Proceedings of Symposia in
PURE MATHEMATICS

Volume 100

Topological Recursion and its Influence in Analysis, Geometry, and Topology

2016 AMS von Neumann Symposium
Topological Recursion and its Influence
in Analysis, Geometry, and Topology
July 4–8, 2016
Charlotte, North Carolina

Chiu-Chu Melissa Liu
Motohico Mulase
Editors

AMERICAN
MATHEMATICAL
SOCIETY

Topological Recursion and its Influence in Analysis, Geometry, and Topology

Proceedings of Symposia in
PURE MATHEMATICS

Volume 100

Topological Recursion and its Influence in Analysis, Geometry, and Topology

2016 AMS von Neumann Symposium
Topological Recursion and its Influence
in Analysis, Geometry, and Topology
July 4–8, 2016
Charlotte, North Carolina

Chiu-Chu Melissa Liu
Motohico Mulase
Editors

AMS AMERICAN MATHEMATICAL SOCIETY
Providence, Rhode Island

2010 *Mathematics Subject Classification*. Primary 14H10, 14H15, 14H60, 14H60, 14J33, 14N10, 14N35, 53D45, 81T45.

The AMS von Neumann Symposium was supported by a grant from the National Science Foundation, NSF DMS-1619760.

Library of Congress Cataloging-in-Publication Data

Names: AMS von Neumann Symposium (2016: Charlotte, North Carolina) | Liu, Chiu-Chu Melissa, 1974– editor. | Mulase, Motohico, 1954– editor.

Title: Topological recursion and its influence in analysis, geometry, and topology: 2016 AMS von Neumann Symposium, July 4–8, 2016, Charlotte, North Carolina / Chiu-Chu Melissa Liu, Motohico Mulase, editors.

Description: Providence, Rhode Island: American Mathematical Society, [2018] | Series: Proceedings of symposia in pure mathematics; volume 100 | Includes bibliographical references.

Identifiers: LCCN 2018033790 | ISBN 9781470435417 (alk. paper)

Subjects: LCSH: Topology–Congresses.

Classification: LCC QA611.A1 A47 2018 | DDC 516.3/5–dc23

LC record available at https://lccn.loc.gov/2018033790

DOI: http://dx.doi.org/10.1090/pspum/100

10 9 8 7 6 5 4 3 2 1   23 22 21 20 19 18

Contents

Preface

This volume is intended for presenting a snapshot of rapid and rich development in an emerging research field, *topological recursion.* As we review below, topological recursion was conceived in 2003–2004, appearing completely independently in two disparate contexts: One is random matrix theory/matrix models, where the recursion structure was discovered in calculating multi-resolvent correlation functions of random matrices [**3**, **13**, **34**]. The other is in geometry. Working on her Harvard thesis, Mirzakhani encountered the same recursive structure in calculating the Weil-Petersson volume of moduli spaces of bordered hyperbolic surfaces. Her thesis of 2004 was later published as [**71**, **72**]. Although the shape of formulas of Eynard [**34**] and Mirzakhani [**71**] are similar, they are not exactly the same. Surprisingly, they become *identical* after the Laplace transform.

Once the mathematics behind the scene was understood, topological recursion began to play an unexpected role in connecting a wide area of mathematics and theoretical physics through its universal recursive structures. The exact same formulas are found in such topics, not only as random matrix theory/matrix models and the Weil-Petersson volume of moduli spaces as mentioned above but also in topological quantum field theory and cohomological field theory, enumeration of various Hurwitz numbers, counting Grothendieck's dessins d'enfants and more general graphs drawn on surfaces, intersection numbers of cohomology classes on the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable curves, Gromov-Witten theory, A-polynomials and polynomial invariants of hyperbolic knots and algebro-geometric invariants of algebraic knots, WKB analysis of classical ordinary differential equations, Painlevé equations, moduli spaces of Higgs bundles and character varieties of surface groups, and *opers.*

1. The 2016 AMS von Neumann Symposium

Several experts of topological recursion convened at the 2016 American Mathematical Society von Neumann Symposium, *The topological recursion and its influence in analysis, geometry, and topology*, held in Charlotte, North Carolina, during the week of July 4–8, 2016. The conference was organized by Bertrand Eynard, Chiu-Chu Melissa Liu, and Motohico Mulase serving as chair of the organizing committee. The meeting was planned right at the time when many discoveries and crucial theorems were established, and at the same time, numerous new mysteries were arising.

The symposium featured four mini-courses to illuminate the state of the art of topological recursion, from its origin to the most significant achievements and to a glimpse of future developments. The speakers and titles are as follows:

- Jørgen Ellegaard Andersen: *Geometric quantisation of moduli spaces and topological recursion.*
- Petr Dunin-Barkowski: *Topological recursion and Givental's formalism: spectral curves for Gromov-Witten theories.*
- Bertrand Eynard: *An introduction to topological recursion.*
- Bohan Fang: *Topological recursion and mirror symmetry.*

We also invited many key players of the field to present the most recent developments as plenary lecturers:

- Gaëtan Borot: 2*d CFTs, Verlinde formula and indices via topological recursion.*
- Tom Bridgeland: *Quadratic differentials and wall-crossing.*
- Leonid Chekhov: *Topological recursion for classical ensembles and cohomological field theories.*
- Alessandro Chiodo: *Generalized Borcea-Voisin mirror duality in any dimension.*
- Laura Fredrickson: *From the Hitchin component to opers.*
- Kohei Iwaki: *Exact WKB analysis, Painlevé equations and the Stokes phenomenon.*
- Felix Janda: *Frobenius manifolds near the discriminant.*
- Rinat Kashaev: *Invariants of finite cyclic covers of knot complements from Teichmüller TQFT.*
- Paul Norbury: *Primary invariants of Frobenius manifolds and periods on Riemann surfaces.*
- Nicolas Orantin: *Topological recursion, twisted periods and almost Frobenius manifolds.*
- Masa-Hiko Saito: *An explicit geometry of moduli spaces of Higgs bundles and singular connections on a smooth curve and differential equations of Painlevé types.*
- Yan Soibelman: (1) *Holomorphic Floer theory and Riemann-Hilbert correspondence.* (2) *Airy structures and topological recursion.*
- Piotr Sułkowski: *Matrix models, quantum curves, and conformal field theory.*
- Ravi Vakil: *Topological recursions in the Grothendieck ring of varieties.*

Several contributed short talks were also given:

- Bojko Bakalov: *Vertex operators in Gromov-Witten theory.*
- Vincent Bouchard: *An elliptic quantum curve.*
- Andrea Brini: *Quantum invariants at large N and the topological recursion.*
- Qingtao Chen: *Congruent relations, cyclotomic expansions and volume conjectures for various quantum invariants.*
- João N. Esteves: *Hopf algebras and topological recursion.*
- Taro Kimura: *Double quantization of Seiberg-Witten curve and W-algebras.*
- Dmitry Korotkin: *Periods of meromorphic quadratic differentials and Goldman bracket.*
- Karol Kozlowski: *Partition function of the* sinh*-model with varying interactions.*
- Danilo Leo Lewanski: *Ramifications of Hurwitz theory, KP integrability and quantum curves.*

- Olivier Marchal: *A lonely runner problem, asymptotics of Toeplitz determinants and topological recursion.*
- Ran Tessler: *Intersection theory on moduli of bordered Riemann surfaces, and related integrable systems.*
- Zhengyu Zong: *Some applications of the Remodeling Conjecture.*

After the conclusion of the symposium, the invited speakers and participants, including those originally invited but could not participate due to unanticipated circumstances, were invited to submit their papers, which are related to the theme of the conference for publication in this volume. The submitted papers were then sent to referees for peer review. We are pleased to present the papers recommended for publication in this volume.

