Harmonic Analysis
A Comprehensive Course in Analysis, Part 3
Barry Simon

\[ |f - f_Q|_Q = \frac{1}{|Q|} \int_Q |f(x) - f_Q| d^\nu x \]

\[ |\{ x \mid M_{HL} f(x) > \alpha \}| \leq \frac{3^\nu}{\alpha} \| f \|_{L^1(\mathbb{R}^\nu, d^\nu x)} \]
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Barry Simon
To the memory of Cherie Galvez

extraordinary secretary, talented helper, caring person

and to the memory of my mentors,

Ed Nelson (1932-2014) and Arthur Wightman (1922-2013)

who not only taught me Mathematics
but taught me how to be a mathematician
Contents

Preface to the Series xi
Preface to Part 3 xvii

Chapter 1. Preliminaries 1
§1.1. Notation and Terminology 1
§1.2. Some Results for Real Analysis 3
§1.3. Some Results from Complex Analysis 12
§1.4. Green’s Theorem 16

Chapter 2. Pointwise Convergence Almost Everywhere 19
§2.1. The Magic of Maximal Functions 22
§2.2. Distribution Functions, Weak-L^1, and Interpolation 26
§2.3. The Hardy–Littlewood Maximal Inequality 41
§2.4. Differentiation and Convolution 52
§2.5. Comparison of Measures 60
§2.6. The Maximal and Birkhoff Ergodic Theorems 65
§2.7. Applications of the Ergodic Theorems 92
§2.8. Bonus Section: More Applications of the Ergodic Theorems 102
§2.9. Bonus Section: Subadditive Ergodic Theorem and Lyapunov Behavior 133
§2.10. Martingale Inequalities and Convergence 147
§2.11. The Christ–Kiselev Maximal Inequality and Pointwise Convergence of Fourier Transforms 168
Chapter 3. Harmonic and Subharmonic Functions

§3.1. Harmonic Functions 177
§3.2. Subharmonic Functions 202
§3.3. Bonus Section: The Eremenko–Sodin Proof of Picard’s Theorem 213
§3.4. Perron’s Method, Barriers, and Solution of the Dirichlet Problem 220
§3.5. Spherical Harmonics 232
§3.6. Potential Theory 252
§3.7. Bonus Section: Polynomials and Potential Theory 278
§3.8. Harmonic Function Theory of Riemann Surfaces 298

Chapter 4. Bonus Chapter: Phase Space Analysis 319

§4.1. The Uncertainty Principle 320
§4.2. The Wavefront Sets and Products of Distributions 345
§4.3. Microlocal Analysis: A First Glimpse 352
§4.4. Coherent States 373
§4.5. Gabor Lattices 390
§4.6. Wavelets 407

Chapter 5. $H^p$ Spaces and Boundary Values of Analytic Functions on the Unit Disk 437

§5.1. Basic Properties of $H^p$ 439
§5.2. $H^2$ 444
§5.3. First Factorization (Riesz) and $H^p$ 450
§5.4. Carathéodory Functions, $h^1$, and the Herglotz Representation 459
§5.5. Boundary Value Measures 464
§5.6. Second Factorization (Inner and Outer Functions) 468
§5.7. Conjugate Functions and M. Riesz’s Theorem 472
§5.8. Homogeneous Spaces and Convergence of Fourier Series 493
§5.9. Boundary Values of Analytic Functions in the Upper Half-Plane 498
§5.10. Beurling’s Theorem 515
§5.11. $H^p$-Duality and BMO 517
§5.12. Cotlar’s Theorem on Ergodic Hilbert Transforms 539
Chapter 6. Bonus Chapter: More Inequalities

§6.1. Lorentz Spaces and Real Interpolation

§6.2. Hardy-Littlewood–Sobolev and Stein–Weiss Inequalities

§6.3. Sobolev Spaces; Sobolev and Rellich–Kondrachov Embedding Theorems

§6.4. The Calderón–Zygmund Method

§6.5. Pseudodifferential Operators on Sobolev Spaces and the Calderón–Vaillancourt Theorem

§6.6. Hypercontractivity and Logarithmic Sobolev Inequalities


§6.8. Restriction to Submanifolds

§6.9. Tauberian Theorems

Bibliography

Symbol Index

Subject Index

Author Index

Index of Capsule Biographies
Preface to the Series

Young men should prove theorems, old men should write books.
—Freeman Dyson, quoting G. H. Hardy

Reed–Simon starts with “Mathematics has its roots in numerology, geometry, and physics.” This puts into context the division of mathematics into algebra, geometry/topology, and analysis. There are, of course, other areas of mathematics, and a division between parts of mathematics can be artificial. But almost universally, we require our graduate students to take courses in these three areas.

