha}$, $d = 0, 1, 2, \ldots,$, $\alpha = 0, \ldots, n-1$ in the following way:

(254)

$$\begin{aligned}\omega_{g,k} &= \sum_{\substack{i_1,\ldots,i_k \\ d_1,\ldots,d_k}} \int_{\overline{\mathcal{M}}_{g,k}} \alpha_{g,k}\left(\Delta_{i_1}^{\frac{1}{2}}\frac{\partial}{\partial u_{i_1}}, \ldots, \Delta_{i_k}^{\frac{1}{2}}\frac{\partial}{\partial u_{i_k}}\right) \prod_{j=1}^{k} \psi_j^{d_j} d\left(\left(\frac{d}{dx}\right)^{d_j} \xi_{i_j}\right). \\ &= \sum_{\substack{\alpha_1,\ldots,\alpha_k \\ d_1,\ldots,d_k}} \langle \tau_{d_1\alpha_1} \cdots \tau_{d_k\alpha_k}\rangle_{g,k}(t) \prod_{j=1}^{k} d\left(\left(\frac{d}{dx}\right)^{d_j} \frac{-1}{p^{\alpha_j+1}}\right) + O(\epsilon).\end{aligned}$$

(by $\langle \tau_{d_1\alpha_1} \cdots \tau_{d_k\alpha_k}\rangle_{g,k}(t)$ we denote the coefficients of the expansion of $\log \mathcal{A}_t$). Indeed, let us expand the vector $\sum_{i=1}^n \xi_i(p)\Delta_i^{\frac{1}{2}}\frac{\partial}{\partial u_i}$ near $p = \infty$. Recall that we denote by $c_1, \ldots, c_n$ the positions of the critical points of function $f(p,\tau)$. We have:

$$\begin{aligned}\sum_{i=1}^{n} \xi_i(p)\Delta_i^{\frac{1}{2}}\frac{\partial}{\partial u_i} &= \sum_{i=1}^{n}\left(\frac{1}{z-p}\frac{dz}{d\sqrt{f(p,\tau)-u_i}}\right)\Bigg|_{z=c_i} \Delta_i^{\frac{1}{2}}\frac{\partial}{\partial u_i} \\ &= -\sum_{k=0}^{\infty}\frac{1}{p^{k+1}}\sum_{i=1}^{n} c_i^k \frac{\partial}{\partial u_i} = -\sum_{k=0}^{n-1}\frac{1}{p^{k+1}}\frac{\partial}{\partial t_k} + O(\epsilon).\end{aligned} \tag{255}$$

(we use Equation (253) and the fact that $c_i^n = O(\epsilon)$ for the last equality).

Recall [**56**] that the correlators of the ancestor potential $\mathcal{A}_t$ are represented in terms of the correlators of the descendant potential (which is exactly the string solution of the $(n+1)$-Gelfand-Dickey hierarchy) as

$$\langle \tau_{d_1\alpha_1} \cdots \tau_{d_k\alpha_k} \rangle_{g,k}(t) = \langle \tau_{d_1\alpha_1} \cdots \tau_{d_k\alpha_k} \rangle_{g,k} + O(\epsilon). \tag{256}$$

Thus we see that

$$\omega_{g,k} = \sum_{\substack{\alpha_1,\dots,\alpha_k \\ d_1,\dots,d_k}} \langle \tau_{d_1\alpha_1} \cdots \tau_{d_k\alpha_k} \rangle_{g,k} \prod_{j=1}^{k} d\left(\left(\frac{d}{dx}\right)^{d_j} \frac{-1}{p^{\alpha_j+1}}\right) + O(\epsilon). \tag{257}$$

In the limit $\epsilon \to 0$ we get exactly Equation (251). □

4.7. Frobenius manifolds for hypermaps. In this Section we construct a global spectral curve for the Frobenius manifold given by the superpotential $\lambda = f(p,a)$, where $a = (a_0, \dots, a_{n+1})$, $n \geq 1$, and

$$f(p,a) = p^n + a_2 p^{n-2} + a_3 p^{n-3} + \dots + a_n + \frac{a_{n+1}}{p - a_1}. \tag{258}$$

This superpotential defines a semi-simple Frobenius manifold (this Frobenius manifold is studied in [**12**, Section 5]). Furthermore the spectral curve

$$(\Sigma, x, y, B) = \left(\mathbb{CP}^1, f(p,a), p, \frac{dp_1 dp_2}{(p_1 - p_2)^2}\right)$$

satisfies equations (210)-(213) hence it stores the correlators of a CohFT via Equation (215). The following theorem answers the question of whether these two CohFTs coincide.

THEOREM 4.19. *The CohFT associated to the Frobenius manifold given by the superpotential $\lambda = f(p,a)$ coincides with the one reconstructed from the spectral curve $(\mathbb{CP}^1, f(p,a), p, dp_1dp_2/(p_1-p_2)^2)$.*

REMARK 4.20. The correlation differentials for this spectral curve considered for the particular values of the parameters a enumerate hypermaps on the curves. This is proved in [**20**], see also [**9**], where some special case of that was conjectured.

So, Theorem 4.19 is to be used in the converse way: We start with a combinatorial problem that is known to be solved by global topological recursion. It appears that the correlators of this global topological recursion are expressed in terms of a CohFT. This CohFT appears to be homogeneous, so it is associated to a Frobenius manifold, and this Theorem describes precisely the underlying Frobenius manifold.

