10.1090/pspum/102/10

Proceedings of Symposia in Pure Mathematics
Volume **102**, 2019
https://doi.org/10.1090/pspum/102/01811

A Heegaard Floer analog of algebraic torsion

Çağatay Kutluhan, Gordana Matić, Jeremy Van Horn-Morris, and Andy Wand

Abstract. This article provides the origin story for *spectral order*, an invariant of contact structures on closed 3-manifolds defined by the authors. This invariant can be thought of as an assignment of a 'degree of tightness' to a contact structure. The principal motivation for the development of this invariant was work of Hutchings establishing a filtration on the differential in embedded contact homology. We explain how this filtration may be translated into the language of Heegaard Floer homology.

1. Introduction

In the '80s, Bennequin [**Ben83**] initiated and Eliashberg [**Eli87**, **Eli88**] modernized the study of contact structures on 3-manifolds, demonstrating first that they fall into two categories: *overtwisted* and *tight*. Overtwisted contact structures on closed 3-manifolds contain a specific embedded submanifold, an *overtwisted disk*, which enables complete 'flexiblity.' On a closed 3-manifold, each overtwisted contact structure is uniquely determined up to contact deformation by its structure as an oriented 2-plane field up to homotopy [**Eli89**], and every oriented 2-plane field is homotopic to an overtwisted contact structure [**Mar71**]. Tight contact structures, on the other hand, exhibit 'rigidity': they do not exist in every homotopy class of oriented 2-plane fields, and when they do, they are not necessarily unique [**Eli96**] [**LM96**].

A fundamental problem in 3-dimensional contact topology is the classification of tight contact structures: *which 3-manifolds carry tight contact structures, and for those that do, how do we distinguish distinct tight contact structures?* Eliashberg showed that the contact 3-manifold that naturally arises on the boundary of a compact symplectic 4-manifold is tight [**Eli90**] (see also [**Gro85**]), which yielded a wealth of constructions. However, not every tight contact structure arises this

2010 *Mathematics Subject Classification.* Primary 57M17, 57R58.

The first author was supported in part by NSF grants DMS-1128155 and DMS-1360293, and a Simons Foundation grant No. 519352.

The second author was supported in part by Simons Foundation grant No. 246461.

The third author was supported by Simons Foundation grant No. 279342 and NSF grant DMS-1612412.

The fourth author was supported in part by ERC grant GEODYCON and EPSRC EP/P004598/1.

way, [**EH02**, **LS03**].[1] Such phenomena sparked a search for invariants of contact structures that could detect tightness and fillability properties.

The foundational invariant of contact manifolds is *symplectic field theory* (SFT) of Eliashberg, Givental, and Hofer [**EGH00**]. It conjecturally detects tight contact structures on closed 3-manifolds. In [**LW11**], Latschev and Wendl investigated a property of the SFT algebra, its *algebraic torsion*, which measures the torsion of a certain element in the ground ring. They found out that if SFT algebraic torsion is finite, it obstructs the existence of symplectic fillings. Furthermore, it is non-decreasing under Legendrian surgery, making it a numerical measurement that further partitions the class of non-fillable contact structures.

Similar in construction to symplectic field theory is Hutchings's *embedded contact homology* (ECH) [**Hut10**] of closed contact 3-manifolds. In [**LW11**, Appendix], Hutchings described a method to define an analog of the SFT algebraic torsion in the framework of embedded contact homology. In the ECH analog of algebraic torsion, Hutchings uses a normalized Euler characteristic of the pseudoholomorphic curves counted in defining the ECH differential to introduce a filtration on it. He then uses this filtration to measure the 'degree of vanishing' of a canonical element in ECH.

In [**KMVWc**], the authors define the *spectral order* of a closed contact 3-manifold. It arises as an analog of the SFT algebraic torsion in the context of *Heegaard Floer homology*, a topological invariant of closed oriented 3-manifolds introduced by Ozsváth and Szabó [**OS04**], which is shown to be isomorphic to ECH [**CGH12b**, **CGH12c**, **CGH12a**]. Given a contact structure ξ on a closed 3-manifold Y, Ozsváth and Szabó specify a distinguished homology class $\hat{c}(\xi)$ (well defined up to ± 1) in the Heegaard Floer homology group $\widehat{HF}(Y)$. This class, known as the *Ozsváth–Szabó contact class*, yields an invariant of contact structures on closed 3-manifolds. The vanishing or non-vanishing of this class has been a powerful instrument to detect or obstruct tightness and fillability properties of contact structures on closed 3-manifolds. Ozsváth and Szabó proved that $\hat{c}(\xi)$ vanishes for overtwisted contact structures and is non-vanishing for symplectically fillable ones. However, it does not always detect tightness even in its strongest form (see, for example, [**Ghi06**]). Spectral order attempts to refine the information that one could extract from the Ozsváth–Szabó contact class.

The definition of spectral order is algebraic in nature. It uses a version of the usual Heegaard Floer chain complex associated to an open book decomposition with twisted coefficients resulting from the normalized Euler characteristic used by Hutchings. It takes values in the set $\mathbb{Z}_{\geq 0} \cup \{\infty\}$ and it has many desirable properties, perhaps foremost that it vanishes for overtwisted contact structures and is non-decreasing under Legendrian surgery and contact connected sum. These latter two properties are, for example, sufficient for it to give a very general obstruction to existence of Stein cobordisms between two closed contact 3-manifolds, answering [**LW11**, Question 1]. Furthermore, in [**KMVWc**], the authors give examples of families of contact 3-manifolds for which $\hat{c}(\xi) = 0$ but with non-zero spectral order and realizing infinitely many distinct positive integer values. These examples prove

[1] Indeed, there are many different flavors of 'fillings' of contact 3-manifolds by compact symplectic 4-manifolds. For ease of reading, in most cases throughout the article 'symplectically fillable' will mean 'strongly symplectically fillable.'

that spectral order detects non-trivial information about contact structures that the Ozsváth–Szabó contact class could not detect.