2. What is topological recursion?

Since Eynard's excellent survey of topological recursion based on his invited address at the ICM-2014 in Seoul is available [**38**], we refer to it for a discoverer's own description of the subject. In particular, his paper presents how it was encountered in random matrix theory and how it is appearing in various problems in geometry, and also cites many original contributions to topological recursion by researchers in a wide variety of comminities. Therefore, in this preface we restrict ourselves to pure mathematical side of the story relevant to the papers collected in this volume, and we refer readers who wish to know other sides of the story, such as mathematical and theoretical physics points of view, to [**38**, **40**].

Sir George Biddel Airy [**2**] devised a function

$$Ai(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(ipx + i\frac{p^3}{3}\right) dp,$$

which he called the *rainbow integral*, to explain the diffraction pattern of the rainbow. Since the diffraction phenomena are a quantum mechanical process, his method based on classical analysis does not explain the physics of rainbows. Phenomenologically, still the location of the local maxima and minima of the Airy function on the $x < 0$ side tells us where we see the sequence of arches of the rainbow, and their brightness. The asymptotic formula

$$Ai(x) \sim \frac{1}{\sqrt{\pi}x^{\frac{1}{4}}} \exp\left(-\frac{2}{3}x^{\frac{3}{2}}\right), \qquad |\mathrm{Arg}(x)| < \frac{\pi}{3} \tag{2.1}$$

tells us that the Airy function exponentially decays on the right side $x > 0$. Therefore, underneath the brightest arch there are no rainbows.

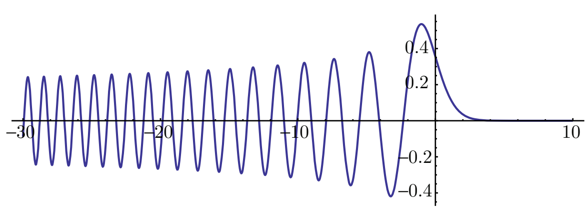

FIGURE 2.1. The Airy function

The above asymptotic formula (2.1) has a refined expansion in terms of the intersection numbers of $\overline{\mathcal{M}}_{g,n}$:

$$
\begin{aligned}
(2.2)\quad Ai(x) &= \exp\left(-\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{4}\log x - \frac{1}{2}\log \pi\right)\\
&\cdot\exp\left(\sum_{2g-2+n>0} \frac{(-1)^n}{n!}\left(\frac{1}{2}\right)^{2g-2+n} x^{-\frac{(6g-6+3n)}{2}} \sum_{\substack{d_1+\cdots+d_n\\=3g-3+n}} \langle\tau_{d_1}\cdots\tau_{d_n}\rangle_{g,n}\prod_{i=1}^{n}|2d_i-1|!!\right)\\
&= \exp\left(-\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{4}\log x - \frac{1}{2}\log\pi - \frac{5}{48}x^{-\frac{3}{2}} + \frac{5}{64}x^{-3} + \cdots\right),
\end{aligned}
$$

where the coefficients

$$\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\cdots\psi_n^{d_n} \tag{2.3}$$

are the cotangent class intersection numbers on the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable curves of genus g with n non-singular marked points. Let us introduce an $\hbar$-dependence by defining

$$Ai(x,\hbar) := Ai\left(\frac{x}{\hbar^{2/3}}\right).$$

Then this new function satisfies a stationary *Schrödinger equation*

$$\left(\left(\hbar\frac{d}{dx}\right)^2 - x\right)Ai(x,\hbar) = 0. \tag{2.4}$$

This equation is referred to as a *quantum curve* in many articles in this volume. The *semi-classical limit* of this quantum curve is a quadratic equation, called a *spectral curve*:

$$y^2 - x = 0. \tag{2.5}$$

Topological recursion is a powerful induction process to calculate every single intersection number $\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_{g,n}$ from the spectral curve (2.5). For this particular case, the recursion is the same as the Virasoro constraint condition of these intersection numbers predicted by Dijkgraaf, Verlinde, and Verlinde [**20**]. We also note that the differential operator of (2.4) at $\hbar = 1$ is the initial value of the Lax operator which contains information of all cotangent class intersection numbers. That this Lax operator satisfies the KdV equation was conjectured by Witten [**87**] and proved by Kontsevich [**64**].

3. The origin

Topological recursion, as we call today, was first devised by Chekhov, Eynard, and Orantin in 2006 [**13**] as a *universal* analytic formula to calculate the asymptotic expansions of the *multi-resolvent correlation functions* of 2-matrix models. The derivation of this formula is very simple, which is the idea of Virasoro constraint condition [**20**] mentioned above. A definite integral, such as a convergent Hermitian matrix integral, is invariant under a change of integration parameters. From an infinitesimal change of variables, one obtains a series of equations as a result of invariance of the integral. This technique is known as *loop equations* in physics. In random matrix theory, the main concern has been the asymptotic behavior of correlation functions as the size of the matrix N tends to infinity. Identities coming from loop equations produce an infinite series of equations in terms of power series in $1/N$ for an arbitrary N as a variable. The degree of powers of $1/N$ in the series expansion takes the form $2g-2+n$, where n is the number of resolvents in the n-point correlation function, and g is the genus parameter of the loop expansion. The formula provides an effective method for calculating correlation functions of the resolvent of random matrices. The new observation here is that the coefficients of the $1/N$-expansion can be calculated in a closed formula in terms of complex analysis on the Riemann surface, which appears as the maximum domain of holomorphy of the resolvent. This Riemann surface is commonly called the spectral curve of the starting matrix integral.

A year later in 2007, Eynard and Orantin [**40**] changed the point of view 180°. They propose the induction formula as an axiom to compute a series of *invariants* from a given algebraic equation in 2-variables, such as (2.5), that determines the spectral curve of the theory. It was noticed that many matrix models appearing in topological string theory can be solved, in a perturbative way, by topological recursion [**19**, **83**].

Question 3.1. *What does the abstract topological recursion calculate when it is not related to matrix models? What is the geometric meaning of the invariants?*

In geometry, topological recursion has a very different origin. It first appeared in 2004 as an integral recursion formula discovered by Mirzakhani [**71**, **72**] in her work on the Weil-Petersson volume of the moduli space of bordered hyperbolic surfaces. Eynard [**35**] noticed that the *Laplace transform* of the Mirzakhani recursion is an example of topological recursion of [**40**]. The spectral curve for this case is the sine curve that was discovered by Mulase and Safnuk [**74**] as the intertwining operator between two Virasoro representations, one for the intersection numbers (2.3) of [**20**], and the other for the Weil-Petersson volumes. Mirzakhani recursion is based on pair-of-pants decompositions of bordered topological surfaces. Her formula for the volume of moduli spaces satisfies an integral recursion equation representing removal of a pair-of-pants in all possible ways. Its Laplace transform, as a complex analytic equation, leads to a topological recursion formulated on the sine curve.

The Laplace transform turns out to be the key to understanding the mirror symmetry behind the scene. In 2009, Eynard, Mulase, Safnuk, and Zhang [**39**, **77**] solved a conjecture for simple Hurwitz numbers predicted by the physicists Bouchard and Mariño [**11**]. The key discovery of [**39**] is that the conjectural topological recursion for Hurwitz numbers is a consequence of the Laplace transform of a combinatorial equation known as the *cut-and-join equation* of simple Hurwitz numbers [**55**, **56**, **84**] (see also [**46**]).

The Laplace transform appearing in these examples is then identified as mirror symmetry, connecting the A-model side of enumerative geometry with a B-model of holomorphic geometry. Evidence for the importance of topological recursion is seen in the work of [**77**]. It is shown that both Witten's conjecture [**87**] and the λ_g conjecture (first proved in [**42**]) are straightforward consequences of the formula established in [**77**], which is the differential equation obtained via the Laplace transform of the cut-and-join equation. The same method is used for proving the λ_g-conjecture for orbifold Hurwitz numbers in [**69**].