PROOF. The proof is completely parallel to the proof of Theorem 4.16. Note that as in the case of A_n-singularity, we claim that the spectral curve is the superpotential itself. We use Theorem 4.10, so it is enough to show that we can reproduce the superpotential $\lambda = f(p,a)$ via Dubrovin's construction from Section 2.3.

The canonical coordinates $u_1, \dots, u_{n+1}$ of this Frobenius manifold are the critical values of $f(p,a)$; the Euler vector field is given by

$$E = \sum_{i=1}^{n} u_i \frac{\partial}{\partial u_i} = \sum_{i=1}^{n+1} \frac{i}{n} a_i \frac{\partial}{\partial a_i}; \tag{259}$$

the unit vector field is equal to

$$e = \sum_{i=1}^{n} \frac{\partial}{\partial u_i} = \frac{\partial}{\partial a_n}; \tag{260}$$

and the constant d is equal to $(n-2)/n$. Note that

$$Ef(p,a) = f(p,a) - \frac{p}{n}\frac{df(p,a)}{dp}. \tag{261}$$

As in the case of A_n singularity, the solutions to the Equation (8) are given by the integrals over the cycles $\beta \in H_0(f^{-1}(\lambda), \mathbb{C})$, where the components of the solutions are given by

$$\phi_i^\beta := \int_\beta \frac{\Delta_i^{\frac{1}{2}}}{\sqrt{2}} \frac{\partial f(p,a)}{\partial u_i} \cdot \left(\frac{df(p,a)}{dp}\right)^{-1}. \tag{262}$$

Consider the λ-plane as the image of the map $\lambda = f(p,\tau)$. Recall that $u_1, \dots, u_{n+1}$ are the critical values of $f(p,\tau)$. We can alway choose a system of cuts L_i, $i = 1, \dots, n+1$, from u_i to infinity such that the preimage $f^{-1}(\mathbb{C} \setminus \cup_{i=1}^n L_i)$ is a union of $n+1$ disks, $D_0, D_1, \dots, D_n$, glued along the boundary cuts in the following way:

- D_0 is glued to D_i, $i = 2, \dots, n$, along the boundary that is a double cover of L_i; in particular, their common boundary contains the critical preimage of u_i;
- D_0 is glued to D_1 along two components of the boundary that are double covers of L_1 and L_{n+1} and these boundary components have common point $p = a_1$. In particular, these boundary components contain the critical preimages of u_1 and u_{n+1};
- All other lifts of the cut L_i are just cuts inside D_j, $j \neq 0, i$ for $i = 2, \dots, n$ and $j \neq 0, 1$ for $i = 1, n+1$; the endpoints of these cuts are non-critical preimages of u_i.

For $n = 1, 2, 3$ we give the corresponding pictures for a real orientable blow-up at infinity (that is, the external boundary circle on the picture corresponds to the infinity point of the source sphere, and the internal circle corresponds to $p = a_1$). The domain D_0 is shadowed.

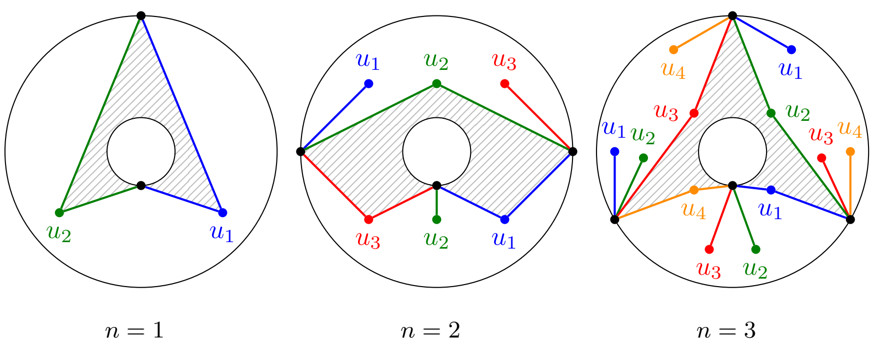

FIGURE 2. Hypermaps $z^n + \frac{1}{z}$, $1 \leq n \leq 3$

Consider the vanishing cycles $\beta_i \in H_0(f^{-1}(\lambda), \mathbb{C})$ given by $\beta_i := p_i - p_0$, where $\lambda = f(p_0,\tau) = f(p_i,\tau)$, and $p_0 \in D_0$, $p_i \in D_i$. Then the system of solutions of Equation (8) given by $\phi^{(i)}(\lambda,\tau) := \phi^{\beta_i}(\lambda,\tau)$, $i = 1, \dots, n$, $\phi^{(n+1)} = \phi^{(1)}$, satisfies the properties given by Equations (10)–(13). In particular, $G^{ii} = 1$ for $i = 1, \dots,$

$n+1$, $G^{1,n+1} = G^{n+1,1} = 1$, and for all other $i \neq j$ $G^{ij} = 1/2$. So, this matrix is degenerate.