The goal of this article is to provide the origin story for spectral order. In this article, we describe in detail how a filtration on the ECH differential as defined by Hutchings can be ported to the setting of Heegaard Floer homology, details missing from the paper [**KMVWc**]. The main tool is the isomorphism between the latter and a variant of embedded contact homology for certain stable Hamiltonian structures, denoted *ech*, by Lee, Taubes, and the first named author (see [**KLT10a**, **KLT10b**, **KLT10c**], or [**Kut13**], for an outline).

Organization. In Section 2, we outline how an open book decomposition supporting a contact structure is used to construct an auxiliary manifold with a stable Hamiltonian structure on it, and how this is used to define *ech*. In Section 3, we describe Hutchings's filtration adapted to *ech* and define an analog of Latschev and Wendl's algebraic torsion in the context of *ech*. In Section 4, we translate the latter to Heegaard Floer homology.

2. The *ech* chain complex

In order to define a Heegaard Floer analog of Latschev and Wendl's algebraic torsion, we first define a filtered chain complex using the *ech* chain complex starting from an open book decomposition. In this regard, let M be a closed, connected, and oriented 3-manifold with a co-oriented contact structure ξ, and (S,ϕ) an abstract open book decomposition of M supporting ξ. Here, S (the *page*) is a compact oriented surface of genus g with B boundary components, and ϕ (the *monodromy*) is an orientation preserving diffeomorphism of S which restricts to identity in a neighborhood of the boundary. The manifold M is then diffeomorphic to $S\times[0,1]/\sim$ where $(p,1)\sim(\phi(p),0)$ for any $p\in S$ and $(p,t)\sim(p,t')$ for any $p\in\partial S$ and $t,t'\in[0,1]$. Let $\mathrm{G}=2g+\mathrm{B}-1$, and fix a self-indexing Morse function on S that has a single index-2 critical point, no index-0 critical points, and attains its minimum along ∂S. Then a suitably chosen pseudo-gradient vector field for the latter defines a basis of arcs $\mathbf{a}=\{a_1,\dots,a_{\mathrm{G}}\}$ on S; i.e. a disjoint collection of properly embedded arcs which cut S into a polygon. Each arc belonging to $\mathbf{a}$ is the descending manifold of an index-1 critical point of the Morse function for the chosen pseudo-gradient vector field. As in [**HKM09b**, §3.1], this basis together with the monodromy ϕ determines a Heegaard diagram $(\Sigma,\{\beta_1,\dots,\beta_{\mathrm{G}}\},\{\alpha_1,\dots,\alpha_{\mathrm{G}}\})$ for $-M$. To be more explicit, let $b=\{b_1,\dots,b_{\mathrm{G}}\}$ be a collection of arcs on S such that each b_i is obtained from a_i by an isotopy pushing each endpoint slightly along ∂S in the direction of the standard (counter-clockwise) orientation (see Figure 1). Then we have $\Sigma=S\times\{\frac12\}\cup_{\partial S}-S\times\{0\}$, $\alpha_i=a_i\times\{\frac12\}\cup a_i\times\{0\}$, and $\beta_i=b_i\times\{\frac12\}\cup\phi(b_i)\times\{0\}$.

The pseudo-gradient vector field on S and the monodromy ϕ can be used to define a self-indexing Morse function f and a pseudo-gradient vector field v on $-M$ which yield the Heegaard diagram $(\Sigma,\{\beta_1,\dots,\beta_{\mathrm{G}}\},\{\alpha_1,\dots,\alpha_{\mathrm{G}}\})$. Each α-curve appears as the intersection of Σ with the descending manifold of an index-1 critical point of f for v, while each β-curve appears as the intersection of Σ with the ascending manifold of an index-2 critical point of f for v. Therefore, the canonical pairing of the α- and β-curves indicated by their indexing induces a canonical pairing of the index-1 and index-2 critical points of f. With this understood, we construct a new manifold Y out of M by attaching *handles* following [**KLT10b**, §1a]. Each handle is a smooth manifold diffeomorphic to $[-1,1]\times S^2$ and it is

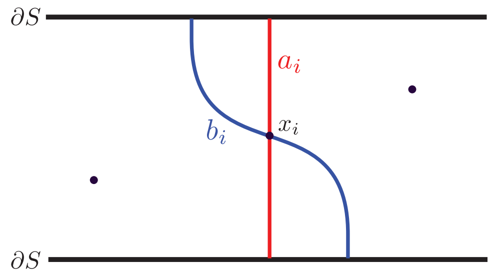

FIGURE 1. The arcs a_i and b_i on the surface S.

attached along its boundary to the complement of some regular coordinate balls centered at a distinguished pair of critical points of f. The resulting manifold Y is diffeomorphic to $M\#_{G+1}S^1 \times S^2$ via an orientation preserving diffeomorphism[2]. To be more explicit, one handle, denoted $\mathcal{H}_0$, is attached along the index-0 and index-3 critical points of f, and G additional handles are attached along pairs of index-1 and index-2 critical points of f. We denote the latter kind of handles by $\mathcal{H}_i$ for $i \in \{1, \dots, G\}$.

