

Proceedings of Symposia in  
**PURE MATHEMATICS**

---

Volume 103.2

Integrability, Quantization,  
and Geometry

II. Quantum Theories and  
Algebraic Geometry

Dedicated to the Memory of Boris Dubrovin  
1950–2019

Sergey Novikov  
Igor Krichever  
Oleg Ogievetsky  
Senya Shlosman  
Editors



AMERICAN  
MATHEMATICAL  
SOCIETY

Integrability, Quantization,  
and Geometry

II. Quantum Theories and  
Algebraic Geometry



Proceedings of Symposia in  
**PURE MATHEMATICS**

---

Volume 103.2

Integrability, Quantization,  
and Geometry  
II. Quantum Theories and  
Algebraic Geometry

Dedicated to the Memory of Boris Dubrovin  
1950–2019

Sergey Novikov  
Igor Krichever  
Oleg Ogievetsky  
Senya Shlosman  
Editors

---



AMERICAN  
MATHEMATICAL  
SOCIETY

Providence, Rhode Island

2020 *Mathematics Subject Classification*. Primary 14J81, 16T20, 20C15, 20C35, 52C07, 53D45, 53D55, 81T13, 81T30, 81T40.

---

### Library of Congress Cataloging-in-Publication Data

Names: Novikov, S., editor.

Title: Integrability, quantization, and geometry / Sergey Novikov, Igor Krichever, Oleg Ogievetsky, Senya Shlosman, editors.

Description: Providence, Rhode Island : American Mathematical Society, [2021] | Series: Proceedings of symposia in pure mathematics, 0082-0717 ; 103.1, 103.2 | Includes bibliographical references.

Identifiers: LCCN 2020043148 | ISBN 9781470455927 (paperback) | ISBN 9781470464356 (ebook)

Subjects: LCSH: Dubrovin, B. A. (Boris Anatolievich)—Influence. | Geometry, Algebraic. | Topology. | Homology theory. | Quantum theory. | AMS: Algebraic geometry – (Co)homology theory – Sheaves, derived categories of sheaves and related constructions. | Algebraic geometry – (Co)homology theory – Differentials and other special sheaves; D-modules; Bernstein-Sato ideals and polynomials. | Algebraic geometry – Arithmetic problems. Diophantine geometry – Zeta-functions and related questions. | Algebraic geometry – Surfaces and higher-dimensional varieties – Families, moduli, classification: algebraic theory. | Algebraic geometry – Surfaces and higher-dimensional varieties – Mirror symmetry. | Algebraic geometry – Surfaces and higher-dimensional varieties – Vector bundles on surfaces and higher-dimensional varieties, and their moduli | Special functions | Differential geometry – Classical differential geometry – Higher-dimensional and -codimension | Differential geometry – Symplectic geometry, contact geometry | Differential geometry – Symplectic geometry, contact geometry | Algebraic geometry – Surfaces and higher-dimensional varieties – Relationships with physics. | Associative rings and algebras – Hopf algebras, quantum groups and related topics – Ring-theoretic aspects of quantum groups. | Group theory and generalizations – Representation theory of groups – Ordinary representations and characters. | Group theory and generalizations – Representation theory of groups – Applications of group representations to physics. | Convex and discrete geometry – Discrete geometry – Lattices and convex bodies in  $n$  dimensions. | Differential geometry – Symplectic geometry, contact geometry | Differential geometry – Symplectic geometry, contact geometry | Quantum theory – Quantum field theory; related classical field theories – Yang-Mills and other gauge theories. | Quantum theory – Quantum field theory; related classical field theories – String and superstring theories; other extended objects (e.g., branes). | Quantum theory – Quantum field theory; related classical field theories – Two-dimensional field theories, conformal field theories, etc..

Classification: LCC QA564 .I54 2021 | DDC 516.3/5–dc23

LC record available at <https://lccn.loc.gov/2020043148>

---

**Color graphic policy.** Any graphics created in color will be rendered in grayscale for the printed version unless color printing is authorized by the Publisher. In general, color graphics will appear in color in the online version.

**Copying and reprinting.** Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy select pages for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication is permitted only under license from the American Mathematical Society. Requests for permission to reuse portions of AMS publication content are handled by the Copyright Clearance Center. For more information, please visit [www.ams.org/publications/pubpermissions](http://www.ams.org/publications/pubpermissions).

Send requests for translation rights and licensed reprints to [reprint-permission@ams.org](mailto:reprint-permission@ams.org).

© 2021 by the American Mathematical Society. All rights reserved.

The American Mathematical Society retains all rights  
except those granted to the United States Government.  
Printed in the United States of America.

∞ The paper used in this book is acid-free and falls within the guidelines  
established to ensure permanence and durability.

Visit the AMS home page at <https://www.ams.org/>

10 9 8 7 6 5 4 3 2 1      26 25 24 23 22 21

IN MEMORY  
of  
BORIS DUBROVIN (April 6, 1950 – March 19, 2019)





## Contents

### VOLUME 2. Quantum Theories and Algebraic Geometry

Geometry from Donaldson-Thomas invariants T. BRIDGELAND	1
Fricke identities, Frobenius $k$ -characters and Markov equation V. M. BUCHSTABER and A. P. VESELOV	67
On Harish-Chandra theory of global nonsymmetric functions I. CHEREDNIK	79
Symmetries of genus zero modular operad N. C. COMBE and Y. I. MANIN	101
On Schur problem and Kostka numbers R. COQUEREAUX and J.-B. ZUBER	111
An analytic version of the Langlands correspondence for complex curves P. ETINGOF, E. FRENKEL, and D. KAZHDAN	137
A note on some minimally supersymmetric models in two dimensions D. GAIOTTO, T. JOHNSON-FREYD, and E. WITTEN	203
A moment map interpretation of the Ricci form, Kähler–Einstein structures, and Teichmüller spaces O. GARCÍA-PRADA and D. SALAMON	223
Global gauge conditions in the Batalin-Vilkovisky formalism E. GETZLER and S. W. POHORENCE	257
Interpolated Apéry numbers, quasiperiods of modular forms, and motivic gamma functions V. GOLYSHEV and D. ZAGIER	281
Quantum flips I: local model Y.-P. LEE, H.-W. LIN, and C.-L. WANG	303
Aspects of $p$ -adic geometry related to entanglement entropy M. MARCOLLI	353
Grothendieck–Teichmüller group, operads and graph complexes: a survey S. MERKULOV	383
Platonic compounds of cylinders O. OGIEVETSKY and S. SHLOSMAN	447





## Preface

The book that the reader is about to open is a collection of articles written in memory of Boris Dubrovin. By it, the authors express their admiration for his remarkable personality and for the contribution he made to mathematical physics. For many, he was a friend, a colleague, for some an inspiring mentor and teacher. For all who knew him, he was a man in joyful love for life, in all its manifestations.