Around the same time, Norbury [**79**, **80**] showed that the lattice point counting problem of the combinatorial moduli space $\mathcal{M}_{g,n}^{\text{comb}} = \mathcal{M}_{g,n} \times \mathbb{R}_+^n$ can be solved via topological recursion. Following his lead, numerous examples of topological recursion were found in various graph enumeration problems [**12**, **22**, **27**, **73**, **82**]. The example coming out of Catalan numbers [**27**] is particularly simple and elegant. In all of these examples, topological recursion is a consequence of the Laplace transform of edge-contraction operations appearing in graph enumeration. In 2013, Dumitrescu and Mulase noticed that this operation itself is exactly the same as the multiplication and comultiplication operations in a Frobenius algebra, and hence it is equivalent to a two-dimensional topological quantum field theory (2D TQFT) [**25**]. A 2D TQFT is the degree 0 part of a cohomological field theory (CohFT) of Kontsevich and Manin [**65**]. Therefore, it suggests a relation between topological recursion and CohFT.

Double Hurwitz numbers exhibits a rather different nature compared to simple Hurwitz numbers. A topological recursion for a restricted case of orbifold Hurwitz numbers is established in [**8**, **21**]. In the paper by Do and Karev in this volume, a conjecture regarding double Hurwitz numbers in a more general context is proposed with abundant evidence.

4. Topological recursion and semi-simple cohomological field theories

The genus zero Gromov-Witten theory of a projective manifold (or, more generally, a compact symplectic manifold) defines a Frobenius manifold, to which the associated 2D TQFT is the degree zero part of the CohFT defined by the Gromov-Witten theory of the projective manifold. The genus zero Gromov-Witten theory of a target with generically semi-simple quantum cohomology defines a semi-simple Frobenius manifold. For any semi-simple Frobenius manifold, Givental [**54**] conjectured a formula of higher genus potentials in terms of functions on the semi-simple Frobenius manifold and the Kontsevich-Witten τ-function (which is equal to the all-genus total descendant Gromov-Witten potential of a point); the formula implies the Virasoro constraint condition. This is known as Givental's higher genus reconstruction conjecture, and has been proved in full generality by Teleman [**86**].

The topological recursion determines higher genus invariants $\omega_{g,n}$ recursively from genus zero invariants $\omega_{0,1}$ and $\omega_{0,2}$. It is natural to speculate a relation between the topological recursion and Givental's formula for higher genus invariants. Dunin-Barkowski, Orantin, Shadrin, and Spitz showed that any semi-simple CohFT, and in particular Gromov-Witten theory of a target with semi-simple quantum cohomology, can be reproduced by a *local* version of topological recursion [**31**]. This volume contains a thorough treatment of the equivalence between topological recursion and Givental's formula based on Dunin-Barkowski's mini-course on the results in [**30**, **31**]. Conversely, in their contribution to this volume, Dunin-Barkowski, Norbury, Oranin, Popolitov, and Shadrin show that the topological recursion on a *global* compact spectral curve computes invariants of the CohFT associated with Dubrovin's Hurwitz Frobenius manifold.

5. Topological recursion and toric mirror symmetry

As an application of their main result, the authors of [**31**] proved the Norbury-Scott conjecture [**82**], which provides a precise correspondence between stationary Gromov-Witten invariants of the complex projective line $\mathbb{P}^1$ and invariants obtained by applying topological recursion to the spectral curve $x = z + z^{-1}$, $y = \log z$. This proof is also explained in Dunin-Barkowski's contribution to this volume. It is interesting to observe that the holomorphic Morse function $W : \mathbb{C}^* \to \mathbb{C}$ given by $z \mapsto z + z^{-1}$ is the superpotential of the Landau-Ginzburg mirror of $\mathbb{P}^1$ at the stationary point. Therefore, the Norbury-Scott conjecture can be viewed as a version of all-genus mirror symmetry. Similar correspondence is expected to hold for higher dimensional compact toric manifolds such as complex projective spaces. An equivariant version of the Norbury-Scott conjecture, which can be interpreted as an equivariant all-genus mirror symmetry for $\mathbb{P}^1$, is formulated and proved by Fang, Liu, and Zong in [**47**] using the main result in [**31**] and a formula for the total descendant potential of equivariant Gromov-Witten theory of toric manifolds derived by Givental using torus localization [**54**]. From this result, the Bouchard-Mariño conjecture (cf. Section 3 above) and the Norbury-Scott conjecture can be recovered by taking the large radius limit and the non-equivariant limit, respectively.

Topological recursion also arises in higher genus mirror symmetry for toric Calabi-Yau 3-folds. Mirror symmetry relates the A-model on a Calabi-Yau manifold X, defined in terms of the symplectic structure on X, to the B-model on its mirror Calabi-Yau manifold $\check{X}$, defined in terms of the complex structure on $\check{X}$. Gromov-Witten theory can be viewed as a mathematical theory of A-model topological string. Gromov-Witten invariants are defined for any projective manifolds at all genera; for toric Calabi-Yau 3-folds, which are non-compact, we consider equivariant Gromov-Witten invariants defined by torus action. On the B-model side, genus zero (resp., genus one) B-model topological string is defined in terms of period integrals (resp., Ray-Singer analytic torsion), but a mathematical theory of genus $g \geq 2$ B-model on a general Calabi-Yau manifold has not been established. In 2007-08, Mariño [**70**], and then in collaboration with Bouchard, Klemm, and Pasquetti [**9, 10**], proposed a mathematically defined, *remodeled* all-genus B-model on the mirror of a symplectic toric Calabi-Yau 3-manifold/3-orbifold X, in both closed and open string sectors, based on topological recursion on the mirror curve of X. (Indeed, the original topological recursion [**40**], defined for spectral curves in $\mathbb{C}^2$, needs to be suitably modified since the mirror curve is an affine curve in $(\mathbb{C}^*)^2$ instead of $\mathbb{C}^2$.) The *remodeling conjecture* of Bouchard, Klemm, Mariño, and Pasquetti provides a precise correspondence between closed and open Gromov-Witten invariants of a symplectic toric 3-manifold/3-orbifold X and the invariants $\omega_{g,n}$ obtained by applying topological recursion to the mirror curve of X. Here, open Gromov-Witten invariants (corresponding to $\omega_{g,n}$ with $n > 0$) count parametrized holomorphic curves in X with boundaries in Aganagic-Vafa Lagrangian branes, and may also be interpreted as certain relative Gromov-Witten invariants.

In 2009, Lin Chen proved the remodeling conjecture for $\mathbb{C}^3$ in the open string sector based on the method of [**39**]. Because of the tragic death of the author (in December 2009) shortly after the completion of the paper (in October 2009), Chen's paper has never been submitted to a journal. We publish Chen's paper in this volume based on strong recommendation by the referee. (An alternative proof of the same result can be found in [**88**].) The closed string sector of the remodeling conjecture for $\mathbb{C}^3$ was proved independently by Bouchard, Catuneanu, Marchal, and Sułkowski [**7**], and by Zhu [**91**].

In 2011, Eynard derived graph sum formulas for the unique solution to the topological recursion on an arbitrary spectral curve [**36, 37**], which are key ingredients in the identification of topological recursion and Givental formalism due to

[**31**] (cf. Section 4 above), and also in the work on the remodeling conjecture for general smooth symplectic toric Calabi-Yau 3-folds by Eynard and Orantin [**41**] of 2012.

Fang, Liu, and Zong proved the remodeling conjecture for affine toric Calabi-Yau 3-orbifolds $[\mathbb{C}^3/G]$ in the open string sector [**44, 45**], and finally for general symplectic toric Calabi-Yau 3-orbifolds (including smooth symplectic toric Calabi-Yau 3-folds) in both open and closed string sectors in [**47**]. Key ingredients of the proof include orbifold version of Givental's formula for higher genus equivariant Gromov-Witten invariants [**92**] (specialized to toric Calabi-Yau 3-orbifolds) and the graph sum formula for invariants defined by topological recursion [**31, 36**] (applied to mirror curves). This volume contains an expository article by Fang and Zong on this proof, which is based on the mini-course by Fang and the contributed talk of Zong at the symposium.