Next fix a basepoint $z \in \Sigma \setminus \bigcup_{i\in\{1,\dots,G\}}(\alpha_i \cup \beta_i)$ according to the convention in [**HKM09b**, §3.1], namely, fix a point in $S \times \{\frac{1}{2}\}$ outside the thin strips between a_i and b_i in Figure 1. By changing the monodromy in its isotopy class, one can make sure that the pointed Heegaard diagram $(\Sigma, \{\beta_1, \dots, \beta_G\}, \{\alpha_1, \dots, \alpha_G\}, z)$ is *strongly admissible* for any given $Spin^c$ structure. As stated in [**KLT10b**, Lemma 1.1], this means that there exists an area form w_Σ on Σ with total area equal to 2, and the signed area of each periodic domain $\mathcal{P}$ on this Heegaard diagram is equal to the pairing $\langle c_1(\mathfrak{s}_\xi), \mathcal{H}(\mathcal{P})\rangle$, where $\mathcal{H}(\mathcal{P}) \in H_2(M;\mathbb{Z})$ is the homology class corresponding to $\mathcal{P}$. Use v and w_Σ, as in [**KLT10b**, §1a-1d], to construct a stable Hamiltonian structure on Y. The latter is a pair (a, w) where a is a smooth 1-form and w is a smooth closed 2-form such that $da = hw$ for some smooth function $h : Y \to \mathbb{R}$, and $a \wedge w$ is nowhere zero. Associated to this stable Hamiltonian structure is a vector field R satisfying $w(R, \cdot) = 0$ and $a(R) = 1$, which agrees with the pseudo-gradient vector field v on $Y \setminus \bigcup_{i\in\{0,1,\dots,G\}} \mathcal{H}_i$. Of interest are periodic orbits of the vector field R. A complete description of periodic orbits of R is provided in [**KLT10b**, §2]. In particular, there is a unique embedded periodic orbit passing through the basepoint z, denoted by γ_z. This orbit intersects every cross-sectional sphere of $\mathcal{H}_0$ exactly once. Meanwhile, for each $i \in \{1, \dots, G\}$, there are exactly two distinct embedded periodic orbits inside $\mathcal{H}_i$, denoted by γ_i^+ and γ_i^-, which are hyperbolic and homologically trivial.

Periodic orbits of the Reeb vector field and pseudo-holomorphic curves in $\mathbb{R}\times Y$ for suitably generic almost complex structure are used in [**KLT10b**, Appendix A] to define a variant of Hutchings's embedded contact homology for certain homology classes $\Gamma \in H_1(Y;\mathbb{Z})$. First, we recall some basic definitions. An *orbit set* is a finite collection of the form $\{(\gamma, m)\}$ where γ are distinct embedded periodic orbits of the Reeb vector field and m is a positive integer. An orbit set $\{(\gamma, m)\}$ with the

[2]Note that this construction is equivalent to attaching 4-dimensional 1-handles on the outgoing boundary of the product cobordism $I \times M$, and Y is the outgoing boundary of the resulting cobordism.

extra requirement that $m = 1$ when γ is hyperbolic is called an *admissible orbit set*. The homology class of an orbit set $\Theta = \{(\gamma, m)\}$ is defined by $\sum m[\gamma]$. Of interest here are admissible orbit sets with homology class $\Gamma \in H_1(Y;\mathbb{Z})$ such that $\langle PD(\Gamma), [S^2]\rangle = 0$ if $[S^2]$ is a generator of $H_2(\mathcal{H}_0;\mathbb{Z})$, $\langle PD(\Gamma), [S^2]\rangle = 1$ if $[S^2]$ is the positive generator of $H_2(\mathcal{H}_i;\mathbb{Z})$ for $i \in \{1, \ldots, \mathrm{G}\}$, and $\mathfrak{s} - \mathfrak{s}_\xi = PD(\Gamma|_M)$ where $\mathfrak{s}_\xi$ is the canonical Spin^c structure on M defined by the contact structure ξ, and $\mathfrak{s}$ is a Spin^c structure on M. Note that the homology class Γ is uniquely determined by these properties once a Spin^c structure $\mathfrak{s}$ on M is fixed. For such $\Gamma \in H_1(Y;\mathbb{Z})$, it follows from [**KLT10b**, §2] that any γ that belongs to an admissible orbit set with homology class Γ is hyperbolic. Hence, we may drop the integer m from the notation. The free $\mathbb{Z}$-module generated by admissible orbit sets with homology Γ will be denoted by $\widehat{ecc}(Y, \Gamma)$.

Having fixed an almost complex structure J on $\mathbb{R} \times Y$ satisfying the conditions in [**KLT10b**, §3a], the *ECH index* of a J-holomorphic curve in $\mathbb{R} \times Y$ asymptotic to orbit sets representing the same homology class at $\pm\infty$ is an additive quantity that depends only on the relative homology class of the curve, and it is equal to the Fredholm index for embedded curves (see [**Hut02**]). The differential $\widehat{\partial}_{ech}$ on $\widehat{ecc}(Y, \Gamma)$ is defined to be the endomorphism of $\widehat{ecc}(Y, \Gamma)$ sending a generator Θ_+ to

$$\sum \sigma(\Theta_+, \Theta_-)\Theta_-,$$

where $\sigma(\Theta_+, \Theta_-)$ is a signed count, modulo $\mathbb{R}$-translation, of ECH index-1 J-holomorphic curves asymptotic to Θ_+ at $+\infty$ and to Θ_- at $-\infty$, and disjoint from $\mathbb{R} \times \gamma_z$. Note that $\mathbb{R} \times \gamma_z$ is a J-holomorphic cylinder. Hence, by additivity and positivity of intersections of J-holomorphic curves, it follows from [**KLT10c**] (cf. [**HT07**, **HT09**]) that $\widehat{\partial}_{ech} \circ \widehat{\partial}_{ech} = 0$, and

$$\widehat{ech}(Y, \Gamma) := H_*(\widehat{ecc}(Y, \Gamma), \widehat{\partial}_{ech}).$$