Boris passed away on March 19, 2019, at the age of 68 (he was born on April 6, 1950), after a long and courageous battle against a rare and terrible disease (ALS).

Boris graduated from Moscow State University. He was a student of Sergey P. Novikov. In 1974 they created the “Periodic Inverse Scattering transform” for the Korteweg-de Vries (KdV) equation.

Boris’s first steps in mathematics came at a remarkable time of the birth of a new field of mathematical physics. After the discovery by Kruskal and Zabuski of the infinite number of integrals of the KdV equation the “Inverse Scattering Transform” was created in the famous work (1967) by Gardner, Green, Kruskal, Miura, clarified later by Lax. After works by Zakharov-Shabat and Ablowitz-Kaup-Newal-Segur it became clear that a wide class of fundamental nonlinear equations of mathematical physics are within the reach of analytic methods – and not only the computer simulations.

The mathematics needed was already in the ready-to-use state in the case of rapidly decreasing initial data, but far from being effective enough in the case of periodic problems. A breakthrough in periodic problems was done by Novikov (1974) who found the “finite-gap” solutions of the KdV equation and proved that stationary solutions to higher KdV equations are finite-gap Schrodinger potentials (whose degenerate limit are rapidly decreasing reflectionless potentials). Dubrovin’s first result was an “inverse theorem”: finite-gap potentials are stationary solutions of higher KdV equations. The proof was based on ideas and methods of classical algebraic geometry. Establishing new unexpected connections between seemingly unrelated areas of mathematics was a characteristic feature of Boris’s scientific style.

Periodic (2+1)-dimensional problems were initiated by Krichever (1976) and developed by Dubrovin, Krichever, Novikov for 2D Schrodinger operators (1976). A lot of famous mathematicians developed periodic soliton theory since 1975 – Lax, McKean, Marchenko, and their schools. Its and Matveev interacted with Novikov and Dubrovin since 1974 and made significant contributions.

Among Boris’s numerous works on the finite gap theory are the spectral theory of finite-gap operators with matrix coefficients and his work with Natanzon on the periodic solutions to the famous sine-Gordon equation. The latter problem turned out to be very difficult and its solution was completed by Novikov and Grinevich in 2003.

It is hard to overestimate the influence of his paper “Hydrodynamics of weakly deformed soliton lattices. Differential geometry and Hamiltonian theory”, written together with S. Novikov, where the foundations of the Hamiltonian theory of the Whitham perturbation theory of finite gap solutions were laid and connections with the theory of  $n$ -orthogonal curvilinear coordinate systems, or flat diagonal metrics were established. For more than a century since the famous work of Dupin and Binet, the problem of the construction of such systems was one of the most important problems of differential geometry. Treated as a classification problem, it was mainly solved by G. Darboux at the beginning of the 20th century.

These connections were central to the most famous concept introduced by Boris at the beginning of the 90s: the notion of the Frobenius manifolds which turned out to be a proper geometric language for the celebrated Witten-Dijkgraaf-Verlinde-Verlinde equations introduced in an attempt to classify topological quantum field models. From the geometrical point of view, an identification of a space of the deformation parameters of such theory with the ring of primary fields can be seen as a compatible multiplication on the tangent space that has the structure of a Frobenius algebra. Boris observed that the WDVV equations are equivalent to the flatness condition of a certain connection with a “spectral” parameter on the tangent space. The most important Boris’s addition to the WDVV equations was the requirement of quasi-homogeneity of their solutions, which became a part of the axiomatics of Frobenius manifolds. Under the assumption of quasi-homogeneity, the WDVV equations can be identified with the equations of isomonodromic deformations of the ordinary equation with rational coefficients. Thus, quasi-homogeneous solutions of the WDVV equations are uniquely determined by the monodromy data of the corresponding equation. In essence, this result is the answer to the problem of classifying topological quantum field theories.

Connections of the theory of Frobenius manifolds with the classical theory of isomonodromy deformations have proven to be beneficial for both sides. Motivated by the study of algebraic Frobenius manifolds, Boris Dubrovin and M. Mazzocco proposed a new approach to the problem of classification of algebraic solutions of the Schlesinger equations. In particular, they classified the algebraic solutions of a certain case of the Painleve-VI equation. Later, Dubrovin - Mazzocco’s approach was extended by O. Lysovij et al. to a complete classification of algebraic solutions of the general Painleve-VI equation.

Frobenius manifolds play a central role in the mathematical formulation of the mirror symmetry, which is one of the most influential ideas brought to mathematics by theoretical physics. At the end of 90s Boris proved that quantum cohomology of Fano varieties produce a Frobenius manifold and connect it with the Frobenius manifold produced by the Saito theory of hypersurface singularity.

The classification of semi-simple Frobenius manifolds which are in one-to-one correspondence dispersionless limits of integrable PDE was a starting point of Boris approach towards the classification problem of integrable evolution equations. In a series of works, Boris and Youjin Zhang were able to reconstruct, using Frobenius manifolds, the integrable dispersive PDEs from the integrable systems of hydrodynamic type. They have shown that a particular solution to those PDEs matches the generating function of the intersection numbers of given cohomological field theory (CohFTs). An important application of the Dubrovin-Zhang approach to CohFT is the completion of the Gromov–Witten invariants of the complex projective line

in terms of a new integrable hierarchy, the so-called extended Toda lattice (jointly with G. Carlet and Y. Zhang). This work has completed the analysis initiated earlier by T. Eguchi et al and by A. Okounkov and R. Pandharipande. Later, B. Dubrovin, in collaboration with D. Yang and M. Bertola, developed the concept of topological ODEs, in order to compute in a simple and very efficient way the correlators of cohomological field theories and random matrices.

