

# Proceedings of Symposia in PURE MATHEMATICS

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Volume 104

## Nine Mathematical Challenges An Elucidation

Linde Hall Inaugural Math Symposium  
February 22–24, 2019  
California Institute of Technology,  
Pasadena, California

A. Kechris  
N. Makarov  
D. Ramakrishnan  
X. Zhu  
Editors



AMERICAN  
MATHEMATICAL  
SOCIETY

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Challenges  
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## Preface

Problems are the lifeblood of mathematics. These are the immortal words of Hilbert, uttered during the 1900 International Congress of Mathematicians, which are well known and oft repeated. This volume is a similar venture, albeit more modest, but still with the same purpose. What we have here are nine beautiful articles by distinguished mathematicians, each discussing a specific open problem, or a set of such, in nine different fields within mathematics, each very important in its own right.

The idea for this volume began with the plans for a special event at Caltech celebrating the opening of the newly remodeled Ron and Maxine Linde Hall of Mathematics and Physics, which mainly houses the math department. There was an Inaugural Math Symposium, which was held at Caltech during February 22–24, 2019. Nine distinguished mathematicians representing different areas of the subject spoke, each giving a one-hour lecture during the symposium. There were two lectures on the afternoon of Friday, February 22, five lectures during Saturday, with three in the morning and two in the afternoon, and two lectures on the morning of Sunday, February 24. During lunch on Saturday there was a poster session for students and postdocs. The speakers were, in alphabetical order, M. Aschbacher, H. Esnault, B. Gross, U. Hamenstädt, P. Michel, A. Poltoratski, B. Simon, T. Tao, and W.H. Woodin. The program of the Symposium can be found right after this preface.

The Lectures were such a resounding success, giving rare and exciting glimpses into a slew of mathematical challenges, that it was decided by the organizing committee—composed of A. Kechris, N. Makarov, D. Ramakrishnan (chair), and X. Zhu—to try to bring out a special volume presenting the contents of the Lectures with suitable expansion. It was enthusiastically supported by Elena Mantovan, then the Executive Officer, a.k.a. chair, of the math department. We asked the lecturers if they would contribute expanded write-ups of their lectures, but unfortunately B. Gross and B. Simon were unable to provide articles for the volume. It was decided that the other speakers could write about anything they wanted, preferably related to the subject of their talks.

We approached the American Mathematical Society about the possibility of bringing out the volume in the Proceedings of the Symposia in Pure Mathematics series. After due evaluation, they agreed and we began putting together the volume with invitations to the authors. Hélène Esnault decided to write about something completely different and submitted a joint article with V. Srinivas on ramification in covers and curves. Meanwhile, the organizing committee decided to ask two distinguished mathematicians to supply articles in the general areas of B. Gross and B. Simon. Fortunately for us, Chris Skinner agreed to write about the same



subject matter as Gross's talk (on the Birch and Swinnerton-Dyer Conjecture) in collaboration with A. Burungale and Ye Tian. And R. Frank agreed to write a survey on a different topic from the lecture of B. Simon, but still in the general area of Mathematical Physics.

Now we will briefly touch upon the contents. In the first article, M. Aschbacher explains the problem of classification of finite groups and explains where it stands. In the second, A. Burungale, C. Skinner, and Ye Tian explain the recent progress on the Birch and Swinnerton-Dyer Conjecture for rational elliptic curves  $E$ , which relates the order of vanishing of its  $L$ -function at  $s = 1$  to the rank  $r$  of  $E(\mathbb{Q})$  as an abelian group, focusing on  $r = 0, 1$ . In the third article, H. Esnault and V. Srinivas analyze  $\ell$ -adic local systems of bounded rank and ramification on a smooth variety in characteristic  $p$  (different from  $\ell$ ), then introduce and study ramification outside codimension 2 by a finite separable extension of bounded degree. In the fourth, R. Frank gives a survey of problems and results concerning the Lieb-Thirring inequalities. In the fifth article, U. Hamenstädt studies topological properties of surface bundles over surfaces which result from the Milnor-Wood inequality for the Euler class of flat  $S^1$ -bundles over surfaces. In the sixth contribution, P. Michel explains substantial recent extensions of the fundamental equidistribution theorem of W. Duke. The seventh article, by A. Poltoratski, describes the Gap and Type problems in Fourier analysis with new results. In the following (eighth) article, T. Tao discusses quantitative bounds for higher regularity norms of the classical solutions to three-dimensional Navier-Stokes equations. In the final (ninth) article, Hugh Woodin gives a survey of Cantor's Continuum Hypothesis starting from the work of Gödel and Cohen, and leading to the current developments and approach of the author on this problem.

The editors would like to thank Michelle Vine and Stephanie Cha-Ramos for their help with the Linde Inaugural Math Symposium at Caltech, and also acknowledge invaluable help from Christine Thivierge of the AMS in the putting together of this volume.

## The Linde Hall Inaugural Math Symposium at Caltech

The symposium will be held on February 22–24, 2019, to celebrate the opening of the newly remodeled Ron and Maxine Linde Hall of Mathematics and Physics. Nine distinguished speakers representing different areas of Mathematics will each give a one-hour lecture during the symposium. There will be two lectures on the afternoon of Friday, February 22, five lectures on Saturday, with three in the morning and two in the afternoon, and two lectures on the morning of Sunday, February 24. During lunch on Saturday there will be a poster session for students and postdocs.

### Lectures

(with authors in alphabetical order)

- (1) Michael Aschbacher, Caltech  
*The finite simple groups and their classification*
- (2) Hélène Esnault, Free University of Berlin  
*Vanishing Theorems for étale sheaves*
- (3) Benedict Gross, UCSD  
*On the conjecture of Birch and Swinnerton-Dyer*
- (4) Ursula Hamenstädt, University of Bonn  
*Amenable actions and rigidity*
- (5) Philippe Michel, École Polytechnique Fédérale de Lausanne  
*L-functions, moments and subconvexity*
- (6) Alexei Poltoratski, Texas A & M  
*Gap and Type problems in Fourier analysis*
- (7) Barry Simon, Caltech  
*Fifty Years of the Spectral Theory of Schrödinger Operators*
- (8) Terence Tao, UCLA  
*The global regularity problem for Navier-Stokes*
- (9) W. Hugh Woodin, Harvard  
*Cantor's Continuum Hypothesis*

Organizing Committee:

Dinakar Ramakrishnan (chair), Alexander Kechris, Nikolai Makarov, and Xinwen Zhu



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This volume stems from the Linde Hall Inaugural Math Symposium, held from February 22–24, 2019, at California Institute of Technology, Pasadena, California.

The content isolates and discusses nine mathematical problems, or sets of problems, in a deep way, but starting from scratch. Included among them are the well-known problems of the classification of finite groups, the Navier-Stokes equations, the Birch and Swinnerton-Dyer conjecture, and the continuum hypothesis. The other five problems, also of substantial importance, concern the Lieb–Thirring inequalities, the equidistribution problems in number theory, surface bundles, ramification in covers and curves, and the gap and type problems in Fourier analysis. The problems are explained succinctly, with a discussion of what is known and an elucidation of the outstanding issues. An attempt is made to appeal to a wide audience, both in terms of the field of expertise and the level of the reader.

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