However, Remark 4.2 specifies the properties of ϕ that are sufficient for Theorems 4.1 and 4.10. Note that

$$\phi := \frac{2}{n+1} \sum_{i=1}^{n} \phi^{(i)} = \phi^{\beta_0} \tag{263}$$

for $\beta_0 := \frac{2}{n+1}\sum_{i=0}^{n} p_i - 2p_0$ satisfies all condition of Remark 4.2. With this choice of ϕ and, therefore, β_0, Dubrovin's superpotential can be presented as

$$\frac{\sqrt{2}}{1-d}\phi^T (U-\lambda)\Psi \mathbb{1} = \frac{n}{2}\int_{\beta_0} (E-\lambda e) f(p,a) \cdot \left(\frac{df(p,a)}{dp}\right)^{-1}. \tag{264}$$

Using Equation (261), we have:

$$\begin{aligned}\frac{\sqrt{2}}{1-d}\phi^T (U-\lambda)\Psi \mathbb{1} &= \frac{n}{2}\int_{\beta_0} \left(f(p,a) - \frac{p}{n}\frac{df(p,a)}{dp} - \lambda\right)\cdot\left(\frac{df(p,a)}{dp}\right)^{-1} \\ &= \int_{\beta_0} -\frac{p}{2} = p_0 - \frac{1}{n+1}\sum_{i=0}^{n} p_i = \begin{cases} p_0 - \frac{\lambda + a_1}{2} & n=1; \\ p_0 - \frac{a_1}{n+1} & n>1. \end{cases}\end{aligned} \tag{265}$$

(in the last equality we used that we know the sum of all roots of the equation $f(p,a) = \lambda$).

Let us now discuss the cases that we get. For $n>1$ Dubrovin's function $p_{\mathrm{Dub}} = p_{\mathrm{Dub}}(\lambda, a)$ is the branch D_0 of the inverse function of $\lambda = f(p,a)$ shifted by a constant. Obviously, $d_\lambda p_{\mathrm{Dub}}$ has no zeros in $\mathbb{C}\setminus\cup_{i=1}^{n+1} L_i$, and we can choose as the analytic extension of $\lambda|_{D_0}$ the function $\lambda = f(p,a)$ defined on $\mathbb{CP}^1$. Then Theorem 4.10 is applied. We get, therefore, not precisely the statement that we want to prove, but we have instead $y = p_{\mathrm{Dub}} = p - a_1/(n+1)$ (the Bergman kernel is still the same). However, it doesn't change anything in topological recursion if we shift y by a constant.

The case $n=1$ is even more interesting. One can easily check by direct computation that Dubrovin's function $p_{\mathrm{Dub}} = p_{\mathrm{Dub}}(\lambda, a)$ is equal to $\sqrt{(\lambda-u_1)(\lambda-u_2)}/2$. Further construction of the curve gives the following equation:

$$p_{\mathrm{Dub}}^2 - \frac{1}{4}(\lambda - a_1)^2 + a_2 = 0$$

It is a rational curve, and it has a global coordinate $p = p_{\mathrm{Dub}} + (\lambda + a_1)/2$, which is our original coordinate p, that is, $\lambda = p + a_2/(p-a_1)$. Theorem 4.10 can not be applied directly, but in this case we can just check by hand that we get the statement that we want to prove.

Note that Theorem 4.1 suggests that the right choice of function y is $y = p_{\mathrm{Dub}} = p - (\lambda + a_1)/2$ rather than $y = p$. However, it doesn't change anything in topological recursion if we shift y by a function of $x = \lambda$, so there is no contradiction. □

Since in this example we rather start from a combinatorial problem of enumeration of hypermaps and use Theorem 4.19 in order to clarify the structure of the ELSV-type formula (215) for this combinatorial problem, it is interesting to have a description of the underlying Frobenius manifolds (given by superpotentials) in terms of their prepotentials. We know an algorithm which can produce this prepotential for any given n (this algorithm follows from Dubrovin's construction found in [**12**]), but we do not know a general formula which would describe these prepotentials for all $n \geq 1$. Here we list the formulas for cases $n = 1, 2, 3$:

$$
\begin{aligned}
n = 1: &\quad \frac{a_1^2 a_2}{2} + \frac{a_2^2}{2}\log a_2;\\
n = 2: &\quad \frac{a_1^3}{6} + a_1 a_2 a_3 + \frac{a_3^2}{2}\log a_3 + \frac{a_1^3 a_3}{6} - \frac{3a_3^2}{4};\\
n = 3: &\quad \frac{a_1^2 a_4}{2} + a_1 a_2 a_3 - \frac{3a_2^2}{4} + \frac{a_2^2}{2}\log(a_2) + \frac{a_2 a_3^4}{4} + \frac{3a_2 a_3^2 a_4}{2} + \frac{3a_2 a_4^2}{2} - \frac{3a_4^4}{8}
\end{aligned}
$$

Note that in the case $n = 1$ the corresponding combinatorial problem has also interpretation in terms of the discrete volumes of the moduli space of curves [**51**] and discrete surfaces/generalized Catalan numbers [**1, 16, 30**]. The relation of these combinatorial problems to a CohFT is also discussed in [**5, 33**], though it is not mentioned there that the underlying Frobenius manifold is given by the prepotential $a_1^2 a_2/2 + a_2^2/2 \cdot \log a_2$.

4.8. Elliptic example. In this section we give an example of a superpotential that satisfies the conditions of Theorem 4.13.

Consider the spectral curve defined by the Weierstrass $\wp$-function

$$\lambda = \wp(z), \quad p = z, \quad B(z, z') = (\wp(z - z') + b)\, dzdz' \tag{266}$$

where $b \in \mathbb{C}$ and p is only defined locally—it is the primitive of a holomorphic differential on the curve—which is sufficient for topological recursion. The compatibility condition (221) is satisfied by Proposition 4.6. It is equivalent to the elliptic identity:

$$\frac{\wp''(z)}{\wp'(z)^2} = \sum_{i=1}^{3} \frac{\wp(z - \omega_i)}{\wp''(\omega_i)} \tag{267}$$

where the sum is over the zeros ω_i of $\wp'(z)$. Hence the spectral curve defines a CohFT.