Another important achievement of Boris Dubrovin is related to the universality of the critical behavior of solutions of Hamiltonian PDEs, developed together with C. Klein, A. Moro, and T. Grava. The study of properties of solutions of systems of nonlinear Hamiltonian PDEs with slowly varying initial conditions gave rise to the remarkable discovery of the phenomenon of the universality of the behavior of a generic solution at the point of phase transition from regular to the oscillatory regime.

Nowadays, about 25 years after the introduction of Frobenius manifolds by Dubrovin, one finds in Google Scholar many thousands of citations on this subject.

Boris became a full Professor of mathematics at Moscow State University in 1988. He was elected as Distinguished Professor of Mathematical Physics at Scuola Internazionale Superiore di Studi Avanzati (SISSA), Trieste, Italy, in 1993 and he served as director of the Mathematical Physics Group and the Mathematics area for several years. Since 2010 Boris has been the Director (and organizer) of the new Bogolyubov Laboratory Geometrical methods in Mathematical Physics of Moscow University. This Laboratory has become one of the most active mathematical centers in Moscow. He was a member of editorial boards of the Journal of High Energy Physics, Letters in Mathematical Physics, and Journal of Integrable Systems.

For his scientific achievements, Boris received a prize of Moscow Mathematical Society in 1976, together with A. Its and I. Krichever. He gave an invited talk at the International Congress of Mathematical Physicists at Swansea (1988), plenary talk at the European Congress of Mathematicians at Budapest (1996), invited talk at the International Congress of Mathematicians in Berlin (1998) and a plenary talk at the International Congress of Mathematical Physicists in Rio de Janeiro (2006).

The contributions to this collection of papers are split into two volumes. The volume titles – “Integrable Systems” and “Quantum Theories and Algebraic Geometry” – reflect the areas of main scientific interests of Boris. Chronologically, works of Boris may be divided into several parts: integrable systems, integrable systems of hydrodynamic type, WDVV equations (a.k.a Frobenius manifolds), isomonodromy equations (flat connections), quantum cohomology. The articles included in the first volume are those which are more or less directly devoted to these areas (primarily with the first three listed above). The second volume contains articles devoted mostly to quantum theories, algebraic geometry and is less directly connected with the early interests of Boris. Of course, this splitting is by no means a clear-cut;

several papers are written by former coauthors of Boris and contain new results in related areas.

All of the authors of these volumes was influenced by Boris in some way during his prolific scientific life. This influence will grow with time.

Sergey Novikov  
Igor Krichever  
Oleg Ogievetsky  
Senya Shlosman

## Selected Papers of Boris Dubrovin

### Articles

- [1] Dubrovin, B. A., ‘*Generalized Witt groups*’. Mathematical notes of the Academy of Sciences of the USSR, **13(3)** (1973), 253-257.
- [2] Dubrovin B. A., Novikov S. P., ‘*Periodic and conditionally periodic analogs of the many-soliton solutions of the Korteweg–de Vries equation*’. Zh. Eksper. Teoret. Fiz, **67(6)** (1974), 2131-2144.
- [3] Dubrovin B. A., Novikov S. P., ‘*Periodic Korteweg–de Vries and Sturm–Liouville problems. Their connection with algebraic geometry*’. In Sov. Math. Dokl, **Vol. 219** (1974), 3.
- [4] Dubrovin B. A., ‘*Inverse problem for periodic finite-zoned potentials in the theory of scattering*’. Funktsional’nyi Analiz i ego Prilozheniya, **9(1)**, (1975), 65-66.
- [5] Dubrovin B. A., ‘*Periodic problems for the Korteweg–de Vries equation in the class of finite band potentials*’. Functional analysis and its applications, **9(3)** (1975), 215-223.
- [6] Dubrovin B. A., Krichever I. M., Novikov S. P., ‘*The Schrödinger equation in a periodic field and Riemann surfaces*’. In Dokl. Akad. Nauk SSSR, **Vol. 229, No. 1** (1976), 15-18.
- [7] Dubrovin B. A., Matveev V. B., Novikov S. P., ‘*Non-linear equations of Korteweg–de Vries type, finite-zone linear operators, and Abelian varieties*’. Russian mathematical surveys, **31(1)** (1976), 59.
- [8] Dubrovin B. A., ‘*Finite band linear differential operators and Abelian varieties*’. Uspekhi Matematicheskikh Nauk, **31(4)** (1976), 259-260.
- [9] Dubrovin B. A., ‘*Completely integrable Hamiltonian systems associated with matrix operators and Abelian varieties*’. Functional Analysis and Its Applications, **11(4)** (1977), 265-277.
- [10] Dubrovin B. A., ‘*On SP Novikov’s conjecture in the theory of  $\theta$ -functions and nonlinear equations of Korteweg–de Vries and Kadomtsev–Petviashvili type*’. In Doklady Akademii Nauk, **Vol. 251, No. 3** (1980), 541-544. Russian Academy of Sciences.
- [11] Dubrovin B. A., Novikov S. P., ‘*Ground states of a two-dimensional electron in periodic magnetic field*’. (1980): 1006-1016.
- [12] Myasnikov V.P., Fedoryuk M.V., Boldrighini C., Dobrushin R.L., Sukhov Y.M., Schwarz A.S., Ziglin S.L., Dubrovin B.A., Novikov S.P., ‘*Sessions of the Petrovskii Seminar on differential equations and mathematical problems of physics*’. Uspekhi Matematicheskikh Nauk. **35(5)**(1980), 251-256.