A detailed description of J-holomorphic curves in $\mathbb{R} \times Y$ is given in [**KLT10b**, §3 and §4]. In what follows we specialize to a subcomplex $\widehat{ecc}_\circ(Y, \Gamma)$ generated by a smaller set of admissible orbit sets. To describe the latter, note that by [**KLT10b**, Proposition 2.8], there exists a 1–1 correspondence between the set of admissible orbit sets and the set $\mathcal{Z}_{\mathrm{HF}} \times \prod_{i\in\{1,\ldots,\mathrm{G}\}}(\mathbb{Z} \times \mathrm{O})$ where $\mathcal{Z}_{\mathrm{HF}}$ denotes G-tuples of integral curves of the pseudo-gradient vector field v connecting the index-1 critical points of f to its index-2 critical points in such a way that each critical point appears as the limit of a unique integral curve, $\mathrm{O} = \{0, +1, -1, \{+1, -1\}\}$, and $i \in \{1, \ldots, \mathrm{G}\}$ corresponds to the handle $\mathcal{H}_i$. Then $\widehat{ecc}_\circ(Y, \Gamma)$ is the subcomplex generated by admissible orbit sets whose restriction to each $\mathcal{H}_i$ correspond to the pair $(0, 0) \in \mathbb{Z} \times \mathrm{O}$. The fact that $\widehat{ecc}_\circ(Y, \Gamma)$ is a subcomplex of $\widehat{ecc}(Y, \Gamma)$ follows from [**KLT10c**].

3. Hutchings's filtration in *ech*

Given orbit sets Θ_+ and Θ_- with the same homology class, Hutchings defines variants of his ECH index for relative homology classes in $H_2(Y, \Theta_+, \Theta_-)$, denoted by J_0, J_+, and J_- (see [**Hut**, §6]). Among other things, these are additive in the sense that for $Z_1 \in H_2(Y, \Theta_+, \Theta)$ and $Z_2 \in H_2(Y, \Theta, \Theta_-)$, we have

$$J_\circ(Z_1 + Z_2) = J_\circ(Z_1) + J_\circ(Z_2),$$

where $\circ \in \{0,+,-\}$ (see [**Hut**, Proposition 6.5]). If C is an embedded J-holomorphic curve in $\mathbb{R}\times Y$ with positive ends at an admissible orbit set Θ_+ and negative ends at an admissible orbit set Θ_-, then positive, respectively negative, ends of C are at distinct embedded periodic orbits of R, since an admissible orbit set consists only of hyperbolic orbits. Hence,

$$J_+(C) = -\chi(C) + |\Theta_+| - |\Theta_-|,$$

where $|\cdot|$ denotes the cardinality of an admissible orbit set. Decomposing into connected components, we can rewrite the above formula as

$$J_+(C) = \sum_{C_j \subset C} (2g_j - 2 + 2|\Theta_{j+}|), \tag{1}$$

where each C_j denotes a connected component of C, g_j denotes the genus of C_j, and each $\Theta_{j+} \subset \Theta_+$ is the collection of periodic orbits of the Reeb vector field that comprise the positive ends of C_i. Clearly, $2 \mid J_+(C)$.

LEMMA 1. *Let Θ_+ and Θ_- be generators of $\widehat{ecc}_\circ(Y,\Gamma)$ and C be an embedded J-holomorphic curve C with positive ends at Θ_+ and negative ends at Θ_-, and with no closed components. Then $J_+(C) \geq 0$.*

PROOF. Note that no subcollection of periodic orbits from an admissible orbit set Θ can be homologically trivial unless Θ contains one or more of the homologically trivial γ_i^+, $\gamma_i^- \subset \mathcal{H}_i$. Therefore, if Θ_+ and Θ_- are generators of $\widehat{ecc}_\circ(Y,\Gamma)$, then no non-empty subcollection of periodic orbits from Θ_+ or Θ_- can be homologically trivial, and every connected component of an embedded J-holomorphic curve C with positive ends at Θ_+ and negative ends at Θ_- has non-empty positive and negative ends by what is said in [**KLT10b**, §3 and §4]. It then follows from (1) that $J_+(C) \geq 0$. □

Following [**LW11**, Appendix], we decompose the differential $\widehat{\partial}_{ech}$ as

$$\widehat{\partial}_{ech} = \partial_0 + \partial_1 + \cdots + \partial_k + \cdots,$$

where ∂_k counts admissible ECH index-1 J-holomorphic curves C with $J_+(C) = 2k$ and empty intersection with $\mathbb{R}\times\gamma_z$. Since J_+ is additive under gluing of J-holomorphic curves, we can define a filtered chain complex $\widehat{ecc}_\circ(Y,\Gamma)\otimes_{\mathbb{Z}}\mathbb{Z}[t,t^{-1}]$ with differential $\sum_{k=0}^{\infty}\partial_k\otimes t^{-k}$. The spectral sequence associated to this filtered chain complex has pages $E^k(S,\phi,\mathbf{a};J)$.

Note that there is a distinguished point x_i in $\alpha_i\cap\beta_i$ that lies in $S\times\{\frac{1}{2}\}$ for each $i = 1,\ldots,\mathrm{G}$ as depicted in Figure 1. The set of distinguished points $\mathbf{x}_\xi = \{x_1,\ldots,x_{\mathrm{G}}\}$ specifies a unique element of $\mathcal{Z}_{\mathrm{HF}}$. This in turn specifies a distinguished generator Θ_ξ of $\widehat{ecc}_\circ(Y,\Gamma_\xi)$ where Γ_ξ is the unique homology class satisfying $\langle PD(\Gamma),[S^2]\rangle = 0$ if $[S^2]$ is a generator of $H_2(\mathcal{H}_0;\mathbb{Z})$, $\langle PD(\Gamma),[S^2]\rangle = 1$ if $[S^2]$ is the positive generator of $H_2(\mathcal{H}_i;\mathbb{Z})$ for $i\in\{1,\ldots,\mathrm{G}\}$, and $\Gamma|_M = 0$. The generator Θ_ξ is a disjoint union of G embedded periodic orbits of the Reeb vector field. In fact, each of these orbits represents a positive generator of $H_1(S^1\times S^2;\mathbb{Z})$ for a distinguished copy of $S^1\times S^2$ in a connected sum decomposition of $Y \simeq M\#_{\mathrm{G}+1}S^1\times S^2$ determined by the unique collection of integral curves of the pseudo-gradient vector field v passing through the points $\{z,x_1,\ldots,x_{\mathrm{G}}\}$. To define the *ech* analog of algebraic torsion, we keep track of the generator Θ_ξ in the spectral sequence associated to this filtered chain complex $\widehat{ecc}_\circ(Y,\Gamma_\xi)\otimes_{\mathbb{Z}}\mathbb{Z}[t,t^{-1}]$.