This five-volume series began and, to some extent, remains a set of texts for a basic graduate analysis course. In part it reflects Caltech’s three-terms-per-year schedule and the actual courses I’ve taught in the past. Much of the contents of Parts 1 and 2 (Part 2 is in two volumes, Part 2A and Part 2B) are common to virtually all such courses: point set topology, measure spaces, Hilbert and Banach spaces, distribution theory, and the Fourier transform, complex analysis including the Riemann mapping and Hadamard product theorems. Parts 3 and 4 are made up of material that you’ll find in some, but not all, courses—on the one hand, Part 3 on maximal functions and $H^p$-spaces; on the other hand, Part 4 on the spectral theorem for bounded self-adjoint operators on a Hilbert space and det and trace, again for Hilbert space operators. Parts 3 and 4 reflect the two halves of the third term of Caltech’s course.

While there is, of course, overlap between these books and other texts, there are some places where we differ, at least from many:

(a) By having a unified approach to both real and complex analysis, we are able to use notions like contour integrals as Stietljes integrals that cross the barrier.

(b) We include some topics that are not standard, although I am surprised they are not. For example, while discussing maximal functions, I present Garcia’s proof of the maximal (and so, Birkhoff) ergodic theorem.

(c) These books are written to be keepers—the idea is that, for many students, this may be the last analysis course they take, so I’ve tried to write in a way that these books will be useful as a reference. For this reason, I’ve included “bonus” chapters and sections—material that I do not expect to be included in the course. This has several advantages. First, in a slightly longer course, the instructor has an option of extra topics to include. Second, there is some flexibility—for an instructor who can’t imagine a complex analysis course without a proof of the prime number theorem, it is possible to replace all or part of the (non-bonus) chapter on elliptic functions with the last four sections of the bonus chapter on analytic number theory. Third, it is certainly possible to take all the material in, say, Part 2, to turn it into a two-term course. Most importantly, the bonus material is there for the reader to peruse long after the formal course is over.

(d) I have long collected “best” proofs and over the years learned a number of ones that are not the standard textbook proofs. In this regard, modern technology has been a boon. Thanks to Google books and the Caltech library, I’ve been able to discover some proofs that I hadn’t learned before. Examples of things that I’m especially fond of are Bernstein polynomials to get the classical Weierstrass approximation theorem, von Neumann’s proof of the Lebesgue decomposition and Radon–Nikodym theorems, the Hermite expansion treatment of Fourier transform, Landau’s proof of the Hadamard factorization theorem, Wielandt’s theorem on the functional equation for $\Gamma(z)$, and Newman’s proof of the prime number theorem. Each of these appears in at least some monographs, but they are not nearly as widespread as they deserve to be.

(e) I’ve tried to distinguish between central results and interesting asides and to indicate when an interesting aside is going to come up again later. In particular, all chapters, except those on preliminaries, have a listing of “Big Notions and Theorems” at their start. I wish that this attempt to differentiate between the essential and the less essential
didn’t make this book different, but alas, too many texts are monotone listings of theorems and proofs.

(f) I’ve included copious “Notes and Historical Remarks” at the end of each section. These notes illuminate and extend, and they (and the Problems) allow us to cover more material than would otherwise be possible. The history is there to enliven the discussion and to emphasize to students that mathematicians are real people and that “may you live in interesting times” is truly a curse. Any discussion of the history of real analysis is depressing because of the number of lives ended by the Nazis. Any discussion of nineteenth-century mathematics makes one appreciate medical progress, contemplating Abel, Riemann, and Stieltjes. I feel knowing that Picard was Hermite’s son-in-law spices up the study of his theorem.

On the subject of history, there are three cautions. First, I am not a professional historian and almost none of the history discussed here is based on original sources. I have relied at times—horrors!—on information on the Internet. I have tried for accuracy but I’m sure there are errors, some that would make a real historian wince.