- [13] Dubrovin B.A., Novikov S.P., ‘*Ground states in a periodic field. Magnetic Bloch functions and vector bundles*’. In Doklady Akademii Nauk **Vol. 253, No. 6**, (1980), 1293-1297. Russian Academy of Sciences.
- [14] Dubrovin B.A., ‘*The Kadomtsev–Petviashvili equation and the relations between the periods of holomorphic differentials on Riemann surfaces*’. Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya **45, no. 5** (1981), 1015-1028.
- [15] Dubrovin B.A., ‘*Theta functions and non-linear equations*’. Uspekhi Matematicheskikh Nauk **36, no. 2** (1981), 11-80.
- [16] Dubrovin B.A., Natanzon S.M., ‘*Real two-zone solutions of the sine-Gordon equation*’. Functional Analysis and Its Applications **16, no. 1** (1982), 21-33.
- [17] Dubrovin B.A., ‘*Analytical properties of spectral data for nonselfadjoint linear operators connected with real periodic solutions of the sine-Gordon equation*’. In Doklady Akademii Nauk **Vol. 265, No. 4** (1982), 789-793. Russian Academy of Sciences.
- [18] Dubrovin, B.A., Novikov S.P., ‘*Algebraic-geometric Poisson brackets for real finite-gap solutions of the Sine-Gordon equation and the non-linear Schrödinger equation*’. Soviet Scientific Reviews **267:6** (1982), 1295-1300.
- [19] Dubrovin B.A., Krichever I.M. and Novikov S.P., ‘*Topological and algebraic geometrical methods in modern mathematical physics*’. II. Soviet Scientific Reviews **Vol. C3** (1982), 1-150
- [20] Dubrovin B.A., Novikov S.P., ‘*The Hamiltonian formalism of one-dimensional systems of hydrodynamic type and the Bogolyubov–Whitham averaging method*. Soviet Math. Dokl. **270:4** (1983), 781-785.
- [21] Dubrovin B.A., ‘*Matrix finite-gap operators*’. Itogi Nauki i Tekhniki. Seriya “Sovremennye Problemy Matematiki. Noveishie Dostizheniya” **23** (1983), 33-78.
- [22] Dubrovin B.A. ‘*Matrix finite-zone operators*’. J Math Sci **28**, 20–50 (1985). <https://doi.org/10.1007/BF02104895>
- [23] Dubrovin B. A., ‘*Nonlinear equations connected with matrix finite-gap operators*’. Nonlinear and Turbulent Processes in Physics (1984), 1319.
- [24] Anosov D, Dubrovin B., Mal’tsev A., Oleinik O., Shiryaev A., Sinai Y. (1988). ‘*Sergei Petrovich Novikov (on his fiftieth birthday)*’. Russian Mathematical Surveys - RUSS MATH SURVEY-ENGL TR. 43
- [25] Dubrovin B. A., Novikov, S.P., ‘*On Poisson brackets of the hydrodynamic type*’. Soviet Math. Dokl. **279:2** (1984): 294-297.
- [26] Dubrovin B.A., Theory of operators and real algebraic geometry. In: Global Analysis—Studies and Applications III 1988 (pp. 42-59). Springer, Berlin, Heidelberg.
- [27] Dubrovin B.A., Natanzon S.M., ‘*Real theta-function solutions of the Kadomtsev–Petviashvili equation*’. Mathematics of the USSR-Izvestiya **32, no. 2** (1989), 269.
- [28] Dubrovin, B. A., Malanyuk T. G., Krichever I. M., Makhankov V. G., ‘*Exact solutions to a time-dependent Schrödinger equation with self-consistent potential*. Soviet J. Part. Nucl. 19:3 (1988) 579-621
- [29] Dubrovin B. A., ‘*Differential-geometric Poisson brackets on a lattice*’. Functional Analysis and Its Applications **23, no. 2** (1989), 131-133.

- [30] Dubrovin B. A., Novikov S. P., ‘*Hydrodynamics of weakly deformed soliton lattices. Differential geometry and Hamiltonian theory*’. Uspekhi Matematicheskikh Nauk **44(6)** (1989), 29-98.
- [31] Dubrovin B. A., Marmo G., Simoni A., ‘*Alternative Hamiltonian description for quantum systems*’. Modern Physics Letters **A 5, no. 15** (1990), 1229-1234.
- [32] Dubrovin B. A., ‘*Weakly perturbed soliton lattices*. Nucl. Phys. B (Proc. Suppl.) **18A** (1990), 23-44.
- [33] Novikov S., Anosov D., Dubrovin B., Mal'tsev A., Oleinik O., Shiryayev A., Sinai Y. (1990). ‘*Vladimir Petrovich Platonov (on his fiftieth birthday)*’. Russian Mathematical Surveys. **45**. 231.
- [34] Dubrovin B.A. ‘*On differential geometry of strongly integrable systems of hydrodynamic type*’. Funktsionaln. Analiz i ego Prilozheniya **24:4** (1990) 25-30. English translation (pdf) in: Funct. Anal. Appl. **24** (1990) 280-285.
- [35] Dubrovin B.A. ‘*Differential geometry of moduli spaces and its applications to soliton equations and to topological conformal field theory*’. In: Surveys in Differential Geometry , Vol. IV (1999), p. 213 - 238.
- [36] Dubrovin B. A., ‘*Hamiltonian formalism of Whitham-type hierarchies and topological Landau-Ginsburg models*’. Communications in mathematical physics **145, no. 1** (1992), 195-207.
- [37] Dubrovin B. A., ‘*Integrable systems in topological field theory*’. Nuclear physics **B 379, no. 3** (1992), 627-689.
- [38] Dubrovin B. A., ‘*Integrable systems and classification of 2-dimensional topological field theories*’. In Integrable systems, Birkhäuser, Boston, MA (1993), 313-359.
- [39] Dubrovin B. A., ‘*Topological conformal field theory from the point of view of integrable systems*’. In Integrable Quantum Field Theories (1993), 283-302. Springer, Boston, MA.
- [40] Dubrovin B.A., Giordano M., Marmo G., Simoni, A., ‘*Poisson brackets on presymplectic manifolds*’. International Journal of Modern Physics **A, 8(21)** (1993.), 3747-3771.
- [41] Dubrovin, B.. (2021). ‘*Quantum cohomology*’: lectures given at the C.I.M.E. Summer School held in Cetrato, Italy, June 30-July 8, 1997 / K. Behrend ... [et al.]. SERBIULA (sistema Librum 2.0).
- [42] Dubrovin B. ‘*Differential geometry of the space of orbits of a Coxeter group*. in: Surveys in Differential Geometry , Vol. IV (1999), p. 181 - 212.
- [43] Dubrovin B., Malanyuk T., Krichever I. and Makhankov V., (1988). ‘*Exact solutions of the time-dependent Schroedinger equation with self-consistent potential*’. Sov. J. Part. Nucl.**19:3** (1988) 579-621
- [44] Dubrovin B. A., ‘*Dispersion Relations for Nonlinear Waves and the Schotky Problem*’. In Important Developments in Soliton Theory 1993, 86-98. Springer, Berlin, Heidelberg.
- [45] Dubrovin B. A., ‘*Geometry and integrability of topological-antitopological fusion*’. Communications in mathematical physics **152, no. 3** (1993), 539-564.
- [46] Diener P., Dubrovin B. A., ‘*Algebraic-geometrical Darboux coordinates in R-matrix formalism*’. Preprint SISSA **88/94/FM** (1994).