6. A knot theory twist and quantum curves

In topology, it has been known since 1994 [**14**] that the $SL(2,\mathbb{C})$-character variety of the fundamental group of the complement of a knot in S^3 determines a plane algebraic curve defined over $\mathbb{Z}$, and that its defining equation, called the *A-polynomial*, is an invariant of the knot. The plane $\mathbb{C}^2$ here is an extension of $\mathbb{C}^* \times \mathbb{C}^*$, which is the $SL(2,\mathbb{C})$-character variety of the fundamental group of the torus T^2 that appears as the boundary of a tubular neighborhood of the knot complement in S^3. By choosing the meridian and longitude of the torus as a basis for $H_1(T^2,\mathbb{Z}) = \pi_1(T^2)$, the plane acquires a canonical coordinate (x, y), and every A-polynomial is a $\mathbb{Z}$-coefficient polynomial in these variables. Based on an earlier work [**50**], Garoufalidis [**52**] conjectured in 2004 that the A-polynomial of a hyperbolic knot should be *quantized* into a *difference operator*, which should then characterize the *colored Jones polynomial* of the knot as a solution to this difference equation. This conjecture is known as the AJ-conjecture, since the quantization of the A-polynomial annihilates the colored Jones polynomial. Around the same time, physicist Gukov [**57**] arrived at the same conjecture independently, and noticed that this was closely related to the notion of *quantum curve* of Aganagic, Dijkgraaf, Klemm, Mariño, and Vafa [**1**]. Physical theory of quantum curves has been further developed by Dijkgraaf, Hollands, Sułkowski, and Vafa [**17, 18, 59**].

Then in 2012, Gukov and Sułkowski [**58**] discovered a conjectural perturbative mechanism to construct the quantization of an A-polynomial so that the AJ-conjecture holds, using a topological recursion on the character variety as its spectral curve. A possible relation between the hyperbolic volume conjecture of Kashaev [**63**] and topological recursion was first pointed out in [**16**]. The mechanism of Gukov and Sułkowski [**58**] is the same as the exact WKB analysis of a Schrödinger equation, and their discovery is a conjectural closed formula for each term of the WKB expansion in terms of the integral of the differential forms determined by topological recursion. An algebraic K-theory condition, that the Steinberg symbol $\{x, y\} \in K_2(F)$ of the K_2 of the function field F of the spectral curve being a 2-torsion element [**14**], is speculated to be the condition for the unique quantizability of the spectral curve in [**58**]. (See also [**6**].)

These developments from physics and topology around the idea of quantum curves energized the topological recursion community. Many rigorous examples of quantum curves have been constructed by Bouchard, Do, Dunin-Barkowski, X. Liu, Manescu, Mulase, Norbury, Popolitov, Shadrin, Spitz, Sułkowski, Zhou, and others [**8**, **22**, **29**, **75**, **76**, **81**, **89**, **90**]. The conjectural WKB-type analysis was first proved for two cases, the Catalan numbers and simple Hurwitz numbers, in [**76**]. Surprisingly, a number of these examples exhibit a deep connection to representation theory of symmetric groups and fermionic operator calculus. In these examples, if the spectral curve is *trigonometric*, i.e., if it is contained in $\mathbb{C}^* \times \mathbb{C}^*$, then the quantum curves are not derived from topological recursion. It is also surprising that there seems to be the unique, or preferred, quantization in each case. The mechanism of this uniqueness is not fully understood as of now. At least the torsion condition of the Steinberg symbol holds for all these examples.

Relations between exact WKB analysis, counting problems, Painlevé equations, and topological recursion has been actively investigated in recent papers [**15**, **60**, **61**, **67**].

Toric mirror symmetry discussed above has a direct relation to torus knots. The topological recursion appearing in the context of mirror to the open Gromov-Witten invariants of the resolved conifold provides all-genera invariants of torus knots [**49**].

7. Relation to the Hitchin theory of Higgs bundles, and quantum curves as opers

The formulation of the original topological recursion is restricted to a plane algebraic (or an analytic) curve. The recursion formula, which is a sequence of residue calculations, is considered as a local operation, and the significance of a global nature of the complete spectral curve has not been fully developed in theory until recently. We note that for many cases of enumerative problems, topological recursion is established by calculating residues explicitly via global contour deformation [**8**, **27**, **39**, **77**]. Thus the global geometry is indeed reflected in topological recursion.

In 2014, Dumitrescu and Mulase [**24**] observed that the notion of the spectral curve of the topological recursion due to Eynard and Orantin [**40**], which comes from spectral curves in random matrix theory, is exactly the same as the notion of the Hitchin spectral curves, and thus a topological recursion can be formulated for *any* Hitchin spectral curve in the cotangent bundle T^*C of a base curve C. The original context is the case of $C = \mathbb{C}$. This unexpected relation between the Hitchin theory of Higgs bundles and topological recursion opened a door to a new uncharted territory (see, for example, [**5**]).

In this geometric framework of spectral curves of topological recursion, the formalism can be further generalized for *singular* spectral curves, as discussed in the paper by Dumitrescu and Mulase in this volume. The Airy function example (2.5) corresponds to a singular spectral curve in the Hirzebruch surface, which comes from a rank 2 vector bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ over $\mathbb{P}^1$ and a Higgs field $\phi = \begin{bmatrix} & x(dx)^2 \\ 1 & \end{bmatrix}$. Although (2.5) is a non-singular curve in $\mathbb{P}^2$, we need to consider it in the compactification of the cotangent bundle $T^*\mathbb{P}^1$, which has a quintic cusp singularity at the infinity. This singularity reflects in the essential singularity of the entire solution $Ai(x)$ of the differential equation (2.4) at ∞. Still the relationship between geometry of Higgs bundles and the quantum invariants that the generalized topological recursion should calculate, such as (2.2), are not fully explored.

After this relation to Hitchin theory is established, the notion of a quantum curve corresponding to a given spectral curve becomes clear in terms of the mathematical language of *opers* [**23**, **26**]. A spectral curve Σ embedded in the cotangent bundle T^*C of a base curve C is called *rational*, contrasting to the trigonometric case $\Sigma \subset \mathbb{C}^* \times \mathbb{C}^*$ and the *elliptic* case, which is a divisor in a $K3$ surface. For the rational case, if we identify quantum curves being opers corresponding to Hitchin spectral curves, then the unique quantization mechanism is well understood in terms of Kostant's three-dimensional subgroups (TDS) [**66**]. Since topological recursion does not refer to the TDS condition in general, it usually does not produce opers. It is unknown at this moment how to implement the TDS condition into topological recursion associated with Hitchin spectral curves for an arbitrary algebraic group so that we can directly identify the two quantization procedures: one through opers and the other through topological recursion.

While [**24**] is the first attempt of establishing an algebro-geometric framework for topological recursion by identifying its spectral curves with Hitchin spectral curves, the geometry of the recursive structure of topological recursion itself is first mathematically identified as CohFT in [**31**]. In more concrete examples, a formula based transformation from a Kontsevich-Manin CohFT [**65**] to the multilinear differential forms on the spectral curve defined by topological recursion has been rigorously obtained. Andersen, Borot, and Orantin report in this volume a different perspective on the relation between CohFT and topological recursion than those found in [**31**]. The authors describe a systematic way of constructing vector bundles on $\overline{\mathcal{M}}_{g,n}$ through modular functors and then define a CohFT using their characteristic classes. They then prove that in certain cases these CohFT directly give rise to a solution to topological recursion. Important examples in their paper not covered in [**31**] and above mentioned work include the Verlinde formula [**85**].