A second caution concerns looking at the history assuming the mathematics we now know. Especially when concepts are new, they may be poorly understood or viewed from a perspective quite different from the one here. Looking at the wonderful history of nineteenth-century complex analysis by Bottazzini–Gray will illustrate this more clearly than these brief notes can.

The third caution concerns naming theorems. Here, the reader needs to bear in mind Arnol’d’s principle. If a notion bears a personal name, then that name is not the name of the discoverer (and the related Berry principle: The Arnol’d principle is applicable to itself). To see the applicability of Berry’s principle, I note that in the wider world, Arnol’d’s principle is called “Stigler’s law of eponymy.” Stigler named this in 1980, pointing out it was really discovered by Merton. In 1972, Kennedy named Boyer’s law Mathematical formulas and theorems are usually not named after their original discoverers after Boyer’s book. Already in 1956, Newman quoted the early twentieth-century philosopher and logician A. N. Whitehead as saying: “Everything of importance has been said before by somebody who

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did not discover it.” The main reason to give a name to a theorem is to have a convenient way to refer to that theorem. I usually try to follow common usage (even when I know Arnol’d’s principle applies).

I have resisted the temptation of some text writers to rename things to set the record straight. For example, there is a small group who have attempted to replace “WKB approximation” by “Liouville–Green approximation”, with valid historical justification (see the Notes to Section 15.5 of Part 2B). But if I gave a talk and said I was about to use the Liouville–Green approximation, I’d get blank stares from many who would instantly know what I meant by the WKB approximation. And, of course, those who try to change the name also know what WKB is! Names are mainly for shorthand, not history.

These books have a wide variety of problems, in line with a multiplicity of uses. The serious reader should at least skim them since there is often interesting supplementary material covered there.

Similarly, these books have a much larger bibliography than is standard, partly because of the historical references (many of which are available online and a pleasure to read) and partly because the Notes introduce lots of peripheral topics and places for further reading. But the reader shouldn’t consider for a moment that these are intended to be comprehensive—that would be impossible in a subject as broad as that considered in these volumes.

These books differ from many modern texts by focusing a little more on special functions than is standard. In much of the nineteenth century, the theory of special functions was considered a central pillar of analysis. They are now out of favor—too much so—although one can see some signs of the pendulum swinging back. They are still mainly peripheral but appear often in Part 2 and a few times in Parts 1, 3, and 4.

These books are intended for a second course in analysis, but in most places, it is really previous exposure being helpful rather than required. Beyond the basic calculus, the one topic that the reader is expected to have seen is metric space theory and the construction of the reals as completion of the rationals (or by some other means, such as Dedekind cuts).

Initially, I picked “A Course in Analysis” as the title for this series as an homage to Goursat’s Cours d’Analyse a classic text (also translated into English) of the early twentieth century (a literal translation would be

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of Analysis” but “in” sounds better). As I studied the history, I learned that this was a standard French title, especially associated with École Polytechnique. There are nineteenth-century versions by Cauchy and Jordan and twentieth-century versions by de la Vallée Poussin and Choquet. So this is a well-used title. The publisher suggested adding “Comprehensive”, which seems appropriate.

It is a pleasure to thank many people who helped improve these texts. About 80% was TeXed by my superb secretary of almost 25 years, Cherie Galvez. Cherie was an extraordinary person—the secret weapon to my productivity. Not only was she technically strong and able to keep my tasks organized but also her people skills made coping with bureaucracy of all kinds easier. She managed to wind up a confidant and counselor for many of Caltech’s mathematics students. Unfortunately, in May 2012, she was diagnosed with lung cancer, which she and chemotherapy valiantly fought. In July 2013, she passed away. I am dedicating these books to her memory.

During the second half of the preparation of this series of books, we also lost Arthur Wightman and Ed Nelson. Arthur was my advisor and was responsible for the topic of my first major paper—perturbation theory for the anharmonic oscillator. Ed had an enormous influence on me, both via the techniques I use and in how I approach being a mathematician. In particular, he taught me all about closed quadratic forms, motivating the methodology of my thesis. I am also dedicating these works to their memory.

After Cherie entered hospice, Sergei Gel’fand, the AMS publisher, helped me find Alice Peters to complete the TeXing of the manuscript. Her experience in mathematical publishing (she is the “A” of A K Peters Publishing) meant she did much more, for which I am grateful.