- [47] Dubrovin B. A., Fokas, A. S., Santini, P. M., *Integrable functional equations and algebraic geometry*. Duke Math. J. 76 (1994) 645-668
- [48] Dubrovin B. A., ‘*Geometry of 2D topological field theories*’. In Integrable systems and quantum groups, Springer, Berlin, Heidelberg **1620** 1996, 120-348.
- [49] Dubrovin B. A., Flickinger R., Segur H., ‘*Three-Phase Solutions of the Kadomtsev–Petviashvili Equation*’. Studies in Applied Mathematics, **99(2)** (1997), 137-203.
- [50] Dubrovin B. A., ‘*Functionals of the Peierls–Fröhlich type and the variational principle for the Whitham equations*’. Translations of the American Mathematical Society - Series 2, **179** (1997), 35-44.
- [51] Dubrovin B. A., Zhang Y., ‘*Extended affine Weyl groups and Frobenius manifolds*’. Compositio Mathematica, **111(2)** (1998), 167-219.
- [52] Dubrovin B. A., Zhang Y., ‘*Bihamiltonian hierarchies in 2D topological field theory at one-loop approximation*’. Communications in mathematical physics **198(2)** (1998), 311-361.
- [53] Dubrovin B. A., *Flat pencils of metrics and Frobenius manifolds*. Proceedings of 1997 Taniguchi Symposium "Integrable Systems and Algebraic Geometry", Editors M.-H. Saito, Y.Shimizu and K.Ueno, 47-72. World Scientific, 1998.
- [54] Dubrovin B. A., *Geometry and analytic theory of Frobenius manifolds*. Proceedings of ICM98 2 (1998) 315-326.
- [55] Dubrovin B. A., Maltsev A.Y., ‘*Recurrent procedure for the determination of the Free Energy  $\epsilon^2$ -expansion in the Topological String Theory*’. arXiv preprint solv-int/9904004 (1999).
- [56] Dubrovin B. A., Zhang Y., ‘*Frobenius manifolds and Virasoro constraints*’. Selecta Mathematica **5(4)** (1999), 423-466.
- [57] Dubrovin B. A., ‘*Painlevé transcendents in Two-Dimensional Topological Field Theory*’. In: Conte R. (eds) The Painlevé Property. CRM Series in Mathematical Physics. Springer, New York, NY.  
[https://doi.org/10.1007/978-1-4612-1532-5\\_6](https://doi.org/10.1007/978-1-4612-1532-5_6)
- [58] Dubrovin B. A., Mazzocco M., ‘*Monodromy of certain Painlevé–VI transcendents and reflection groups*’. Inventiones mathematicae **141, no. 1** (2000), 55-147.
- [59] Dubrovin B.A., Zhang Y., ‘*Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov - Witten invariants*’.  
e-Print: math/0108160 [math.DG]
- [60] Carlet G., Dubrovin B.A., Zhang Y., ‘*The extended Toda hierarchy*’. Mosc. Math. J., 2004, Volume 4 Number 2. Pages 313–332
- [61] Dubrovin B. A., ‘*On almost duality for Frobenius manifolds*’. Translations of the American Mathematical Society-Series 2 **212** (2004), 75-132.
- [62] Dubrovin B. A., Zhang, Y., ‘*Virasoro symmetries of the extended Toda hierarchy*’. Communications in mathematical physics, **250(1)** (2004), 161-193.
- [63] Dubrovin B. A., ‘*On analytic families of invariant tori for PDEs*’. Astérisque **297** (2004), 35-66.

- [64] Dubrovin B. A., Zhang Y., Zuo D., ‘*Extended affine Weyl groups and Frobenius manifolds – II*’, math.DG/0502365
- [65] Dubrovin B. A., Liu S.Q., Zhang Y., ‘*On Hamiltonian perturbations of hyperbolic systems of conservation laws I: Quasi-Triviality of bi-Hamiltonian perturbations*’. Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences **59(4)** (2006), 559-615.
- [66] Dubrovin B. A., ‘*WDVV equations and Frobenius manifolds*’, Encyclopedia of Mathematical Physics, eds. J.-P. Francoise, G.L. Naber and Tsou S.T., Oxford: Elsevier **Vol. 1** (2006), 438-447.
- [67] Dubrovin B. A., ‘*On Hamiltonian perturbations of hyperbolic systems of conservation laws, II: universality of critical behaviour*’. Communications in mathematical physics **267, no. 1** (2006), 117-139.
- [68] Dubrovin B. A., Mazzocco M., ‘*Canonical structure and symmetries of the Schlesinger equations*’. Communications in mathematical physics **271, no. 2** (2007), 289-373.
- [69] Dubrovin B. A., Mazzocco M., ‘*On the reductions and classical solutions of the Schlesinger equations*’, Differential Equations and Quantum Groups. Andrey A. Bolibrukh Memorial Volume (ed. D. Bertrand et al.) - IRMA Lectures in Mathematics and Theoretical Physics. **9** (2007), 157-187.
- [70] Dubrovin B. A., ‘*On universality of critical behaviour in Hamiltonian PDEs*’, Amer. Math. Soc. Transl. 224 (2008) 59-109.
- [71] Dubrovin B. A., ‘*Hamiltonian PDEs and Frobenius manifolds*’. Russian Mathematical Surveys, **63(6)** (2008), 999.
- [72] Dubrovin B. A., Liu, S. Q., Zhang, Y., ‘*Frobenius manifolds and central invariants for the Drinfeld–Sokolov biHamiltonian structures*’. Advances in Mathematics, **219(3)** (2008), 780-837.
- [73] Dubrovin B. A., ‘*Hamiltonian perturbations of hyperbolic PDEs: from classification results to the properties of solutions*’. In New Trends in Mathematical Physics, Springer, Dordrecht (2009), 231-276).
- [74] Dubrovin B. A., Grava T., Klein C., ‘*On universality of critical behavior in the focusing nonlinear Schrödinger equation, elliptic umbilic catastrophe and the tritronquée solution to the Painlevé-I equation*’. Journal of nonlinear science, **19(1)** (2009), 57-94.
- [75] Dubrovin B. A., ‘*Hamiltonian PDEs: deformations, integrability, solutions*’. Journal of Physics A: Mathematical and Theoretical **43(43)** (2010), 434002.
- [76] Dubrovin B. A., Pavlov M. V., Zykov S. A., ‘*Linearly degenerate Hamiltonian PDEs and a new class of solutions to the WDVV associativity equations*’. Functional Analysis and Its Applications, **45(4)** (2011), 278-290.
- [77] Dubrovin B. A., Grava T., Klein C., ‘*Numerical Study of breakup in generalized Korteweg–de Vries and Kawahara equations*’. SIAM Journal on Applied Mathematics, **71(4)** (2011), 983-1008.
- [78] Carlet G., Dubrovin B. A., Mertens L. P., ‘*Infinite-dimensional Frobenius manifolds for  $2+1$  integrable systems*’. Mathematische Annalen, **349(1)** (2011), 75-115.