Introduce three parameters ω, ω' and c into the spectral curve to define the following superpotential taken from [**12**]:

$$\lambda = \wp(z; \omega, \omega') + c, \quad p = \frac{z}{\omega} \tag{268}$$

where

$$\wp(z; \omega, \omega') = \frac{1}{z^2} + \sum_{(m,n) \neq (0,0)} \frac{1}{(z - 2m\omega - 2n\omega')^2} - \frac{1}{(2m\omega + 2n\omega')^2}. \tag{269}$$

The Frobenius manifold structure on $M = \{(\omega, \omega', c)\}$ is given by the formulae (4), (5), (6) where the vector fields ∂ on M are given by, for example ∂_ω, $\partial_{\omega'}$, ∂_c. Note that in (267), $\omega_1 = \omega$, $\omega_2 = \omega'$, $\omega_3 = \omega + \omega'$.

Remark 4.21. Note that we know that the superpotential (268) defines a Frobenius manifold due to the existence of flat coordinates, proven in [**12**], and also given below. The CohFT produced by topological recursion applied to (266) is homogeneous if we choose b in (266) so that $\int_A B(z,z')=0$, i.e. $b=\eta/\omega$ where the A and B periods are $2\omega=\oint_A dz$, $2\omega'=\oint_B dz$ and

$$\eta=-\frac{1}{2}\oint_A \wp(z)dz,\quad \eta'=-\frac{1}{2}\oint_B \wp(z)dz.$$

The periods satisfy Legendre's relation

$$\eta\omega'-\eta'\omega=\frac{i\pi}{2}.$$

The homogeneous CohFT corresponds to a conformal Frobenius manifold which gives rise to a superpotential via Dubrovin's construction (actually, since $d=1$ it is a variant of the construction). What needs to be proven is that the two superpotentials agree.

THEOREM 4.22. *The superpotential* (268) *can be obtained via (a variant of) Dubrovin's construction described in Section* 2.3 *applied to the Frobenius manifold* M. *The conditions of Theorem* 4.13 *are satisfied for this superpotential. Hence the two cohomological field theories—obtained from the superpotential* (268) *and topological recursion applied to the spectral curve* (266) *with* $b=\eta/\omega$*—agree.*

PROOF. To apply Dubrovin's construction to M we construct a solution of the Gauss-Manin system as in the proof of Theorem 4.16.

The flat metric (4) for the superpotential (268) is given by

$$(\partial,\partial'):=\sum_{i=1}^{3}\operatorname*{res}_{z=\omega_i}\frac{\partial\lambda\cdot\partial'\lambda}{\wp'}dp. \tag{270}$$

We use this to construct a vector field $I_\beta(\lambda;u)$ on M for any cycle $\beta\in H_0(\lambda^{-1}(\mathrm{pt}),\mathbb{C})$ specified by:

$$(I_\beta(\lambda;u),\partial):=\int_\beta\frac{\partial(\lambda)}{d_p\lambda/dp}. \tag{271}$$

The elliptic curve (268) is built by gluing two copies of the disk $D=\mathbb{C}\setminus\cup_{i=1}^3 L_i$ in the λ plane along L_i. Choose β to be the cycle given by p_0-p_1, for p_0 and p_1 the pre-images of λ in each of the two disks. In normalized canonical coordinates $I_\beta(\lambda;u)$ is represented by a solution $\phi^\beta(\lambda;u)=\sum_i\phi_i^\beta(\lambda;u)\partial_{v_i}$ of the Gauss-Manin system (8) which has components given by

$$\begin{aligned}\phi_i^\beta(\lambda;u)&=\left(I_\beta(\lambda;u),\frac{1}{\sqrt{2}}\Delta_i^{\frac{1}{2}}\partial_{u_i}\right)\\&=\int_\beta\frac{\Delta_i^{\frac{1}{2}}\partial_{u_i}\lambda}{\sqrt{2}\cdot d_p\lambda/dp}\\&=\frac{\sqrt{2}\cdot\Delta_i^{\frac{1}{2}}\partial_{u_i}\lambda}{d_p\lambda/dp}\end{aligned}$$

where the integral over β simply doubles the integrand since the integrand is skew symmetric. Since $d=1$, we cannot use the inversion formula (9) so we directly check that $\frac{1}{\sqrt{2}}\phi^\beta=\nabla_u p$ as follows.

$$\nabla_u p=\eta^{-1}d_u p=\frac{1}{\omega}\sum_{i=1}^{3}\Delta_i\cdot\frac{\partial_{u_i}\lambda}{\wp'(z)}\partial_{u_i}=\frac{1}{\omega}\sum_{i=1}^{3}\frac{\Delta_i^{\frac{1}{2}}\partial_{u_i}\lambda}{\wp'(z)}\partial_{v_i}=\frac{1}{\sqrt{2}}\phi^\beta.$$