This set of books has about 150 figures which I think considerably add to their usefulness. About half were produced by Mamikon Mnatsakanian, a talented astrophysicist and wizard with Adobe Illustrator. The other half, mainly function plots, were produced by my former Ph.D. student and teacher extraordinaire Mihai Stoiciu (used with permission) using Mathematica. There are a few additional figures from Wikipedia (mainly under WikiCommons license) and a hyperbolic tiling of Douglas Dunham, used with permission. I appreciate the help I got with these figures.

Over the five-year period that I wrote this book and, in particular, during its beta-testing as a text in over a half-dozen institutions, I received feedback and corrections from many people. In particular, I should like to thank (with apologies to those who were inadvertently left off): Tom Alberts, Michael Barany, Jacob Christiansen, Percy Deift, Tal Einav, German Enciso, Alexander Eremenko, Rupert Frank, Fritz Gesztesy, Jeremy Gray,
Preface to the Series


Much of these books was written at the tables of the Hebrew University Mathematics Library. I’d like to thank Yoram Last for his invitation and Naavah Levin for the hospitality of the library and for her invaluable help.

This series has a Facebook page. I welcome feedback, questions, and comments. The page is at www.facebook.com/simon.analysis.

Even if these books have later editions, I will try to keep theorem and equation numbers constant in case readers use them in their papers.

Finally, analysis is a wonderful and beautiful subject. I hope the reader has as much fun using these books as I had writing them.
Preface to Part 3

I don’t have a succinct definition of harmonic analysis or perhaps I have too many. One possibility is that harmonic analysis is what harmonic analysts do. There is an active group of mathematicians, many of them students of or grandstudents of Calderón or Zygmund, who have come to be called harmonic analysts and much of this volume concerns their work or the precursors to that work. One problem with this definition is that, in recent years, this group has branched out to cover certain parts of nonlinear PDE’s and combinatorial number theory.

Another approach to a definition is to associate harmonic analysis with “hard analysis,” a term introduced by Hardy, who also used “soft analysis” as a pejorative for analysis as the study of abstract infinite-dimensional spaces. There is a dividing line between the use of abstraction, which dominated the analysis of the first half of the twentieth century, and analysis which relies more on inequalities, which regained control in the second half. And there is some truth to the idea that Part 1 in this series of books is more on soft analysis and Part 3 on hard, but, in the end, both parts have many elements of both abstraction and estimates.

Perhaps the best description of this part is that it should really be called “More Real Analysis.” With the exception of Chapter 5 on $H^p$-spaces, any chapter would fit with Part 1—indeed, Chapter 4, which could be called “More Fourier Analysis,” started out in Part 1 until I decided to move it here.

The topics that should be in any graduate analysis course and often are, are the results on Hardy–Littlewood maximal functions and the Lebesgue
differentiation theorem in Chapter 2 the very basics of harmonic and sub-harmonic functions, something about $H^p$-spaces and about Sobolev inequalities.

The other topics are exceedingly useful but are less often in courses, including those at Caltech. Especially in light of Calderón’s discovery of its essential equivalence to the Hardy–Littlewood theorem, the maximal ergodic theorem should be taught. And wavelets have earned a place, as well. In any event, there are lots of useful devices to add to our students’ toolkits.
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Symbol Index

A∗, adjoint of A, 3
Ac, complement, 2
A ∩ B, intersection, 2
A Δ B, symmetric difference, 2
A†, transpose of A, 3
A ∪ B, union, 2
A(z(x)), Segal–Bargmann transform, 8
Bf, Segal–Bargmann transform, 8
C±, upper half-plane, 2
C∞(X), continuous function vanishing at infinity, 2
D, unit disk in C, 2
Dν f, ∂βν f/∂x1...∂νxν, 6
∂D, unit circle in C, 2
E(μ), Coulomb energy, 253
f*(t), equimeasurable decreasing rearrangement, 30
f**(t), Muirhead maximal function, 36
f̂, Fourier transform, 7
f̃, inverse Fourier transform, 7
f̂ν, Fourier series coefficient, 7
fQ, average over Q, 519
Gε, Green’s function, 253
H, Hilbert space, 8
H+, right half-plane, 2
Hν f, Perron solution, 224
Hν,d, homogeneous polynomials of degree d, 284
Hν, Hardy space, 439
hν, real harmonic function obeying Hν condition, 439
Hν-p(Rν), generalized Sobolev space of order s and index p, 508
K, placeholder for R or C, 22
L(X), bounded linear transformation, 3
Log(z), natural logarithm, 2
Lp w, weak-Lp space, 31
mμ(t), distribution function of μ, 26
(MRadf)(e iθ), radial maximal function, 444
Mν f, given by (6.1.1), 439
N, Nevanlinna class, 439
N, natural numbers, 2
∥g∥p,w, weak-Lp “norm”, 30
♯(A), number of elements in A, 2
Op(a) f, pseudodifferential operator, 353
O∂(ε), outer boundary of ε, 258
Φε, equilibrium potential, 253
Φμ, potential of a measure, μ, 206
φx0,p0, a(x), Gaussian function with center at (x0, p0) in phase space, 321
Symbol Index