- [79] Dubrovin B. A., Skrypnyk T. V., ‘*Classical double, R-operators, and negative flows of integrable hierarchies*’. Theoretical and Mathematical Physics, **172(1)** (2012), 911-931.
- [80] Dubrovin B. A., Liu S.Q., Zhang Y., ‘*On the genus two free energies for semisimple Frobenius manifolds*’. Russian Journal of Mathematical Physics, **19(3)** (2012), 273-298.
- [81] Dubrovin B. A., Elaeva M., ‘*On the critical behavior in nonlinear evolutionary PDEs with small viscosity*’. Russian Journal of Mathematical Physics, **19(4)** (2012), 449-460.
- [82] Vershik A.M., Veselov A.P., Gaifullin A.A., Dubrovin B.A., Zhizhchenko A.B., Krichever I.M., Mal'tsev A.A.E., Millionshchikov D.V., Novikov S.P., Panov T.E., Sergeev A.G., ‘*Viktor Matveevich Buchstaber (on his 70th birthday)*’. Russian Mathematical Surveys, **68(3)** (2013), 581-590.
- [83] Tamizhmani K.M., Ilangovane R., Dubrovin, B. (2013). ‘*Symmetries and Casimir of an extended classical long wave system*’. Pramana. 80. 10.1007/s12043-012-0497-9.
- [84] Dubrovin B.A., Kapaev A., ‘*On an isomonodromy deformation equation without the Painlevé property*’. Russian Journal of Mathematical Physics, **21(1)** (2014), 9-35.
- [85] Belavin A., Dubrovin B. A., Mukhametzhanov B., ‘*Minimal Liouville Gravity correlation numbers from Douglas string equation*’. Journal of High Energy Physics, **2014(1)** (2014), 156.
- [86] Dubrovin B.A., ‘*Gromov–Witten invariants and integrable hierarchies of topological type*’. Topology, Geometry, Integrable Systems, and Mathematical Physics, Amer. Math. Soc. Transl. Ser, 2, (2014), 141-171.
- [87] Dubrovin B.A., Grava T., Klein C., Moro A., ‘*On critical behaviour in systems of Hamiltonian partial differential equations*’. Journal of nonlinear science, **25(3)** (2015), 631-707.
- [88] Dubrovin B.A., ‘*Symplectic field theory of a disk, quantum integrable systems, and Schur polynomials*’. In Annales Henri Poincaré **Vol. 17, No. 7** (2016), 1595-1613. Springer International Publishing.
- [89] Buryak A., Dubrovin B., Guéré J., Rossi P., 2018. ‘*Integrable systems of double ramification type*’. International Mathematics Research Notices, Volume 2020, Issue 24, December 2020, Pages 10381–10446, <https://doi.org/10.1093/imrn/rnz029>
- [90] Dubrovin B.A., Liu S. Q., Yang D., Zhang Y., ‘*Hodge integrals and tau-symmetric integrable hierarchies of Hamiltonian evolutionary PDEs*’. Advances in Mathematics, **293** (2016), 382-435.
- [91] Bertola M., Dubrovin B.A., Yang D., ‘*Correlation functions of the KdV hierarchy and applications to intersection numbers over  $\overline{\mathcal{M}}_{g,n}$* ’, Physica D: Nonlinear Phenomena, **327** (2016), 30-57.
- [92] Dubrovin B.A., Grava T., Klein C., ‘*On critical behaviour in generalized Kadomtsev–Petviashvili equations*’. Physica D: Nonlinear Phenomena, **333** (2016), 57-170.
- [93] Adler V.E., Berest Y.Y., Buchstaber V.M., Grinevich P.G., Dubrovin B.A., Krichever I.M., Novikov S.P., Sergeev A.N., Feigin M.V., Felder J., Ferapontov E.V., ‘*Alexander Petrovich Veselov*’. Russian Mathematical Surveys, **71(6)** (2016), 1159.