Using

$$\partial_{u_i}\lambda = \frac{1}{2\wp''(\omega_i)}\frac{\wp'(z)^2}{\wp(z)-\wp(\omega_i)} + \frac{z\wp(\omega_i)+\zeta(z)}{\wp''(\omega_i)}\wp'(z)$$

which can be proven from the known variations $\sum_i u_i^k \partial_{u_i}$, $k=0,1,2$, we see that the solution $\phi^\beta(\lambda;u)$ satisfies

$$\begin{aligned}\frac{1}{\sqrt{2}}\phi^T\Psi\mathbb{1} &= \frac{1}{\omega}\sum_{i=1}^{3}\frac{\partial_{u_i}\lambda}{\wp'(z)}\\ &= \frac{1}{\omega}\sum_{i=1}^{3}\frac{1}{2\wp''(\omega_i)}\frac{\wp'(z)}{\wp(z)-\wp(\omega_i)} + \frac{z\wp(\omega_i)+\zeta(z)}{\wp''(\omega_i)}\\ &= \frac{1}{\omega\wp'(z)} = \frac{dp}{d\lambda}\end{aligned}$$

which is (218) and hence via Remark 4.2 we see that the properties of ϕ are sufficient for Theorems 4.1 and 4.10. Hence the theorem follows. □

Theorem 4.22 states that we can study the CohFT obtained from the superpotential (268) via topological recursion applied to the spectral curve (266). We will need the three-point function of this CohFT in calculations below. We calculate it in two ways to demonstrate the proof, although we know from the theorem that they coincide.

4.8.1. *Three-point function. Superpotential.* Introduce the canonical coordinates

$$u_i = \wp(\omega_i) + c, \quad i = 1,2,3$$

where, as usual, $\omega_1=\omega$, $\omega_2=\omega'$, $\omega_3=\omega+\omega'$. The three-point calculations take place in the ring $\mathbb{C}[E]/\wp' = \mathbb{C}[\wp]/\wp'$ and we have

$$\frac{\partial\lambda}{\partial u_1} \equiv \frac{(\lambda-u_2)(\lambda-u_3)}{(u_1-u_2)(u_1-u_3)}, \quad \frac{\partial\lambda}{\partial u_1}\frac{\partial\lambda}{\partial u_j} \equiv \delta_{1j}\frac{\partial\lambda}{\partial u_1},\ j=2,3.$$

and cyclic permutation of the above. This is quite general and also can be proven via elliptic identities. Hence the three-point function for the superpotential is

$$\left\langle\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_i}\right\rangle = \left\langle\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_i}\right\rangle = \left\langle\frac{\partial}{\partial u_i}\right\rangle = \sum_{j=1}^{3}\operatorname*{res}_{z=\omega_j}\frac{\frac{\partial\lambda}{\partial u_i}}{\wp'(z)}\frac{dz}{\omega^2} = \frac{1}{\omega^2\wp''(\omega_i)}.$$

Thus

$$\left\langle\frac{\partial}{\partial v_i},\frac{\partial}{\partial v_i},\frac{\partial}{\partial v_i}\right\rangle = \omega\sqrt{\wp''(\omega_i)} \tag{272}$$

where $\frac{\partial}{\partial v_i} = \Delta_i^{\frac{1}{2}}\frac{\partial}{\partial u_i}$ for $\Delta_i^{-1} = \left\langle\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_i}\right\rangle = \omega^2\cdot\wp''(\omega_i)$ give the normalized canonical coordinates.

Topological recursion. The three-point function obtained via topological recursion is

$$\begin{aligned}\omega_{0,3}(z_1,z_2,z_3) &= \sum_{j=1}^{3} \operatorname*{res}_{z=\omega_j} \frac{\omega dz}{\wp'(z)} \prod_{i=1}^{3} (\wp(z_i - z) + b) dz_i \\ &= \sum_{j=1}^{3} \frac{\omega}{\wp''(\omega_j)} \prod_{i=1}^{3} (\wp(z_i - \omega_j) + b) dz_i \\ &= \sum_{j=1}^{3} \omega \sqrt{\wp''(\omega_j)} V_0^j(z_1) V_0^j(z_2) V_0^j(z_3)\end{aligned}$$

for

$$V_0^j(z) = \left. \frac{(\wp(z-\omega_j)+b)dzdz_j}{ds_j}\right|_{s_j=0} = \frac{\wp(z-\omega_j)+b}{\sqrt{\wp''(\omega_i)}} dz$$

where $\lambda = \wp(z) + c = \frac{1}{2}s_j^2 + \wp(\omega_j) + c$ defines the local coordinate s_j. The coefficients of $V_0^j(z_i)$ define the three-point function of the cohomological field theory which agree with (272).

4.8.2. *Flat coordinates.* The cohomological field theory is defined on the three-dimensional vector space $\mathbb{C}[\wp]/\wp'$ equipped with its natural ring structure and gives rise to a Frobenius manifold structure on the family M of such rings parametrized by $\{\omega, \omega', c\}$. It will be convenient to express the metric on M with respect to a natural basis of vector fields on M corresponding to the basis $\{1, \wp, \wp^2\}$ of $\mathbb{C}[\wp]/\wp'$ since the metric requires knowledge of the variation of $\wp$ under the action of vector fields on the Frobenius manifold. We will see that $\{\omega, \omega', c\}$ are not flat coordinates and find in Lemma 4.24 flat coordinates $\{t_1, t_2, t_3\}$ on M, i.e. so that the metric on M is constant with respect to them.

Recall that correlation functions of the cohomological field theory arising from topological recursion applied to a spectral curve appear as coefficents of auxiliary differentials on the spectral curve. Proposition 4.26 gives the auxiliary differentials that correspond to the flat basis for the metric.