\[ \mathcal{P}(\nu, d) \], polynomials of total degree at most \( d \).

\( \Psi^{-\infty} \), negligible pseudodifferential operators.

\( \Psi_\mu \), antipotential of a measure, \( \mu \).

\( \Psi_{\rho_\delta} \), space of Hörmander pseudodifferential operators.

\( \mathbb{Q} \), rational numbers.

\( \mathbb{R} \), real numbers.

\( \text{Ran}(f) \), range of a function \( f \).

\( R(\varepsilon) \), Robin constant.

\( \lfloor \cdot \rfloor \), restriction.

\( \rho_e \), equilibrium measure.

\( \mathcal{S}(\mathbb{R}^\nu) \), Schwartz space.

\( s\text{-supp}(\tau) \), singular support of a distribution.

\( \sigma(X, Y) \), \( Y \)-weak topology.

\( \sigma_\nu \), surface area of \( S^{\nu-1} = 2(\pi^{\nu/2})/[\Gamma(\frac{\nu}{2})]^{-1} \).

\( \text{SL}(2, \mathbb{R}) \), \( 2 \times 2 \) real matrices of determinant 1.

\( S(\nu, d) \), harmonic homogeneous polynomials of degree \( d \).

\( \text{SO}(\nu) \), group of rotations in \( \nu \)-dimensions.

\( \mathcal{S}'(\mathbb{R}^\nu) \), tempered distributions.

\( S_{\rho_\delta}(\mathbb{R}^\nu) \), Hörmander symbol space.

\( \text{Var}_f(X) \), variation of \( f \) in \( x \)-space.

\( \text{WF}(\tau) \), wavefront set.

\( W^{\ell, p}(\mathbb{R}^\nu) \), Sobolev space of order \( \ell \) and index \( p \).

\( \lfloor x \rfloor \), greatest integer less than \( x \).

\( \{ x \} \), fractional part of \( x \).

\( \mathbb{Z} \), integer numbers.

\( \mathbb{Z}^+ \), positive integers.