- [94] Buryak A., Dubrovin B., Guéré J., Rossi P., 2018. ‘*Tau-structure for the double ramification hierarchies*’. Communications in Mathematical Physics, 363(1), pp.191-260.
- [95] Dubrovin B., Yang D., 2017. *Generating series for GUE correlators*. Letters in Mathematical Physics, 107(11), pp.1971-2012.
- [96] Dubrovin, B., Yang, D., *On cubic Hodge integrals and random matrices*, Communications in Number Theory and Physics 11 (2017) 311-336,
- [97] Buryak A., Dubrovin B., Guéré J., Rossi P., *Integrable systems of double ramification type*. International Mathematics Research Notices, 2020(24), pp.10381-10446.
- [98] Bertola M., Dubrovin B.A., Yang D., ‘*Simple Lie algebras, Drinfeld–Sokolov hierarchies, and multi-point correlation functions*’. arXiv preprint arXiv:1610.07534, (2016).
- [99] Dubrovin B., Liu S.Q., Yang D., Zhang Y., 2020. *Hodge–GUE correspondence and the discrete KdV equation*. Communications in Mathematical Physics, 379(2), pp.461-490.
- [100] Dubrovin B., Liu S.Q., Zhang Y., 2018. *Bihamiltonian cohomologies and integrable hierarchies II: the tau structures*. Communications in Mathematical Physics, 361(2), pp.467-524.
- [101] Dubrovin B., Yang D., *On cubic Hodge integrals and random matrices*. Commun. Number Theory Phys. 11 (2017), no. 2, 311-336.
- [102] Dubrovin B.A., Yang D., ‘*On Gromov–Witten invariants of  $\mathbb{P}^1$* ’. arXiv preprint arXiv:1702.01669, (2017).
- [103] Dubrovin B.A., Yang D., Zagier D., ‘*Classical Hurwitz numbers and related combinatorics*’. Moscow Mathematical Journal, **17(4)** (2017), 601-633.
- [104] Cotti G., Dubrovin B., Guzzetti D., 2019. *Isomonodromy deformations at an irregular singularity with coalescing eigenvalues*. Duke Mathematical Journal, 168(6), pp.967-1108.
- [105] Cotti G., Dubrovin B., Guzzetti D., 2020. *Local moduli of semisimple Frobenius coalescent structures*. SIGMA. Symmetry, Integrability and Geometry: Methods and Applications, 16, p.040.
- [106] Dubrovin B.A., Skrypnyk T., ‘*Separation of variables for linear Lax algebras and classical r-matrices*’. Journal of Mathematical Physics, **159(9)** (2018), 091405.
- [107] Dubrovin B.A., Liu S.Q., Zhang Y., ‘*Bihamiltonian cohomologies and integrable hierarchies II: the tau structures*’. Communications in Mathematical Physics, **361(2)** (2018), 467-524.
- [108] Bertola M., Dubrovin B.A., Yang D., ‘*Simple Lie algebras and topological ODEs*’. International Mathematics Research Notices, **2018(5)** (2018),1368-1410.
- [109] Dubrovin B.A., ‘*Algebraic spectral curves over  $\mathbb{Q}$  and their tau-functions*’. arXiv:1807.11258 math.AG
- [110] Dubrovin B.A., Kapaev A., ‘*A Riemann–Hilbert approach to the Heun equation*’. SIGMA. Symmetry, Integrability and Geometry: Methods and Applications, **14** (2018), 093.

- [111] Buryak A., Dubrovin B.A., Guéré J., Rossi P., ‘*Tau-structure for the double ramification hierarchies*’. Communications in Mathematical Physics, **363(1)** (2018), 191-260.
- [112] Dubrovin B., Yang D., ‘*On Gromov-Witten invariants of  $P^1$* ’. Math. Res. Lett. 26 (2019), no. 3, 729-748.
- [113] Dubrovin B.A., Skrypnyk T., ‘*Separation of variables for quadratic algebras and skew-symmetric classical  $r$ -matrices*’. Journal of Mathematical Physics, **60(9)** (2019), 093506.
- [114] Dubrovin B.A., Strachan I.A., Zhang Y., Zuo D., ‘*Extended affine Weyl groups of BCD-type: their Frobenius manifolds and Landau–Ginzburg superpotentials*’. Advances in Mathematics, **351** (2019), 897-946.
- [115] Dubrovin B., ‘*Approximating tau-functions by theta-functions*’. Commun. Number Theory Phys. 13 (2019), no. 1, 203-223.
- [116] Dubrovin B.A., Yang D., Zagier D., ‘*Gromov–Witten invariants of the Riemann sphere*’. Pure Appl.Math.Quart. **16** (2020) 1, 153-190.
- [117] Cotti G., Dubrovin B.A., Guzzetti D., ‘*Isomonodromy deformations at an irregular singularity with coalescing eigenvalues*’. Duke Mathematical Journal, **168(6)** (2019), 967-1108.
- [118] Dubrovin B.A., Strachan I. A. B., Zhang Y., Zuo D., ‘*Extended affine Weyl groups of BCD type, Frobenius manifolds and their Landau - Ginzburg superpotentials*’ arXiv:1510.08690
- [119] Bertola M., Dubrovin B., Yang, D., ‘*Simple Lie algebras and topological ODEs*’. International Mathematics Research Notices, 2018(5), pp.1368-1410.
- [120] Dubrovin B., Minakov A., ‘*On a class of compact perturbations of the special pole-free joint solution of KdV and  $P_I^2$* ’. arXiv:1901.07470 math-ph
- [121] Cotti G., Dubrovin B., Guzzetti D., ‘*Helix Structures in Quantum Cohomology of Fano Varieties*’. e-Print: 1811.09235 [math.AG]
- [122] Dubrovin B., Yang D., Zagier D., ‘*On tau-functions for the KdV hierarchy*’. e-Print: 1812.08488 [math-ph]
- [123] Dubrovin B., Yang D., ‘*Matrix resolvent and the discrete KdV hierarchy*’. Commun.Math.Phys. **377** (2020) 3, 1823-1852.
- [124] Dubrovin B., Yang D., ‘*Remarks on intersection numbers and integrable hierarchies. I. Quasi-triviality*’. e-Print: 1905.08106 [math-ph]
- [125] Dubrovin B., Yang D., Zagier D., ‘*Geometry and arithmetic of integrable hierarchies of KdV type. I. Integrality*’. e-Print: 2101.10924 [math.AG]

#### Books

- Dubrovin, B.A., Fomenko, A.T. and Novikov, S.P., 2012. ‘*Modern geometry – methods and applications*’. (Vol. 104). Springer Science & Business Media.
- Dubrovin, B.A., Fomenko, A.T. and Novikov, S.P., ‘*Modern geometry – methods and applications. Introduction to homology theory*’. Graduate Texts in Mathematics, 124.
- Dubrovin B.A., ‘*Riemann surfaces and nonlinear equations*’. Izhevsk НИИ ПХД 2001, ISBN 5-93972-079-X