In the following lemma we calculate the vector fields on M that correspond to the basis elements $1, \wp, \wp^2$ of $\mathbb{C}[\wp]/\wp'$. This uses $g_2 = g_2(\omega, \omega')$ defined by $\wp'(z)^2 = 4\wp(z) - g_2\wp - g_3$.

LEMMA 4.23. *Under the map $TM \to \mathbb{C}[\wp]/\wp'$ defined by $\partial \mapsto \partial\lambda (mod\ \wp')$ for $\lambda = \wp(z; \omega, \omega') + c$*

$$\partial_c \mapsto 1, \quad -\tfrac{1}{2}\left(\omega\partial_\omega + \omega'\partial_{\omega'}\right) \mapsto \wp, \quad -\tfrac{1}{2}\left(\eta\partial_\omega + \eta'\partial_{\omega'}\right) + \tfrac{1}{6}g_2\partial_c \mapsto \wp^2. \tag{273}$$

PROOF. The variation $\partial_c\lambda = 1$ is obvious. The identity

$$\omega\partial_\omega\wp(z) + \omega'\partial_{\omega'}\wp(z) + z\wp'(z) = -2\wp(z) \tag{274}$$

follows immediately from the expansion (269) of $\wp$ and yields $-\frac{1}{2}\left(\omega\partial_\omega + \omega'\partial_{\omega'}\right) \mapsto \wp$. The final identification uses the identity proven in [**34**]

$$\eta\partial_\omega\wp(z) + \eta'\partial_{\omega'}\wp(z) + \zeta(z)\wp'(z) = -2\wp(z)^2 + \tfrac{1}{3}g_2 \tag{275}$$

where $\zeta(z)$ is the Weierstrass ζ-function

$$\zeta(z; \omega, \omega') = \frac{1}{z} + \sum_{(m,n)\neq(0,0)} \frac{1}{z - 2m\omega - 2n\omega'} + \frac{1}{2m\omega + 2n\omega'} + \frac{z}{(2m\omega + 2n\omega')^2}$$

which is not an elliptic function (C.63 in [**12**]). Note that $\eta = \zeta(\omega)$, $\eta' = \zeta(\omega')$. □

The metric

$$\langle \wp^j, \wp^k \rangle = \sum_{i=1}^{3} \operatorname*{res}_{z=\omega_i} \frac{\wp^{j+k}}{\wp'(z)} \frac{dz}{\omega^2} = -\operatorname*{res}_{z=0} \frac{\wp^{j+k}}{\wp'(z)} \frac{dz}{\omega^2}$$

is given by

	1	$\wp$	$\wp^2$
1	0	0	$1/2\omega^2$
$\wp$	0	$1/2\omega^2$	0
$\wp^2$	$1/2\omega^2$	0	$g_2/8\omega^2$

LEMMA 4.24 (Dubrovin [**12**]). *Flat coordinates for the metric are given by*

$$t_1 = c - \frac{\eta}{\omega}, \quad t_2 = \frac{1}{\omega}, \quad t_3 = \frac{\omega'}{\omega}.$$

PROOF. This is (5.95) in [**12**]. We simply use change of coordinates given by (273) and the metric calculated above. We have $\partial_c = \partial_{t_1}$. From the identity

$$(\omega\partial_\omega + \omega'\partial_{\omega'})\frac{\eta}{\omega} = -2\frac{\eta}{\omega}$$

which uses the fact that $\omega\partial_\omega + \omega'\partial_{\omega'}$ is the degree operator, $\frac{\eta}{\omega}$ is homogeneous of degree -2 we have $\omega\partial_\omega + \omega'\partial_{\omega'} = \frac{2\eta}{\omega}\partial_{t_1} - \frac{1}{\omega}\partial_{t_2}$. The identity

$$(\eta\partial_\omega + \eta'\partial_{\omega'})\frac{\eta}{\omega} = -\frac{1}{12}g_2 - \frac{\eta^2}{\omega^2}$$

appearing as (C.69) in [**12**] gives $\eta\partial_\omega + \eta'\partial_{\omega'} = \left(\frac{1}{12}g_2 + \frac{\eta^2}{\omega^2}\right)\partial_{t_1} - \frac{\eta}{\omega^2}\partial_{t_2} - \frac{i\pi}{2\omega^2}\partial_{t_3}$. Hence we have

(276)

$$\partial_{t_1} \mapsto 1, \quad -\frac{\eta}{\omega}\partial_{t_1} + \frac{1}{2\omega}\partial_{t_2} \mapsto \wp, \quad \left(\frac{1}{8}g_2 - \frac{\eta^2}{2\omega^2}\right)\partial_{t_1} + \frac{\eta}{2\omega^2}\partial_{t_2} + \frac{i\pi}{4\omega^2}\partial_{t_3} \mapsto \wp^2.$$

Hence the metric is given by:

	∂_{t_1}	∂_{t_2}	∂_{t_3}
∂_{t_1}	0	0	$2/i\pi$
∂_{t_2}	0	2	0
∂_{t_3}	$2/i\pi$	0	0

which is constant so that $\{t_1, t_2, t_3\}$ are flat coordinates. □

REMARK 4.25. We can choose a different $(0, 2)$ term $B(z, z')$ on the spectral curve (266) which still satisfies the compatibility condition (221) by varying $b \in \mathbb{C}$. For each b it gives rise to a CohFT with the same genus 0 three-point function since ancestor invariants are coefficients of B-dependent differentials. When b is chosen so that $B(z, z')$ is normalised along a choice of cycle, e.g. $b = \eta'/\omega'$ so $\int_B B(z, z') = 0$, then the CohFT is homogeneous and hence the same CohFT as for $b = \eta/\omega$. Other choices of b gives rise to non-homogeneous CohFTs.