A striking byproduct of this line of research is a new proof [**28**, **68**] of the ELSV-type formulae for various Hurwitz numbers, originally due to Ekedahl, Lando, Shapiro, and Vainshtein [**33**], and its generalizations such as the one by Johnson, Pandharipande, and Tseng [**62**] for orbifold Hurwitz numbers. Lewanski provides a concise article for this volume reviewing recent developments in this direction generalizing the ELSV formula to several cases.

8. New developments

We now briefly mention other papers of this volume that are not cited in the above sections.

Many geometric works on topological recursion, as described above, are trying to identify the recursion itself as a mechanism of quantization, such as CohFT. A totally new point of view is proposed by Kontsevich and Soibelman in this volume. The authors propose a different formulation, which include a much larger context of validity, for topological recursion. They put together topological recursion, quantum curves, and WKB analysis from the perspective of (holomorphic) symplectic geometry. The recursion formula of [**40**] does not exhibit manifest symmetry of the n-point function. For enumerative geometry problems, these n-point functions are obtained by the Laplace transform of symmetric functions, such as Hurwitz numbers and the Weil-Petersson volume of moduli spaces of bordered hyperbolic surfaces. The fundamental point of departure of Kontsevich-Soibelman is the manifest symmetry of the quantities to be calculated. They treat topological recursion as a quantization mechanism from the point of view of *deformation quantization*. Then they produce families of cyclic $\mathcal{D}$-modules, which are their definition of what others call quantum curves. The influence of this work can be seen, for example, in [**4**]. More concrete relations between the new framework and other computational aspect of topological recursion is scheduled to be discussed elsewhere by the authors.

A quantum curve is a $\mathcal{D}$-module over a compact Riemann surface, and these $\mathcal{D}$-modules with singularities are of great interest. Although this is a classical problem since the time of Painlevé, recently there have been explosive developments in this direction. In a paper contained in this volume, Korotkin proves a striking result that directly relates the standard symplectic structure of $T^*\mathcal{M}_{g,n}$ and the Goldman bracket on the $SL(2,\mathbb{C})$-character variety of n-punctured surface group, using a second order linear differential equation on a Riemann surface with singularities.

Extending the physical investigation of quantum curves developed in [**17**, **18**, **58**], Ciosmak, Hadasz, Manabe, and Sułkowski present a new algebraic way of viewing quantum curves in this volume. They consider both Virasoro and super Virasoro representations, and they identify quantum curves as singular vectors of Virasoro operators. In their definition, quantum curves can also be constructed via topological recursion. A particularly interesting direction is the supersymmetric generalization. So far the known relation between topological recursion and CohFT is based on commutative Frobenius algebras, corresponding to Gromov-Witten theory of a manifold without odd cohomologies. The super quantum curves of this paper are expected to be related to Gromov-Witten theory where odd degree cohomologies play an essential role.

Brini gives a concise survey of his new results concerning the Gopakumar-Vafa and Ooguri-Vafa correspondences. This physical theory relates Reshetikhin-Turaev-Witten invariants of knots, Gromov-Witten and Donaldson-Thomas invariants of local Calabi-Yau 3-folds, and topological recursion. Brini discusses the most general framework in which this correspondence holds.

Chiodo and Nagel present a cohomological Landau-Ginzburg/Calabi-Yau (LG/CY) correspondence which says the orbifold Chen-Ruan cohomology $H^{p,q}_{\mathrm{CR}}$ of a Calabi-Yau complete intersection in a weighted projective space is isomorphic, as a bi-graded vector space, to the state space of a hybrid Landau-Ginzburg model (which is more general than the hybrid Landau-Ginzburg model in the terminology of Fan-Jarvis-Ruan [**43**]). This state space isomorphism is the first step toward the LG/CY correspondence.

The article by Esteves explores intriguing algebraic structures lying at the heart of topological recursion. By using the Loday-Ronco Hopf algebra of planar rooted trees, he constructs a representation of the space of genus zero correlation functions that enjoys a Hopf algebra structure compatible with the residue formula appearing in topological recursion.

Kimura gives a review of remarkable relations between W-algebras, quiver gauge theories, qq-characters, and integrable systems. An important aspect of this work is the construction of W-algebras out of quivers, as well as an elliptic deformation of this construction. Since it has been speculated that quantum curves would appear in the construction of W-algebra representations in the work of Nakajima [**78**], we expect to see a new discovery in this direction of research.

At the symposium, we had a talk on opers by Fredrickson solving a conjecture of Gaiotto [**51**]. We refer to her paper [**23**] on this development. We also had a talk by Iwaki on Painlevé equations and topological recursion. On this important contribution, we refer to his papers [**60**, **61**].

Chiu-Chu Melissa Liu
Motohico Mulase

Bibliography

[1] Mina Aganagic, Robbert Dijkgraaf, Albrecht Klemm, Marcos Mariño, and Cumrun Vafa, *Topological strings and integrable hierarchies*, Comm. Math. Phys. **261** (2006), no. 2, 451–516, DOI 10.1007/s00220-005-1448-9. MR2191887

[2] G.B. Airy, *On the intensity of light in the neighbourhood of a caustic*, Transactions of the Cambridge Philosophical Society **6**, 379–402 (1838).

[3] A. Alexandrov, A. Mironov, and A. Morozov, *Solving Virasoro constraints in matrix models*, Fortschr. Phys. **53** (2005), no. 5-6, 512–521, DOI 10.1002/prop.200410212. MR2145334

[4] J.E. Andersen, G. Borot, and N. Orantin, *Geometric recursion*, arXiv:1711.04729v2 [math.GT] (2017).

[5] D. Baraglia and Z. Huang, *Special Kähler geometry of the Hitchin system and topological recursion*, arXiv:1707.04975v2 [math.DG] (2017).

[6] Gaëtan Borot and Bertrand Eynard, *All order asymptotics of hyperbolic knot invariants from non-perturbative topological recursion of A-polynomials*, Quantum Topol. **6** (2015), no. 1, 39–138, DOI 10.4171/QT/60. MR3335006

[7] Vincent Bouchard, Andrei Catuneanu, Olivier Marchal, and Piotr Sułkowski, *The remodeling conjecture and the Faber-Pandharipande formula*, Lett. Math. Phys. **103** (2013), no. 1, 59–77, DOI 10.1007/s11005-012-0588-z. MR3004817

[8] Vincent Bouchard, Daniel Hernández Serrano, Xiaojun Liu, and Motohico Mulase, *Mirror symmetry for orbifold Hurwitz numbers*, J. Differential Geom. **98** (2014), no. 3, 375–423. MR3263522

[9] Vincent Bouchard, Albrecht Klemm, Marcos Mariño, and Sara Pasquetti, *Remodeling the B-model*, Comm. Math. Phys. **287** (2009), no. 1, 117–178, DOI 10.1007/s00220-008-0620-4. MR2480744

[10] Vincent Bouchard, Albrecht Klemm, Marcos Mariño, and Sara Pasquetti, *Topological open strings on orbifolds*, Comm. Math. Phys. **296** (2010), no. 3, 589–623, DOI 10.1007/s00220-010-1020-0. MR2628817
78, 263–283 (2008).

[11] Vincent Bouchard and Marcos Mariño, *Hurwitz numbers, matrix models and enumerative geometry*, From Hodge theory to integrability and TQFT tt*-geometry, Proc. Sympos. Pure Math., vol. 78, Amer. Math. Soc., Providence, RI, 2008, pp. 263–283, DOI 10.1090/pspum/078/2483754. MR2483754

[12] Kevin M. Chapman, Motohico Mulase, and Brad Safnuk, *The Kontsevich constants for the volume of the moduli of curves and topological recursion*, Commun. Number Theory Phys. **5** (2011), no. 3, 643–698, DOI 10.4310/CNTP.2011.v5.n3.a3. MR2864463

[13] Leonid Chekhov, Bertrand Eynard, and Nicolas Orantin, *Free energy topological expansion for the 2-matrix model*, J. High Energy Phys. **12** (2006), 053, 31, DOI 10.1088/1126-6708/2006/12/053. MR2276699
118, 47–84 (1994).