Definition 2. Let $o(S,\phi,\mathbf{a};J)$ be the smallest non-negative integer k such that the orbit set Θ_ξ represents the trivial class in $E^{k+1}(S,\phi,\mathbf{a};J)$.

4. Porting Hutchings's filtration to Heegaard Floer homology

Next we translate the J_+ filtration in *ech* to Heegaard Floer homology. In order to do so, we make use of the construction of the isomorphism between *ech* and Heegaard Floer homology [**KLT10b**, **KLT10c**]:

$$\widehat{ech}(Y,\Gamma)\cong \widehat{HF}(-M,\mathfrak{s}_\xi+PD(\Gamma|_M))\otimes \mathbb{Z}^{2^{\mathrm{G}}}.$$

The construction of this isomorphism exploits the cylindrical reformulation of Heegaard Floer homology due to Lipshitz [**Lip06**]. In Lipshitz's reformulation, the Heegaard Floer chain complex $\widehat{CF}(-M,\mathfrak{s})$ is generated by G-tuples of Reeb chords of the form $\{x_1,\dots,x_{\mathrm{G}}\}\times[0,1]\subset\Sigma\times[0,1]$ where each x_i is a point in $\alpha_i\cap\beta_{\sigma(i)}$ for some $\sigma\in S_{\mathrm{G}}$. In comparison, $\{x_1,\dots,x_{\mathrm{G}}\}\in Sym^{\mathrm{G}}(\Sigma)$ is a generator of $\widehat{CF}(-M,\mathfrak{s})$ in the original formulation of Heegaard Floer homology by Ozsváth and Szabó [**OS04**]. Each such G-tuple of Reeb chords corresponds to a unique element of $\mathcal{Z}_{\mathrm{HF}}$. As such, the set of such G-tuples of Reeb chords is in 1–1 correspondence with the set $\mathcal{Z}_{\mathrm{HF}}$. Meanwhile, given an almost complex structure J_{HF} on $\Sigma\times[0,1]\times\mathbb{R}$ satisfying conditions (**J1**)–(**J5**) in [**Lip06**, §1], there exists an almost complex structure J on $\mathbb{R}\times Y$ satisfying the conditions in [**KLT10b**, §3a]. In the present context, where we restrict ourselves to the subcomplex $\widehat{ecc}_\circ(Y,\Gamma)$, there is a 1–1 correspondence between ECH index-1 J-holomorphic curves in $\mathbb{R}\times Y$ with ends at generators of this subcomplex and Fredholm index-1 J_{HF}-holomorphic curves in $\Sigma\times[0,1]\times\mathbb{R}$ that satisfy conditions (**M0**)–(**M6**) in [**Lip06**, §1]. Therefore, $(\widehat{ecc}_\circ(Y,\Gamma),\widehat{\partial}_{ech})$ and $(\widehat{CF}(-M,\mathfrak{s}),\widehat{\partial}_{\mathrm{HF}})$ are canonically isomorphic as chain complexes via the aforementioned 1–1 correspondence between generators (see [**KLT10c**] for details).

Now let C_L be a Fredholm index-1 J_{HF}-holomorphic curve in $\Sigma\times[0,1]\times\mathbb{R}$ satisfying conditions (**M0**)–(**M6**) in [**Lip06**, §1], and $C\subset\mathbb{R}\times Y$ be the corresponding ECH index-1 J-holomorphic curve with positive ends at an admissible orbit set Θ_+ and negative ends at an admissible orbit set Θ_-, both of which are generators of $\widehat{ecc}_\circ(Y,\Gamma)$. Due to the aforementioned 1–1 correspondence, we can write $J_+(C_L)$ for $J_+(C)$. Topologically, C is obtained from C_L by adding 2-dimensional 1-handles in such a way that for each $i=1,\dots,\mathrm{G}$ a 1-handle is attached along a point in α_i and a point in β_i (See Figure 2). Therefore, $-\chi(C)=-\chi(C_L)+\mathrm{G}$, and

$$J_+(C_L)=-\chi(C_L)+\mathrm{G}+|\Theta_+|-|\Theta_-|. \tag{2}$$

Lipshitz proves that $\chi(C_L)$ is determined by the relative homology class $[C_L]\in\widehat{\pi}_2(\mathbf{x}_+,\mathbf{x}_-)$ (see [**Lip06**, Proposition 4.2], cf. [**Lip14**, Proposition 4.2′]); more specifically,

$$\chi(C_L)=\mathrm{G}-n_{\mathbf{x}_+}(\mathcal{D}([C_L]))-n_{\mathbf{x}_-}(\mathcal{D}([C_L]))+e(\mathcal{D}([C_L])). \tag{3}$$

Here, $\mathcal{D}([C_L])$ denotes the domain in the Heegaard diagram associated to C_L, $\mathbf{x}_+$ denotes the Heegaard Floer generator that corresponds to Θ_+, $\mathbf{x}_-$ denotes the Heegaard Floer generator that corresponds to Θ_-, $n_p(\mathcal{D}([C_L]))$ denotes the *point measure*, namely, the average of the coefficients of $\mathcal{D}([C_L])$ for the four regions with corners at $p\in\alpha_i\cap\beta_j$, and $e(\mathcal{D}([C_L]))$ is the *Euler measure* of $\mathcal{D}([C_L])$. Meanwhile, keeping the labeling of the α and β curves in mind, each Heegaard Floer generator