PROPOSITION 4.26. *The flat coordinates correspond to the following auxiliary differentials:*

$$dt_1 \longleftrightarrow T_0^1 = (\omega\wp + b)dz - 2\omega d\left(\frac{\wp^2}{\wp'}\right) + 2\eta d\left(\frac{\wp}{\wp'}\right) + \left(\frac{\omega g_2}{4} + \frac{\eta^2}{\omega}\right) d\left(\frac{1}{\wp'}\right)$$
$$dt_2 \longleftrightarrow T_0^2 = -d\left(\frac{\wp}{\wp'}\right) - \frac{\eta}{\omega} d\left(\frac{1}{\wp'}\right)$$
$$dt_3 \longleftrightarrow T_0^3 = -\frac{i\pi}{2\omega} d\left(\frac{1}{\wp'}\right).$$

PROOF. The auxiliary differentials on the spectral curve corresponding to the normalized canonical basis are straighforward. They are given by $V_k^i dz$ where

$$V_0^i = \frac{\wp(z-\omega_i)}{\sqrt{\wp''(\omega_i)}}$$

and for $k > 0$, V_k^i is the principal part of the kth derivative of V_0^i with respect to $\wp(z)$. We also have the canonical basis $U_0^i = \omega\wp(z-\omega_i)$. The auxiliary differentials $T_k^i dz$ corresponding to flat coordinates are linear combinations of $V_k^i dz$

$$V_k^i = \Psi_\mu^i \cdot T_k^\mu$$

where we recall that Ψ_μ^i is the transition matrix from flat coordinates labeled by μ to normalized canonical coordinates labeled by i. We can calculate Ψ via

$$\begin{pmatrix} 1 & 1 & 1 \\ \wp(\omega_1) & \wp(\omega_2) & \wp(\omega_3) \\ \wp(\omega_1)^2 & \wp(\omega_2)^2 & \wp(\omega_3)^2 \end{pmatrix} \begin{pmatrix} \partial_{u_1} \\ \partial_{u_2} \\ \partial_{u_3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{\eta}{\omega} & \frac{1}{2\omega} & 0 \\ \frac{1}{8}g_2 - \frac{\eta^2}{2\omega^2} & \frac{\eta}{2\omega^2} & \frac{i\pi}{4\omega^2} \end{pmatrix} \begin{pmatrix} \partial_{t_1} \\ \partial_{t_2} \\ \partial_{t_3} \end{pmatrix}$$

which we write as $M\frac{\partial}{\partial u} = T\frac{\partial}{\partial t}$ hence $M^{-1}T = \Delta^{1/2}\Psi^T$. The auxiliary differentials corresponding to $1, \wp, \wp^2$ in the Landau-Ginzburg model are:

$$[U^1, U^2, U^3] \cdot M^{-1} =$$
$$2\omega dz \left[\frac{\wp(z-\omega_1)+b}{\wp''(\omega_1)},\ \frac{\wp(z-\omega_2)+b}{\wp''(\omega_2)},\frac{\wp(z-\omega_3)+b}{\wp''(\omega_3)}\right] \begin{pmatrix} \wp(\omega_1)^2 - \frac{1}{4}g_2 & \wp(\omega_1) & 1 \\ \wp(\omega_2)^2 - \frac{1}{4}g_2 & \wp(\omega_2) & 1 \\ \wp(\omega_3)^2 - \frac{1}{4}g_2 & \wp(\omega_3) & 1 \end{pmatrix}$$
$$= \left[-2\omega d\left(\frac{\wp^2}{\wp'}\right) + (\omega\wp + b)dz + \frac{\omega g_2}{2} d\left(\frac{1}{\wp'}\right), -2\omega d\left(\frac{\wp}{\wp'}\right), -2\omega d\left(\frac{1}{\wp'}\right)\right]$$

which is proven using the elliptic identities

$$\frac{\wp(z)^k}{\wp'(z)^2} = \sum_{i=1}^{3} \frac{\wp(\omega_i)^k \wp(z-\omega_i)}{\wp''(\omega_i)^2}, \quad k = 0, 1, 2$$

and slight generalisations for $k > 2$. Hence

$$[T^1, T^2, T^3] = [U^1, U^2, U^3] \cdot M^{-1} \cdot T$$
$$= \Big[- 2\omega d\left(\frac{\wp^2}{\wp'}\right) + (\omega\wp + b)dz + \left(\frac{\omega g_2}{4} + \frac{\eta^2}{\omega}\right) d\left(\frac{1}{\wp'}\right) + 2\eta d\left(\frac{\wp}{\wp'}\right),$$
$$- d\left(\frac{\wp + \eta/\omega}{\wp'}\right), -\frac{i\pi}{2\omega} d\left(\frac{1}{\wp'}\right) \Big]$$

□

The following lemma allows us to apply equation (215) to obtain ancestor invariants for the CohFT.