\( Z_f \), Zak transform.
Subject Index

a.e. boundary values, 457
Abelian theorem, 686
absolutely continuous, 265
accumulation function, 37
adapted, 148
adjoint pseudodifferential operator, 364
admissible vector, 379, 381
affine group, 383
affine Heisenberg–Weyl group, 321
Ahnors function, 269
almost Matthieu operator, 295
amplitude, 354, 361
Amrein–Berthier theorem, 329
analytic continuation, 288
analytic function, 205, 279, 310
analytic function theory, 391
Anderson model, 295
antipotential, 206, 211, 217, 220, 232
approximate identity, 9, 495
arithmetic combinatorics, 685
Arnold cat map, 132
Ascoli–Arzelà theorem, 192
associated Legendre polynomials, 250
atom of $\text{Re } H^1$, 524
atomic decomposition, 520, 528
Aubry duality, 296
bad cubes, 591
Baire category theorem, 25
Baire functions, 4
balayage, 275
Balian–Low theorem, 400, 402, 404
Banach space, 422, 493, 618
Banach–Alaoglu theorem, 467, 577
Bari basis, 106
barrier, 224, 300, 301
Beckner’s inequality, 652
Benedicks set, 328, 337
Benedicks theorem, 328
Benedicks–Amrein–Berthier theorem, 324
Benford’s law, 97, 99, 100
Berezin–Lieb inequality, 438, 438, 439
Berezin–Weil–Zak transform, 402
Bergman coherent states, 402
Bernoulli shift, 68, 69, 91–93, 97
Bernoulli’s inequality, 641
Bernoulli–Walsh inequality, 291
Bernstein–Walsh lemma, 279
Bessel function, 244, 248
Bessel kernel, 566, 567
Bessel inequality, 541
Bessel potential, 270, 567, 590
Bessel sequence, 395, 398, 399, 401
Bessel transform, 245, 248
best constants, 522
best hypercontractive estimate, 632
Beurling’s theorem, 516, 517
Beurling–Deny criteria, 617, 629, 632
binomial theorem, 424, 611
bipolar Green’s function, 302, 303, 315
<table>
<thead>
<tr>
<th>Term</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>classical Green’s function</td>
<td>182, 183, 208, 222, 224, 227, 231, 268</td>
</tr>
<tr>
<td>classical symbols</td>
<td>353</td>
</tr>
<tr>
<td>clock spacing</td>
<td>292</td>
</tr>
<tr>
<td>closed graph theorem</td>
<td>495</td>
</tr>
<tr>
<td>CLR bounds</td>
<td>658, 665</td>
</tr>
<tr>
<td>CLR inequality</td>
<td>657, 660, 669</td>
</tr>
<tr>
<td>coadjoint orbits</td>
<td>107, 146</td>
</tr>
<tr>
<td>coherent projection</td>
<td>376</td>
</tr>
<tr>
<td>coherent states</td>
<td>320, 374, 376, 380, 382, 385, 407</td>
</tr>
<tr>
<td>Coifman atomic decomposition</td>
<td>140</td>
</tr>
<tr>
<td>compact operator</td>
<td>311, 312</td>
</tr>
<tr>
<td>compact Riemann surface</td>
<td>310</td>
</tr>
<tr>
<td>compactification</td>
<td>277</td>
</tr>
<tr>
<td>complete family</td>
<td>390</td>
</tr>
<tr>
<td>complex analysis</td>
<td>273</td>
</tr>
<tr>
<td>complex interpolation method</td>
<td>350, 550</td>
</tr>
<tr>
<td>complex Poisson formula</td>
<td>143, 173</td>
</tr>
<tr>
<td>complex Poisson kernel</td>
<td>443</td>
</tr>
<tr>
<td>complex Poisson representation</td>
<td>445, 141, 160</td>
</tr>
<tr>
<td>conditional expectation</td>
<td>71, 147</td>
</tr>
<tr>
<td>conjugate</td>
<td>476</td>
</tr>
<tr>
<td>conjugate function</td>
<td>445, 447, 472, 498</td>
</tr>
<tr>
<td>conjugate function duality</td>
<td>477</td>
</tr>
<tr>
<td>conjugate harmonic function</td>
<td>365, 418</td>
</tr>
<tr>
<td>constructive quantum field theory</td>
<td>168, 659</td>
</tr>
<tr>
<td>continued fractions</td>
<td>109</td>
</tr>
<tr>
<td>continuity principle</td>
<td>240, 241</td>
</tr>
<tr>
<td>continuous filtration</td>
<td>168, 358</td>
</tr>
<tr>
<td>continuous wavelets</td>
<td>358</td>
</tr>
<tr>
<td>convergence at large scales</td>
<td>410</td>
</tr>