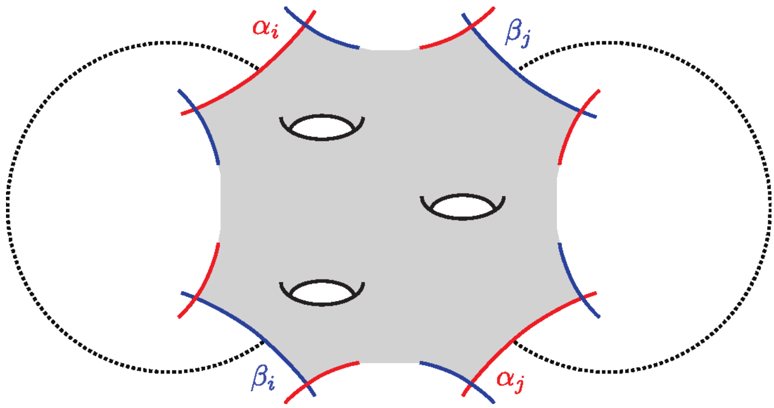

FIGURE 2. An embedded J_{HF}-holomorphic curve C_L in $\Sigma \times [0,1] \times \mathbb{R}$ satisfying conditions (**M0**)–(**M6**) in [**Lip06**, §1] shaded, and cores of some 2-dimensional 1-handles attached to C_L are indicated with dotted lines.

yields an element of the symmetric group, and $|\Theta_\pm|$ is equal to the number of cycles in the element of that symmetric group associated to $\mathbf{x}_\pm$, respectively. We will denote the latter quantity also by $|\cdot|$. With the preceding understood, combine (2) and (3) to obtain the following formula:

$$J_+([C_L]) = n_{\mathbf{x}_+}(\mathcal{D}([C_L])) + n_{\mathbf{x}_-}(\mathcal{D}([C_L])) - e(\mathcal{D}([C_L])) + |\mathbf{x}_+| - |\mathbf{x}_-|. \tag{4}$$

Note that the right hand side of (4) depends only on $\mathcal{D}([C_L])$, and in particular, since a class in $\widehat{\pi}_2(\mathbf{x}_+, \mathbf{x}_-)$ is specified by its domain, $J_+([C_L])$ depends only on the relative homology class $[C_L]$. In fact, we may define J_+ for any domain $\mathcal{D}$ in the Heegaard diagram by the right hand side of (4), which by [**Lip06**, Corollary 4.10] could also be written as

$$J_+(\mathcal{D}) := \mu(\mathcal{D}) - 2e(\mathcal{D}) + |\mathbf{x}_+| - |\mathbf{x}_-|, \tag{5}$$

where $\mu(\mathcal{D})$ is the *Maslov index* of $\mathcal{D}$. If there exists an embedded J_{HF}-holomorphic curve C_L representing $\mathcal{D}$, then the Maslov index of $\mathcal{D}$ agrees with the Fredholm index of C_L ([**Lip14**, Proposition 4.8′]) .

PROPOSITION 3. *The assignment $J_+(\mathcal{D})$ to domains in the Heegaard diagram is additive under concatenation of domains, and it is non-negative on Maslov index-1 domains represented by embedded pseudo-holomorphic curves.*

PROOF. The additivity claim follows from additivity of the Euler measure and the Maslov index under concatenation of domains. As for the non-negativity claim, this follows from Lemma 1 and by the correspondence between moduli spaces of pseudo-holomorphic curves to define the differentials for Heegaard Floer and *ech* chain complexes in [**KLT10c**]. □

For Maslov index-1 domains, it is easier to use the formula

$$J_+(\mathcal{D}) = 2[n_{\mathbf{x}_+}(\mathcal{D}) + n_{\mathbf{x}_-}(\mathcal{D})] - 1 + |\mathbf{x}_+| - |\mathbf{x}_-|. \tag{6}$$

With the preceding understood, we give an obstruction to overtwistedness:

THEOREM 4. *Let* ξ_{OT} *be an overtwisted contact structure on a closed* 3*-manifold* M*. Then there exist an open book decomposition* (S, ϕ) *supporting* ξ*, a basis of arcs* $\mathbf{a}$ *on* S*, and a generic split complex structure* J_{HF} *in the sense of* [**Lip06**] *such that* $o(S, \phi, \mathbf{a}; J_{HF}) = 0$.

PROOF. Note that an overtwisted contact structure is supported by an open book decomposition (S, ϕ) where the monodromy ϕ is not right-veering [**HKM07**, Theorem 1.1]. One can find a basis of arcs $\mathbf{a}$ on S so that in the corresponding Heegaard diagram $\widehat{\partial}_{\mathrm{HF}}\mathbf{y} = \mathbf{x}_{\xi_{\mathrm{OT}}}$ where $\mathbf{y} = \{y_1, x_2, \ldots, x_{\mathrm{G}}\}$ and there is exactly one Maslov index-1 holomorphic domain $\mathcal{D}$, a bigon, that contributes to the differential [**HKM09b**, Lemma 3.2] as defined by a split complex structure J_{HF} on $\Sigma \times [0, 1] \times \mathbb{R}$. Therefore, $n_{\mathbf{y}}(\mathcal{D}) = \frac{1}{4}$, $n_{\mathbf{x}_{\xi_{\mathrm{OT}}}}(\mathcal{D}) = \frac{1}{4}$, $|\mathbf{y}| = \mathrm{G}$, and $|\mathbf{x}_{\xi_{\mathrm{OT}}}| = \mathrm{G}$. Applying (6), we find $J_+(\mathcal{D}) = 0$. As a result, $o(S, \phi, \mathbf{a}; J_{HF}) = 0$. □