- [•] Dubrovin B.A., Soloviev Yu ‘*Topology*’. Moscow State University Publishing House, 1989
- [•] Dubrovin B.A., Krichever I.M., Novikov S.P., 1990. ‘*Integrable systems. I*’. In Dynamical systems IV (pp. 173-280). Springer, Berlin, Heidelberg.
- [•] Dubrovin B.A., Krichever I.M., Novikov S.P., 2004. ‘*Topological and algebraic geometry methods in contemporary mathematical physics*’. (Vol. 2). Cambridge Scientific Publishers.
- [•] Dubrovin B., ‘*Geometry of Hamiltonian evolutionary systems*’. Bibliopolis; 1991.
- [•] Dubrovin B., Zhang Y., ‘*Normal forms of integrable PDEs, Frobenius manifolds and Gromov–Witten invariants*’. In preparation, first draft in: math/0108160



## Selected Published Titles in This Series

- 103 **Sergey Novikov, Igor Krichever, Oleg Ogievetsky, and Senya Shlosman, Editors**, Integrability, Quantization, and Geometry, 2021
- 102 **David T. Gay and Weiwei Wu, Editors**, Breadth in Contemporary Topology, 2019
- 101 **Avraham Aizenbud, Dmitry Gourevitch, David Kazhdan, and Erez M. Lapid, Editors**, Representations of Reductive Groups, 2019
- 100 **Chiu-Chu Melissa Liu and Motohico Mulase, Editors**, Topological Recursion and its Influence in Analysis, Geometry, and Topology, 2018
- 99 **Vicente Muñoz, Ivan Smith, and Richard P. Thomas, Editors**, Modern Geometry, 2018
- 98 **Amir-Kian Kashani-Poor, Ruben Minasian, Nikita Nekrasov, and Boris Pioline, Editors**, String-Math 2016, 2018
- 97 **Tommaso de Fernex, Brendan Hassett, Mircea Mustață, Martin Olsson, Mihnea Popa, and Richard Thomas, Editors**, Algebraic Geometry: Salt Lake City 2015 (Parts 1 and 2), 2018
- 96 **Si Li, Bong H. Lian, Wei Song, and Shing-Tung Yau, Editors**, String-Math 2015, 2017
- 95 **Izzet Coskun, Tommaso de Fernex, and Angela Gibney, Editors**, Surveys on Recent Developments in Algebraic Geometry, 2017
- 94 **Mahir Bilen Can, Editor**, Algebraic Groups: Structure and Actions, 2017
- 93 **Vincent Bouchard, Charles Doran, Stefan Méndez-Diez, and Callum Quigley, Editors**, String-Math 2014, 2016
- 92 **Kailash C. Misra, Daniel K. Nakano, and Brian J. Parshall, Editors**, Lie Algebras, Lie Superalgebras, Vertex Algebras and Related Topics, 2016
- 91 **V. Sidoravicius and S. Smirnov, Editors**, Probability and Statistical Physics in St. Petersburg, 2016
- 90 **Ron Donagi, Sheldon Katz, Albrecht Klemm, and David R. Morrison, Editors**, String-Math 2012, 2015
- 89 **D. Dolgopyat, Y. Pesin, M. Pollicott, and L. Stoyanov, Editors**, Hyperbolic Dynamics, Fluctuations and Large Deviations, 2015
- 88 **Ron Donagi, Michael R. Douglas, Ljudmila Kamenova, and Martin Roček, Editors**, String-Math 2013, 2014
- 87 **Helge Holden, Barry Simon, and Gerald Teschl, Editors**, Spectral Analysis, Differential Equations and Mathematical Physics: A Festschrift in Honor of Fritz Gesztesy's 60th Birthday, 2013
- 86 **Kailash C. Misra, Daniel K. Nakano, and Brian J. Parshall, Editors**, Recent Developments in Lie Algebras, Groups and Representation Theory, 2012
- 85 **Jonathan Block, Jacques Distler, Ron Donagi, and Eric Sharpe, Editors**, String-Math 2011, 2012
- 84 **Alex H. Barnett, Carolyn S. Gordon, Peter A. Perry, and Alejandro Uribe, Editors**, Spectral Geometry, 2012
- 83 **Hisham Sati and Urs Schreiber, Editors**, Mathematical Foundations of Quantum Field Theory and Perturbative String Theory, 2011
- 82 **Michael Usher, Editor**, Low-dimensional and Symplectic Topology, 2011
- 81 **Robert S. Doran, Greg Friedman, and Jonathan Rosenberg, Editors**, Superstrings, Geometry, Topology, and  $C^*$ -algebras, 2010
- 80 **D. Abramovich, A. Bertram, L. Katzarkov, R. Pandharipande, and M. Thaddeus, Editors**, Algebraic Geometry, 2009

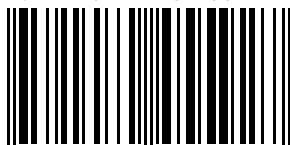
For a complete list of titles in this series, visit the  
AMS Bookstore at [www.ams.org/bookstore/pspumseries/](http://www.ams.org/bookstore/pspumseries/).



This book is a collection of articles written in memory of Boris Dubrovin (1950–2019). The authors express their admiration for his remarkable personality and for the contributions he made to mathematical physics. For many of the authors, Dubrovin was a friend, colleague, inspiring mentor, and teacher.

The contributions to this collection of papers are split into two parts: “Integrable Systems” and “Quantum Theories and Algebraic Geometry”, reflecting the areas of main scientific interests of Dubrovin. Chronologically, these interests may be divided into several parts: integrable systems, integrable systems of hydrodynamic type, WDVV equations (Frobenius manifolds), isomonodromy equations (flat connections), and quantum cohomology. The articles included in the first part are more or less directly devoted to these areas (primarily with the first three listed above). The second part contains articles on quantum theories and algebraic geometry and is less directly connected with Dubrovin’s early interests.

ISBN 978-1-4704-5592-7



9 781470 455927

PSPUM/103.2