[14] D. Cooper, M. Culler, H. Gillet, D. D. Long, and P. B. Shalen, *Plane curves associated to character varieties of* 3*-manifolds*, Invent. Math. **118** (1994), no. 1, 47–84, DOI 10.1007/BF01231526. MR1288467

[15] M. Cutimanco, P. Labelle, and V. Shramchenko, *Topological recursion of Eynard-Orantin and the harmonic oscillator*, arXiv:1701.08913v2 [math-ph] (2017).

[16] Robbert Dijkgraaf, Hiroyuki Fuji, and Masahide Manabe, *The volume conjecture, perturbative knot invariants, and recursion relations for topological strings*, Nuclear Phys. B **849** (2011), no. 1, 166–211, DOI 10.1016/j.nuclphysb.2011.03.014. MR2795276

[17] Robbert Dijkgraaf, Lotte Hollands, and Piotr Sułkowski, *Quantum curves and $\mathcal{D}$-modules*, J. High Energy Phys. **11** (2009), 047, 59, DOI 10.1088/1126-6708/2009/11/047. MR2628886

[18] Robbert Dijkgraaf, Lotte Hollands, Piotr Sułkowski, and Cumrun Vafa, *Supersymmetric gauge theories, intersecting branes and free fermions*, J. High Energy Phys. **2** (2008), 106, 57, DOI 10.1088/1126-6708/2008/02/106. MR2385939

[19] R. Dijkgraaf and C. Vafa, *Two Dimensional Kodaira-Spencer Theory and Three Dimensional Chern-Simons Gravity*, arXiv:0711.1932 [hep-th] (2007).

[20] Robbert Dijkgraaf, Herman Verlinde, and Erik Verlinde, *Loop equations and Virasoro constraints in nonperturbative two-dimensional quantum gravity*, Nuclear Phys. B **348** (1991), no. 3, 435–456, DOI 10.1016/0550-3213(91)90199-8. MR1083914

[21] Norman Do, Oliver Leigh, and Paul Norbury, *Orbifold Hurwitz numbers and Eynard-Orantin invariants*, Math. Res. Lett. **23** (2016), no. 5, 1281–1327, DOI 10.4310/MRL.2016.v23.n5.a3. MR3601067

[22] Norman Do and David Manescu, *Quantum curves for the enumeration of ribbon graphs and hypermaps*, Commun. Number Theory Phys. **8** (2014), no. 4, 677–701, DOI 10.4310/CNTP.2014.v8.n4.a2. MR3318387

[23] O. Dumitrescu, L. Fredrickson, G. Kydonakis, R. Mazzeo, M. Mulase, and A. Neitzke, *From the Hitchin section to opers through nonabelian Hodge*, Journal of Differential Geometry, to appear.

[24] Olivia Dumitrescu and Motohico Mulase, *Quantum curves for Hitchin fibrations and the Eynard-Orantin theory*, Lett. Math. Phys. **104** (2014), no. 6, 635–671, DOI 10.1007/s11005-014-0679-0. MR3200933

[25] Olivia Dumitrescu and Motohico Mulase, *Edge contraction on dual ribbon graphs and 2D TQFT*, J. Algebra **494** (2018), 1–27, DOI 10.1016/j.jalgebra.2017.09.027. MR3723168

[26] O. Dumitrescu and M. Mulase, *Interplay between opers, quantum curves, WKB analysis, and Higgs bundles*, arXiv:1702.00511 (2017).

[27] Olivia Dumitrescu, Motohico Mulase, Brad Safnuk, and Adam Sorkin, *The spectral curve of the Eynard-Orantin recursion via the Laplace transform*, Algebraic and geometric aspects of integrable systems and random matrices, Contemp. Math., vol. 593, Amer. Math. Soc., Providence, RI, 2013, pp. 263–315, DOI 10.1090/conm/593/11867. MR3087960

[28] P. Dunin-Barkowski, M. Kazarian, N. Orantin, S. Shadrin, and L. Spitz, *Polynomiality of Hurwitz numbers, Bouchard-Mariño conjecture, and a new proof of the ELSV formula*, Adv. Math. **279** (2015), 67–103, DOI 10.1016/j.aim.2015.03.016. MR3345179

[29] Petr Dunin-Barkowski, Motohico Mulase, Paul Norbury, Alexander Popolitov, and Sergey Shadrin, *Quantum spectral curve for the Gromov-Witten theory of the complex projective line*, J. Reine Angew. Math. **726** (2017), 267–289, DOI 10.1515/crelle-2014-0097. MR3641659

[30] P. Dunin-Barkowski, P. Norbury, N. Orantin, A. Popolitov, and S. Shadrin, *Dubrovin's superpotential as a global spectral curve*, Journal of the Institute of Mathematics of Jussieu, https://doi.org/10.1017/S147474801700007X (2017).

[31] P. Dunin-Barkowski, N. Orantin, S. Shadrin, and L. Spitz, *Identification of the Givental formula with the spectral curve topological recursion procedure*, Comm. Math. Phys. **328** (2014), no. 2, 669–700, DOI 10.1007/s00220-014-1887-2. MR3199996

[32] Petr Dunin-Barkowski, Sergey Shadrin, and Loek Spitz, *Givental graphs and inversion symmetry*, Lett. Math. Phys. **103** (2013), no. 5, 533–557, DOI 10.1007/s11005-013-0606-9. MR3041758

[33] Torsten Ekedahl, Sergei Lando, Michael Shapiro, and Alek Vainshtein, *Hurwitz numbers and intersections on moduli spaces of curves*, Invent. Math. **146** (2001), no. 2, 297–327, DOI 10.1007/s002220100164. MR1864018

[34] Bertrand Eynard, *Topological expansion for the 1-Hermitian matrix model correlation functions*, J. High Energy Phys. **11** (2004), 031, 35 pp. (2005), DOI 10.1088/1126-6708/2004/11/031. MR2118807

[35] Bertrand Eynard, *Recursion between Mumford volumes of moduli spaces*, Ann. Henri Poincaré **12** (2011), no. 8, 1431–1447, DOI 10.1007/s00023-011-0113-4. MR2855174

[36] B. Eynard, *Intersection numbers of spectral curves*, arXiv:1104.0176 (2011).

[37] B. Eynard, *Invariants of spectral curves and intersection theory of moduli spaces of complex curves*, Commun. Number Theory Phys. **8** (2014), no. 3, 541–588, DOI 10.4310/CNTP.2014.v8.n3.a4. MR3282995

[38] Bertrand Eynard, *An overview of the topological recursion*, Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. III, Kyung Moon Sa, Seoul, 2014, pp. 1063–1085. MR3729064

[39] Bertrand Eynard, Motohico Mulase, and Bradley Safnuk, *The Laplace transform of the cut-and-join equation and the Bouchard-Mariño conjecture on Hurwitz numbers*, Publ. Res. Inst. Math. Sci. **47** (2011), no. 2, 629–670, DOI 10.2977/PRIMS/47. MR2849645

[40] B. Eynard and N. Orantin, *Invariants of algebraic curves and topological expansion*, Commun. Number Theory Phys. **1** (2007), no. 2, 347–452, DOI 10.4310/CNTP.2007.v1.n2.a4. MR2346575

[41] B. Eynard and N. Orantin, *Computation of open Gromov-Witten invariants for toric Calabi-Yau 3-folds by topological recursion, a proof of the BKMP conjecture*, Comm. Math. Phys. **337** (2015), no. 2, 483–567, DOI 10.1007/s00220-015-2361-5. MR3339157

[42] C. Faber and R. Pandharipande, *Hodge integrals, partition matrices, and the λ_g conjecture*, Ann. of Math. (2) **157** (2003), no. 1, 97–124, DOI 10.4007/annals.2003.157.97. MR1954265