We see this as the beginning of investigations of a new family of computable invariants of contact structures and related objects via Heegaard Floer homology. In the sequel [**KMVWc**], we adopt Lipshitz's cylindrical reformulation of Heegaard Floer homology, and use the above definition and Proposition 3 along with equation (6) to define the spectral order of a closed contact 3-manifold, then investigate its many properties. Our main result there can be stated as follows:

THEOREM 5. *There exists an invariant of contact structures on closed oriented* 3*-manifolds, denoted* $\mathbf{o}$*, taking values in* $\mathbb{Z}_{\geq 0} \cup \{\infty\}$ *with the following properties:*

- $\mathbf{o}(M, \xi) = 0$ *if* (M, ξ) *is overtwisted.*
- $\mathbf{o}(M, \xi)$ *is non-decreasing under Legendrian surgery.*
- $\mathbf{o}(M, \xi) = \infty$ *if* (M, ξ) *is Stein fillable.*

In later sequels, we will investigate spectral order's ability to detect tightness [**KMVWa**], and introduce a version of spectral order for compact contact manifolds with convex boundary [**KMVWb**] applying Definition 2 to partial open book decompositions with protected boundary. There will be a number of interesting applications of this version of spectral order. For example, in [**JK18**], Juhász and Kang independently observe that the definition of spectral order can be generalized to partial open book decompositions and state that, just like the maps on sutured Floer homology defined in [**HKM09a**], the *sutured* version of $\mathbf{o}$ behaves well under inclusion of contact manifolds—$\mathbf{o}$ of a contact submanifold is an upper bound for $\mathbf{o}$ of the ambient contact manifold. Applying this to the case of a Giroux torsion domain, they give an upper bound of 2 for the spectral order of a contact 3-manifold with positive Giroux torsion.

REMARK. During the course of this project we learned that John Baldwin and David Shea Vela-Vick have independently been working on an invariant of contact 3-manifolds in the framework of Heegaard Floer homology similar in spirit to what is outlined in this article. The definition of their invariant can be found in [**BSVV18**, §1.3].

Acknowledgments

This project was initiated at the "*Interactions between contact symplectic topology and gauge theory in dimensions 3 and 4*" workshop at Banff International Research Station (BIRS) in 2011. We would like to thank BIRS for their hospitality and for creating a wonderful environment for collaboration. We also thank Michael Hutchings for helpful correspondence.

References

[Ben83] Daniel Bennequin, *Entrelacements et équations de Pfaff* (French), Third Schnepfenried geometry conference, Vol. 1 (Schnepfenried, 1982), Astérisque, vol. 107, Soc. Math. France, Paris, 1983, pp. 87–161. MR753131

[BSVV18] John Baldwin and David Vela-Vick, *A note on the knot Floer homology of fibered knots*, Algebr. Geom. Topol. **18** (2018), no. 6, 3669–3690, DOI 10.2140/agt.2018.18.3669. MR3868231

[CGH12a] Vincent Colin, Paolo Ghiggini, and Ko Honda, *The equivalence of Heegaard Floer homology and embedded contact homology III: from hat to plus*, preprint (2012), available at arXiv:1208.1526.

[CGH12b] ———, *The equivalence of Heegaard Floer homology and embedded contact homology via open book decompositions I*, preprint (2012), available at arXiv:1208.1074.

[CGH12c] ———, *The equivalence of Heegaard Floer homology and embedded contact homology via open book decompositions II*, preprint (2012), available at arXiv:1208.1077.

[EGH00] Y. Eliashberg, A. Givental, and H. Hofer, *Introduction to symplectic field theory*, Geom. Funct. Anal. **Special Volume** (2000), 560–673, DOI 10.1007/978-3-0346-0425-3_4. GAFA 2000 (Tel Aviv, 1999). MR1826267

[EH02] John B. Etnyre and Ko Honda, *Tight contact structures with no symplectic fillings*, Invent. Math. **148** (2002), no. 3, 609–626, DOI 10.1007/s002220100204. MR1908061

[Eli87] Ya. M. Eliashberg, *A theorem on the structure of wave fronts and its application in symplectic topology* (Russian), Funktsional. Anal. i Prilozhen. **21** (1987), no. 3, 65–72, 96. MR911776

[Eli88] Ya. M. Eliashberg, *The structure of* 1*-dimensional wave fronts, nonstandard Legendrian loops and Bennequin's theorem*, Topology and geometry—Rohlin Seminar, Lecture Notes in Math., vol. 1346, Springer, Berlin, 1988, pp. 7–12, DOI 10.1007/BFb0082768. MR970069

[Eli89] Y. Eliashberg, *Classification of overtwisted contact structures on* 3*-manifolds*, Invent. Math. **98** (1989), no. 3, 623–637, DOI 10.1007/BF01393840. MR1022310

[Eli90] Yakov Eliashberg, *Filling by holomorphic discs and its applications*, Geometry of low-dimensional manifolds, 2 (Durham, 1989), London Math. Soc. Lecture Note Ser., vol. 151, Cambridge Univ. Press, Cambridge, 1990, pp. 45–67. MR1171908

[Eli96] Yasha Eliashberg, *Unique holomorphically fillable contact structure on the* 3*-torus*, Internat. Math. Res. Notices **2** (1996), 77–82, DOI 10.1155/S1073792896000074. MR1383953

[Ghi06] Paolo Ghiggini, *Infinitely many universally tight contact manifolds with trivial Ozsváth-Szabó contact invariants*, Geom. Topol. **10** (2006), 335–357, DOI 10.2140/gt.2006.10.335. MR2224460