LEMMA 4.27. *The following kernels* K_0^i

$$K_0^1 = y(z), \quad K_0^2 = -2\omega\zeta(z)+2\eta, \quad K_0^3 = \frac{4}{i\pi}\left(\omega z\wp(z)^2 - \left(\frac{\eta^2}{2\omega} + \frac{\omega}{8}g_2\right)z + \eta\zeta(z)\right)$$

are dual (as linear functionals) to T_0^i *for* $i = 1, 2, 3$*, i.e.*

$$\sum_{j=1}^{3} \operatorname*{res}_{z=\omega_j} K_0^j(z) T_k^i(z) = \delta_{ij}\delta_{k0}.$$

PROOF. Each kernel is analytic at $z = \omega_i$, $i = 1, 2, 3$ and hence annihilates differentials analytic at $z = \omega_i$. Consider the action of each kernel on $d(\wp^k/\wp')$ for $k = 0, 1, 2$.

$$\sum_{j=1}^{3} \operatorname*{res}_{z=\omega_j} K_0^i(z) d\left(\frac{\wp^k}{\wp'}\right) = -\sum_{j=1}^{3} \operatorname*{res}_{z=\omega_j} dK_0^i(z)\frac{\wp^k}{\wp'} = \operatorname*{res}_{z=0} dK_0^i(z)\frac{\wp^k}{\wp'}$$

so $K_0^1 = y(z) = z/\omega$ annihilates $d(\wp^k/\wp')$ for $k = 0, 1$ and sends $d(\wp^2/\wp')$ to $-1/2\omega$. Similarly $K_0^2 = \zeta(z)$ annihilates $d(\wp^k/\wp')$ for $k = 0, 2$ and sends $d(\wp/\wp')$ to $1/2$. Apply the kernels to T_0^i given in Proposition 4.26 as linear combinations of $d(\wp^k/\wp')$ (and terms analytic at $z = \omega_i$) to achieve the result.

The kernels K_0^1 and K_0^2 annihilate exact differentials that vanish to order 2 at $z = 0$, in particular T_k^i for $k > 0$ by integration by parts. One can also check that K_0^3 annihilates T_k^i for $k > 0$. □

Remark. One can also produce kernels K_j^i dual to each T_j^i.

The 3-point function in flat coordinates leads to the prepotential given in [**12**] (C.87):

$$F_0 = \frac{1}{i\pi}t_1^2 t_3 + t_1 t_2^2 - \frac{i\pi}{2}t_2^4\left(\frac{1}{24} - \sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\right), \quad q = e^{2\pi i t_3}.$$

PROPOSITION 4.28.

$$\exp F_1 = t_2^{1/8}\eta(q)^{1/4}, \quad \eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n).$$

PROOF. Topological recursion—defined in Section 2.7—applied to the spectral curve (266) uses the kernel

$$\begin{aligned} K(z_1, z) &= \frac{\omega}{2}\frac{\int_{\sigma_i(z)}^{z}(\wp(z_1 - w) + b)dwdz_1}{(z - \sigma_i(z))\wp'(z)dz} \\ &= \frac{\omega}{4}\frac{(\zeta(z_1 + z) - \zeta(z_1 - z) + 2\eta_i + 2b(z - \omega_i))dz_1}{(z - \omega_i)\wp'(z)dz} \end{aligned}$$

where $\sigma_i(z) = 2\omega_i - z$. Hence

$$\omega_{1,1}(z_1) = \sum_{j=1}^{3} \operatorname*{res}_{z=\omega_j} K(z_1,z)\wp(2z) = \sum_{j=1}^{3} \operatorname*{res}_{z=\omega_j} \frac{\omega}{4}\frac{(\zeta(z_1+z)-\zeta(z_1-z))}{(z-\omega_i)\wp'(z)}\wp(2z)dzdz_1$$
$$= \frac{\omega}{8}\left(2\sum_{j=1}^{3}\frac{\wp(\omega_i)\wp(z_0-\omega_i)}{\wp''(\omega_i)} - \sum_{j=1}^{3}\frac{\wp(z_0-\omega_i)^2}{\wp''(\omega_i)}\right)dz_1$$
$$= \frac{\omega}{8}\left(\frac{2\wp\wp''}{(\wp')^2} - \frac{(\wp'')^3}{(\wp')^4} + 10g_2\frac{\wp}{(\wp')^2} + 15g_3\frac{1}{(\wp')^2} + 11\right)dz_1$$

where $\eta_i \in \mathbb{C}$ and b are annihilated by the residues. Integrate the kernels K_j^i against $\omega_{1,1}$ to get

$$\omega_{1,1} = 0\cdot T_0^1 + \frac{\omega}{8}T_0^2 + \frac{i\eta\omega}{4\pi}T_0^3 + \frac{1}{8}T_1^1 + \frac{\eta}{4}T_1^2 + \frac{g_2\omega^2 - 12\eta^2}{48i\pi}T_1^3.$$

The primary part uses only T_0^k and yields

$$F_1 = \frac{1}{8}\log t_2 + f(t_3), \quad f'(t_3) = \frac{i\pi}{2}\left(\frac{1}{24} - \sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\right)$$

which is obtained from $\omega_{1,1}$ since $\frac{\partial F_1}{\partial t_2} = \frac{1}{8t_2} = \frac{\omega}{8}$ agrees with the coefficient of T_0^2 and $\frac{\partial F_1}{\partial t_3} = \frac{i\pi}{2}\left(\frac{1}{24} - \sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\right) = \frac{i\eta\omega}{4\pi}$ agrees with the coefficient of T_0^2. $\square$

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