[43] Huijun Fan, Tyler J. Jarvis, and Yongbin Ruan, *A mathematical theory of the gauged linear sigma model*, Geom. Topol. **22** (2018), no. 1, 235–303, DOI 10.2140/gt.2018.22.235. MR3720344

[44] Bohan Fang, Chiu-Chu Melissa Liu, and Zhengyu Zong, *Equivariant Gromov-Witten theory of affine smooth toric Deligne-Mumford stacks*, Int. Math. Res. Not. IMRN **7** (2016), 2127–2144, DOI 10.1093/imrn/rnv201. MR3509949

[45] Bohan Fang, Chiu-Chu Melissa Liu, and Zhengyu Zong, *All genus mirror symmetry for toric Calabi-Yau 3-orbifolds*, String-Math 2014, Proc. Sympos. Pure Math., vol. 93, Amer. Math. Soc., Providence, RI, 2016, pp. 1–19. MR3524231

[46] Bohan Fang, Chiu-Chu Melissa Liu, and Zhengyu Zong, *The Eynard-Orantin recursion and equivariant mirror symmetry for the projective line*, Geom. Topol. **21** (2017), no. 4, 2049–2092, DOI 10.2140/gt.2017.21.2049. MR3654104

[47] Bohan Fang, Chiu-Chu Melissa Liu, and Zhengyu Zong, *All genus mirror symmetry for toric Calabi-Yau 3-orbifolds*, String-Math 2014, Proc. Sympos. Pure Math., vol. 93, Amer. Math. Soc., Providence, RI, 2016, pp. 1–19. MR3524231

[48] Bohan Fang, Chiu-Chu Melissa Liu, and Zhengyu Zong, *The SYZ mirror symmetry and the BKMP remodeling conjecture*, Adv. Theor. Math. Phys. **20** (2016), no. 1, 165–192, DOI 10.4310/ATMP.2016.v20.n1.a3. MR3531992

[49] B. Fang and Z. Zong, *Topological recursion for the conifold transition of a torus knot*, arXiv:1607.01208v2 [math.AG] (2016).
Soc. **354**, 735–747 (2002).

[50] Charles Frohman, Răzvan Gelca, and Walter Lofaro, *The A-polynomial from the noncommutative viewpoint*, Trans. Amer. Math. Soc. **354** (2002), no. 2, 735–747, DOI 10.1090/S0002-9947-01-02889-6. MR1862565

[51] D. Gaiotto, *Opers and TBA*, arXiv:1403.6137 [hep-th] (2014).

[52] Stavros Garoufalidis, *On the characteristic and deformation varieties of a knot*, Proceedings of the Casson Fest, Geom. Topol. Monogr., vol. 7, Geom. Topol. Publ., Coventry, 2004, pp. 291–309, DOI 10.2140/gtm.2004.7.291. MR2172488

[53] Stavros Garoufalidis and Thang T. Q. Lê, *The colored Jones function is q-holonomic*, Geom. Topol. **9** (2005), 1253–1293, DOI 10.2140/gt.2005.9.1253. MR2174266

[54] Alexander B. Givental, *Semisimple Frobenius structures at higher genus*, Internat. Math. Res. Notices **23** (2001), 1265–1286, DOI 10.1155/S1073792801000605. MR1866444

[55] I. P. Goulden, *A differential operator for symmetric functions and the combinatorics of multiplying transpositions*, Trans. Amer. Math. Soc. **344** (1994), no. 1, 421–440, DOI 10.2307/2154724. MR1249468

[56] I. P. Goulden and D. M. Jackson, *Transitive factorisations into transpositions and holomorphic mappings on the sphere*, Proc. Amer. Math. Soc. **125** (1997), no. 1, 51–60, DOI 10.1090/S0002-9939-97-03880-X. MR1396978

[57] Sergei Gukov, *Three-dimensional quantum gravity, Chern-Simons theory, and the A-polynomial*, Comm. Math. Phys. **255** (2005), no. 3, 577–627, DOI 10.1007/s00220-005-1312-y. MR2134725

[58] Sergei Gukov and Piotr Sułkowski, *A-polynomial, B-model, and quantization*, J. High Energy Phys. **2** (2012), 070, front matter+56. MR2996110

[59] L. Hollands, *Topological strings and quantum curves*, Ph.D. Thesis, University of Amsterdam arXiv:0911.3413 [hep-th] (2009).

[60] K. Iwaki, O. Marchal, and A. Saenz, *Painlevé equations, topological type property and reconstruction by the topological recursion*, J. Geom. Phys. **124** (2018), 16–54, DOI 10.1016/j.geomphys.2017.10.009. MR3754496

[61] Kohei Iwaki and Axel Saenz, *Quantum curve and the first Painlevé equation*, SIGMA Symmetry Integrability Geom. Methods Appl. **12** (2016), Paper No. 011, 24 pages, DOI 10.3842/SIGMA.2016.011. MR3453982

[62] P. Johnson, R. Pandharipande, and H.-H. Tseng, *Abelian Hurwitz-Hodge integrals*, Michigan Math. J. **60** (2011), no. 1, 171–198, DOI 10.1307/mmj/1301586310. MR2785870

[63] R. M. Kashaev, *The hyperbolic volume of knots from the quantum dilogarithm*, Lett. Math. Phys. **39** (1997), no. 3, 269–275, DOI 10.1023/A:1007364912784. MR1434238

[64] Maxim Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys. **147** (1992), no. 1, 1–23. MR1171758

[65] M. Kontsevich and Yu. Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry*, Comm. Math. Phys. **164** (1994), no. 3, 525–562. MR1291244

[66] Bertram Kostant, *The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group*, Amer. J. Math. **81** (1959), 973–1032, DOI 10.2307/2372999. MR0114875

[67] K.G. Krishna, P. Labelle, and V. Shramchenko, *Feynman diagrams, ribbon graphs, and topological recursion of Eynard-Orantin*, arXiv:1802.01773v2 [hep-th] (2018).

[68] Danilo Lewanski, Alexandr Popolitov, Sergey Shadrin, and Dimitri Zvonkine, *Chiodo formulas for the r-th roots and topological recursion*, Lett. Math. Phys. **107** (2017), no. 5, 901–919, DOI 10.1007/s11005-016-0928-5. MR3633029

[69] Wei Luo and Shengmao Zhu, *Hurwitz-Hodge integral identities from the cut-and-join equation*, Ann. Comb. **19** (2015), no. 4, 749–763, DOI 10.1007/s00026-015-0282-9. MR3415010

[70] Marcos Mariño, *Open string amplitudes and large order behavior in topological string theory*, J. High Energy Phys. **3** (2008), 060, 34, DOI 10.1088/1126-6708/2008/03/060. MR2391060

[71] Maryam Mirzakhani, *Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces*, Invent. Math. **167** (2007), no. 1, 179–222, DOI 10.1007/s00222-006-0013-2. MR2264808

[72] Maryam Mirzakhani, *Weil-Petersson volumes and intersection theory on the moduli space of curves*, J. Amer. Math. Soc. **20** (2007), no. 1, 1–23, DOI 10.1090/S0894-0347-06-00526-1. MR2257394

[73] Motohico Mulase and Michael Penkava, *Topological recursion for the Poincaré polynomial of the combinatorial moduli space of curves*, Adv. Math. **230** (2012), no. 3, 1322–1339, DOI 10.1016/j.aim.2012.03.027. MR2921181

[74] Motohico Mulase and Brad Safnuk, *Mirzakhani's recursion relations, Virasoro constraints and the KdV hierarchy*, Indian J. Math. **50** (2008), no. 1, 189–218. MR2379144

[75] M. Mulase, S. Shadrin, and L. Spitz, *The spectral curve and the Schrödinger equation of double Hurwitz numbers and higher spin structures*, Commun. Number Theory Phys. **7** (2013), no. 1, 125–143, DOI 10.4310/CNTP.2013.v7.n1.a4. MR3108774