[Gro85] M. Gromov, *Pseudo holomorphic curves in symplectic manifolds*, Invent. Math. **82** (1985), no. 2, 307–347, DOI 10.1007/BF01388806. MR809718

[HKM07] Ko Honda, William H. Kazez, and Gordana Matić, *Right-veering diffeomorphisms of compact surfaces with boundary*, Invent. Math. **169** (2007), no. 2, 427–449, DOI 10.1007/s00222-007-0051-4. MR2318562

[HKM09a] Ko Honda, William H. Kazez, and Gordana Matić, *The contact invariant in sutured Floer homology*, Invent. Math. **176** (2009), no. 3, 637–676, DOI 10.1007/s00222-008-0173-3. MR2501299

[HKM09b] Ko Honda, William H. Kazez, and Gordana Matić, *On the contact class in Heegaard Floer homology*, J. Differential Geom. **83** (2009), no. 2, 289–311. MR2577470

[HT07] Michael Hutchings and Clifford Henry Taubes, *Gluing pseudoholomorphic curves along branched covered cylinders. I*, J. Symplectic Geom. **5** (2007), no. 1, 43–137. MR2371184

[HT09] Michael Hutchings and Clifford Henry Taubes, *Gluing pseudoholomorphic curves along branched covered cylinders. II*, J. Symplectic Geom. **7** (2009), no. 1, 29–133. MR2491716

[Hut] Michael Hutchings, *The embedded contact homology index revisited*, New perspectives and challenges in symplectic field theory, CRM Proc. Lecture Notes, vol. 49, Amer. Math. Soc., Providence, RI, 2009, pp. 263–297. MR2555941

[Hut02] Michael Hutchings, *An index inequality for embedded pseudoholomorphic curves in symplectizations*, J. Eur. Math. Soc. (JEMS) **4** (2002), no. 4, 313–361, DOI 10.1007/s100970100041. MR1941088

[Hut10] Michael Hutchings, *Embedded contact homology and its applications*, Proceedings of the International Congress of Mathematicians. Volume II, Hindustan Book Agency, New Delhi, 2010, pp. 1022–1041. MR2827830

[JK18] András Juhász and Sungkyung Kang, *Spectral order for contact manifolds with convex boundary*, Algebr. Geom. Topol. **18** (2018), no. 6, 3315–3338, DOI 10.2140/agt.2018.18.3315. MR3868222

[KLT10a] Cagatay Kutluhan, *Floer homology and symplectic forms on S1 x M3*, ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)–University of Michigan. MR2713879

[KLT10b] ———, *$HF = HM$ II : Reeb orbits and holomorphic curves for the ech/Heegaard Floer homology correspondence*, preprint (2010), available at arXiv:1008.1595.

[KLT10c] ———, *$HF = HM$ III : Holomorphic curves and the differential for the ech/Heegaard Floer homology correspondence*, preprint (2010), available at arXiv:1010.3456.

[KMVWa] Çağatay Kutluhan, Gordana Matić, Jeremy Van Horn-Morris, and Andy Wand, *Detecting tightness via spectral order*, in preparation.

[KMVWb] ———, *Spectral order for sutured contact manifolds*, in preparation.

[KMVWc] ———, *Filtering the Heegaard Floer contact invariant*, preprint (2016), available at arXiv:1603.02673.

[Kut13] Çağatay Kutluhan, *Lectures on the equivalence of Heegaard Floer and Seiberg-Witten Floer homologies*, Proceedings of the Gökova Geometry-Topology Conference 2012, Int. Press, Somerville, MA, 2013, pp. 1–42. MR3203356

[Lip06] Robert Lipshitz, *A cylindrical reformulation of Heegaard Floer homology*, Geom. Topol. **10** (2006), 955–1096, DOI 10.2140/gt.2006.10.955. Paging previously given as 955–1097. MR2240908

[Lip14] Robert Lipshitz, *Correction to the article: A cylindrical reformulation of Heegaard Floer homology [MR2240908]*, Geom. Topol. **18** (2014), no. 1, 17–30, DOI 10.2140/gt.2014.18.17. MR3158771

[LM96] Paolo Lisca and Gordana Matić, *On homotopic, non-isomorphic tight contact structures on* 3*-manifolds*, Turkish J. Math. **20** (1996), no. 1, 37–45. MR1392661

[LS03] Paolo Lisca and András I. Stipsicz, *An infinite family of tight, not semi-fillable contact three-manifolds*, Geom. Topol. **7** (2003), 1055–1073, DOI 10.2140/gt.2003.7.1055. MR2026538

[LW11] Janko Latschev and Chris Wendl, *Algebraic torsion in contact manifolds*, Geom. Funct. Anal. **21** (2011), no. 5, 1144–1195, DOI 10.1007/s00039-011-0138-3. With an appendix by Michael Hutchings. MR2846386

[Mar71] J. Martinet, *Formes de contact sur les variétés de dimension* 3 (French), Proceedings of Liverpool Singularities Symposium, II (1969/1970), Springer, Berlin, 1971, pp. 142–163. Lecture Notes in Math., Vol. 209. MR0350771

[OS04] Peter Ozsváth and Zoltán Szabó, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. (2) **159** (2004), no. 3, 1027–1158, DOI 10.4007/annals.2004.159.1027. MR2113019

Department of Mathematics, University at Buffalo, Buffalo, New York 14260
Email address: kutluhan@buffalo.edu

Department of Mathematics, University of Georgia, Athens, Georgia, 30602
Email address: gordana@math.uga.edu

Department of Mathematical Sciences, University of Arkansas, Fayetteville, Arkansas 72701
Email address: jvhm@uark.edu

School of Mathematics and Statistics, University of Glasgow, Glasgow G12 8QW, United Kingdom
Email address: andy.wand@glasgow.ac.uk