<tr>
<td>convergence at small scales</td>
<td>410</td>
</tr>
<tr>
<td>convergence in measure</td>
<td>34, 190</td>
</tr>
<tr>
<td>convergence in probability</td>
<td>31</td>
</tr>
<tr>
<td>convergence of wavelet expansions</td>
<td>490, 431</td>
</tr>
<tr>
<td>convex functions</td>
<td>203</td>
</tr>
<tr>
<td>convolution operator</td>
<td>54, 600</td>
</tr>
<tr>
<td>coordinate disk</td>
<td>298, 305</td>
</tr>
<tr>
<td>coordinate patch</td>
<td>298</td>
</tr>
<tr>
<td>coordinate plane</td>
<td>671</td>
</tr>
<tr>
<td>cotangent bundle</td>
<td>380</td>
</tr>
<tr>
<td>Cotlar’s lemma</td>
<td>618</td>
</tr>
<tr>
<td>Cotlar’s theorem</td>
<td>659, 652</td>
</tr>
<tr>
<td>Cotlar–Knopp–Stein lemma</td>
<td>608</td>
</tr>
<tr>
<td>Cotlar–Stein lemma</td>
<td>613</td>
</tr>
<tr>
<td>Coulomb energy</td>
<td>253</td>
</tr>
<tr>
<td>Coulomb potential</td>
<td>243</td>
</tr>
<tr>
<td>cricket averages</td>
<td>445</td>
</tr>
<tr>
<td>critical Gabor transform</td>
<td>390</td>
</tr>
<tr>
<td>critical Lieb–Thirring inequality</td>
<td>657</td>
</tr>
<tr>
<td>critical LT inequality</td>
<td>668</td>
</tr>
<tr>
<td>Croft–Garsia covering lemma</td>
<td>45, 50, 52</td>
</tr>
<tr>
<td>Cwikel–Lieb–Rosenblum inequality</td>
<td>657</td>
</tr>
<tr>
<td>CZ kernel</td>
<td>605</td>
</tr>
<tr>
<td>Daubechies construction</td>
<td>125, 419</td>
</tr>
<tr>
<td>Daubechies wavelets</td>
<td>419, 428</td>
</tr>
<tr>
<td>Daubechies’ theorem</td>
<td>144</td>
</tr>
<tr>
<td>de la Vallée Poussin’s theorem</td>
<td>660, 471, 472, 151</td>
</tr>
<tr>
<td>de Leeuw–Rudin theorem</td>
<td>456</td>
</tr>
<tr>
<td>de Moivre’s martingale</td>
<td>456</td>
</tr>
<tr>
<td>decreasing rearrangement</td>
<td>291, 301</td>
</tr>
<tr>
<td>Denisov–Rakhmanov theorem</td>
<td>293</td>
</tr>
<tr>
<td>dense orbits</td>
<td>38</td>
</tr>
<tr>
<td>density of states</td>
<td>284</td>
</tr>
<tr>
<td>density of zeros</td>
<td>284</td>
</tr>
<tr>
<td>de Leeuw–Rudin theorem</td>
<td>456</td>
</tr>
<tr>
<td>DFT</td>
<td>339</td>
</tr>
<tr>
<td>diamagnetic inequality</td>
<td>609</td>
</tr>
<tr>
<td>differentiation theorem</td>
<td>609</td>
</tr>
<tr>
<td>dilation</td>
<td>544</td>
</tr>
<tr>
<td>Dini condition</td>
<td>485</td>
</tr>
<tr>
<td>Diophantine approximation</td>
<td>140</td>
</tr>
<tr>
<td>dipolar layer</td>
<td>275</td>
</tr>
<tr>
<td>dipole moment</td>
<td>251</td>
</tr>
<tr>
<td>Dirichlet domain</td>
<td>182, 185</td>
</tr>
<tr>
<td>Dirichlet form</td>
<td>622, 629, 630, 652</td>
</tr>
<tr>
<td>Dirichlet Green’s function</td>
<td>182, 184, 186</td>
</tr>
<tr>
<td>Dirichlet principle</td>
<td>275, 276</td>
</tr>
<tr>
<td>Dirichlet problem</td>
<td>181, 183, 205</td>
</tr>
<tr>
<td>Dirichlet problem for the ball</td>
<td>188</td>
</tr>
<tr>
<td>discontinuous subharmonic function</td>
<td>206</td>
</tr>
<tr>
<td>discrete Hardy inequality</td>
<td>459</td>
</tr>
<tr>
<td>discrete Heisenberg group</td>
<td>403</td>
</tr>
<tr>
<td>discrete Hilbert transform</td>
<td>457, 542</td>
</tr>
<tr>
<td>distribution</td>
<td>335, 445</td>
</tr>
<tr>
<td>distribution function</td>
<td>261, 33</td>
</tr>
<tr>
<td>distribution, positive</td>
<td>210</td>
</tr>
<tr>
<td>distributional derivative</td>
<td>323, 324</td>
</tr>
<tr>
<td>distributional integral kernel</td>
<td>605</td>
</tr>
<tr>
<td>distributions equal near ( x_0 )</td>
<td>845</td>
</tr>
<tr>
<td>domain</td>
<td>614</td>
</tr>
<tr>
<td>dominated convergence theorem</td>
<td>61, 146</td>
</tr>
<tr>
<td>dominated convergence theorem</td>
<td>61, 146</td>
</tr>
<tr>