[76] Motohico Mulase and Piotr Sułkowski, *Spectral curves and the Schrödinger equations for the Eynard-Orantin recursion*, Adv. Theor. Math. Phys. **19** (2015), no. 5, 955–1015, DOI 10.4310/ATMP.2015.v19.n5.a2. MR3487649

[77] Motohico Mulase and Naizhen Zhang, *Polynomial recursion formula for linear Hodge integrals*, Commun. Number Theory Phys. **4** (2010), no. 2, 267–293, DOI 10.4310/CNTP.2010.v4.n2.a1. MR2725053

[78] Hiraku Nakajima, *More lectures on Hilbert schemes of points on surfaces*, Development of moduli theory—Kyoto 2013, Adv. Stud. Pure Math., vol. 69, Math. Soc. Japan, [Tokyo], 2016, pp. 173–205. MR3586508

[79] Paul Norbury, *Counting lattice points in the moduli space of curves*, Math. Res. Lett. **17** (2010), no. 3, 467–481, DOI 10.4310/MRL.2010.v17.n3.a7. MR2653682

[80] Paul Norbury, *String and dilaton equations for counting lattice points in the moduli space of curves*, Trans. Amer. Math. Soc. **365** (2013), no. 4, 1687–1709, DOI 10.1090/S0002-9947-2012-05559-0. MR3009643

[81] Paul Norbury, *Quantum curves and topological recursion*, String-Math 2014, Proc. Sympos. Pure Math., vol. 93, Amer. Math. Soc., Providence, RI, 2016, pp. 41–65. MR3524233

[82] Paul Norbury and Nick Scott, *Gromov-Witten invariants of $\mathbb{P}^1$ and Eynard-Orantin invariants*, Geom. Topol. **18** (2014), no. 4, 1865–1910, DOI 10.2140/gt.2014.18.1865. MR3268770

[83] Hirosi Ooguri, Piotr Sułkowski, and Masahito Yamazaki, *Wall crossing as seen by matrix models*, Comm. Math. Phys. **307** (2011), no. 2, 429–462, DOI 10.1007/s00220-011-1330-x. MR2837121

[84] R. Vakil, *Enumerative Geometry of Curves via Degeneration Methods*, Ph.D. thesis, Harvard University (1997), 149 pp.

[85] Erik Verlinde, *Fusion rules and modular transformations in 2D conformal field theory*, Nuclear Phys. B **300** (1988), no. 3, 360–376, DOI 10.1016/0550-3213(88)90603-7. MR954762

[86] Constantin Teleman, *The structure of 2D semi-simple field theories*, Invent. Math. **188** (2012), no. 3, 525–588, DOI 10.1007/s00222-011-0352-5. MR2917177

[87] Edward Witten, *Two-dimensional gravity and intersection theory on moduli space*, Surveys in differential geometry (Cambridge, MA, 1990), Lehigh Univ., Bethlehem, PA, 1991, pp. 243–310. MR1144529

[88] J. Zhou, *Local Mirror Symmetry for One-Legged Topological Vertex*, arXiv:0910.4320 [math.AG] (2009).

[89] J. Zhou, *Intersection numbers on Deligne-Mumford moduli spaces and quantum Airy curve*, arXiv:1206.5896 [math.AG] (2012).

[90] J. Zhou, *Quantum mirror curves for $\mathbb{C}^3$ and the resolved conifold*, arXiv:1207.0598v1 [math.AG] (2012).

[91] Shengmao Zhu, *On a proof of the Bouchard-Sulkowski conjecture*, Math. Res. Lett. **22** (2015), no. 2, 633–643, DOI 10.4310/MRL.2015.v22.n2.a14. MR3342249

[92] Zhengyu Zong, *Equivariant Gromov-Witten Theory of GKM Orbifolds*, ProQuest LLC, Ann Arbor, MI, 2015. Thesis (Ph.D.)–Columbia University. MR3312922

Selected Published Titles in This Series

100 **Chiu-Chu Melissa Liu and Motohico Mulase, Editors,** Topological Recursion and its Influence in Analysis, Geometry, and Topology, 2018

99 **Vicente Muñoz, Ivan Smith, and Richard P. Thomas, Editors,** Modern Geometry, 2018

98 **Amir-Kian Kashani-Poor, Ruben Minasian, Nikita Nekrasov, and Boris Pioline, Editors,** String-Math 2016, 2018

97 **Tommaso de Fernex, Brendan Hassett, Mircea Mustaţă, Martin Olsson, Mihnea Popa, and Richard Thomas, Editors,** Algebraic Geometry: Salt Lake City 2015 (Parts 1 and 2), 2018

96 **Si Li, Bong H. Lian, Wei Song, and Shing-Tung Yau, Editors,** String-Math 2015, 2017

95 **Izzet Coskun, Tommaso de Fernex, and Angela Gibney, Editors,** Surveys on Recent Developments in Algebraic Geometry, 2017

94 **Mahir Bilen Can, Editor,** Algebraic Groups: Structure and Actions, 2017

93 **Vincent Bouchard, Charles Doran, Stefan Méndez-Diez, and Callum Quigley, Editors,** String-Math 2014, 2016

92 **Kailash C. Misra, Daniel K. Nakano, and Brian J. Parshall, Editors,** Lie Algebras, Lie Superalgebras, Vertex Algebras and Related Topics, 2016

91 **V. Sidoravicius and S. Smirnov, Editors,** Probability and Statistical Physics in St. Petersburg, 2016

90 **Ron Donagi, Sheldon Katz, Albrecht Klemm, and David R. Morrison, Editors,** String-Math 2012, 2015

89 **D. Dolgopyat, Y. Pesin, M. Pollicott, and L. Stoyanov, Editors,** Hyperbolic Dynamics, Fluctuations and Large Deviations, 2015

88 **Ron Donagi, Michael R. Douglas, Ljudmila Kamenova, and Martin Rocek, Editors,** String-Math 2013, 2014

87 **Helge Holden, Barry Simon, and Gerald Teschl, Editors,** Spectral Analysis, Differential Equations and Mathematical Physics: A Festschrift in Honor of Fritz Gesztesy's 60th Birthday, 2013

86 **Kailash C. Misra, Daniel K. Nakano, and Brian J. Parshall, Editors,** Recent Developments in Lie Algebras, Groups and Representation Theory, 2012

85 **Jonathan Block, Jacques Distler, Ron Donagi, and Eric Sharpe, Editors,** String-Math 2011, 2012

84 **Alex H. Barnett, Carolyn S. Gordon, Peter A. Perry, and Alejandro Uribe, Editors,** Spectral Geometry, 2012

83 **Hisham Sati and Urs Schreiber, Editors,** Mathematical Foundations of Quantum Field Theory and Perturbative String Theory, 2011

82 **Michael Usher, Editor,** Low-dimensional and Symplectic Topology, 2011

81 **Robert S. Doran, Greg Friedman, and Jonathan Rosenberg, Editors,** Superstrings, Geometry, Topology, and C^*-algebras, 2010

80 **D. Abramovich, A. Bertram, L. Katzarkov, R. Pandharipande, and M. Thaddeus, Editors,** Algebraic Geometry, 2009

79 **Dorina Mitrea and Marius Mitrea, Editors,** Perspectives in Partial Differential Equations, Harmonic Analysis and Applications, 2008

78 **Ron Y. Donagi and Katrin Wendland, Editors,** From Hodge Theory to Integrability and TQFT, 2008

77 **Pavel Exner, Jonathan P. Keating, Peter Kuchment, Toshikazu Sunada, and Alexander Teplyaev, Editors,** Analysis on Graphs and Its Applications, 2008

For a complete list of titles in this series, visit the
AMS Bookstore at **www.ams.org/bookstore/pspumseries/**.

ISBN 978-1-4704-3541-7

9 781470 435417

PSPUM/100

PSPUM

100

Topological Recursion and its Influence • Liu et al., Editors

AMS