<td>Doob decomposition theorem</td>
<td>155</td>
</tr>
</tbody>
</table>
Doob martingale inequality, 83
Doob maximal inequality, 48
Doob’s inequality, 132, 161, 601
Doob’s upcrossing inequality, 165
double-layer potentials, 275
doubling map, 69
doubly homogeneous space, 496
doubly stochastic map, 70
dressing and undressing, 128
dual indices, 5
duality for $H^p$, 518
Duhamel’s formula, 649
Dunford–Pettis theorem, 617, 626, 662
dyadic cube, 591
dyadic filtration, 592
dyadic Hardy–Littlewood martingale, 151
dyadic Lorentz norm, 557
eigenvalue moment, 557
eigenvalues, 539
elliptic function theory, 391
elliptic PDO, 362
elliptic regularity, 360, 392
elliptic regularity for elliptic $\Psi$DO, 365
elliptic Riemann surface, 367
elliptic symbol of order $m$, 369
entire function, 218
equidistributed, 98, 102, 128
equidistribution, 106
equilibrium measure, 11, 253, 256, 281
258, 259, 260
equimeasurable, 20, 30, 36, 54, 548
equivalence class, 8
equivalence relation, 9
ergodic, 93
ergodic Jacobi matrices, 296, 297
ergodic measurable dynamical system, 89
ergodic measure, 71, 72
ergodic theorem, 72, 73
ergodic theory, 539
ergodicity, 86
essential support, 280
Euler’s formula, 232
extended maximum principle, 264, 266
274
exterior ball condition, 258, 259
exterior cone condition, 230
exterior Dirichlet problem, 266, 307, 317
exterior problem, 266, 267
extreme point, 72, 83, 87
F. and M. Riesz theorem, 153, 156
F. and R. Nevanlinna theorem, 142
150, 170
Faber–Fekete–Szegő theorem, 291
Fatou’s lemma, 5, 273, 283, 150, 152
162, 163
Fatou’s theorem, 150
Fefferman duality, 520
Fefferman duality theorem, 523, 526
Fefferman–Stein decomposition, 521
527, 530, 538
Fejér kernel, 1, 19, 57, 195
Fejér’s theorem, 3, 457
Fejér–Riesz theorem, 426, 434
fermions, 653
filtration, 147
fine topology, 276
finite bordered Riemann subsurface, 301, 311, 315
finite gap set, 289
finite measure, 510, 512
finite simple graph, 631
finite volume, 119
FIO, 367
Fock space, 9, 88, 393, 402
forensic accounting, 100
form domain, 617
formal symbol, 358
Fourier coefficients, 459
Fourier expansion, 147
Fourier integral operator, 320, 352, 366
367
Fourier inversion formula, 8, 503
Fourier multiplier, 598, 599
Fourier series, 6, 21, 502
Fourier series coefficients, 398
Fourier transform, 6, 47, 234, 247, 374
410, 498, 504, 509, 514, 566
599, 600
fractional derivatives, 566
fractional Laplacian, 666
fractional Sobolev space, 566, 569, 582
400, 402, 404
Fréchet space, 6, 442
Fredholm theory, 275
free Green’s function, 181, 202
free Laplacian semigroup, 613
Frostman’s theorem, 266, 274
Fubini’s theorem, 293
Fuchsian group, 120, 132
fundamental theorem of algebra, 280
fundamental theorem of calculus, 581
fundamental theorem of potential theory, 274
Furstenberg’s theorem, 295
Furstenberg–Kesten theorem, 133, 144, 295
Gabor analysis, 386
Gabor frame, 403
Gabor lattice, 385, 390, 394, 396, 397, 401
Gagliardo–Nirenberg inequality, 570, 573, 582, 586, 668, 669
Gauss map, 113, 123, 124
Gauss measure, 109, 112, 113, 123, 124, 652
Gauss semigroup, 630
Gauss’ theorem, 17, 197
Gauss–Kuzmin theorem, 103, 110
Gauss–Kuzmin–Wirsing operator, 111, 125
Gauss-Kuzmin distribution, 103
Gaussian coherent states, 374
Gaussian curvature, 682, 685
Gaussian measure, 641, 643, 655
Gaussians, 566
Gegenbauer polynomial, 241
Gel’fand’s question, 95, 97, 99
generalized Bernoulli shift, 93
generalized Dirichlet problem, 265
generalized Hardy inequality, 551
generalized Sobolev spaces, 556
generalized Stein–Weiss inequality, 508
generator, 616
godesic flow, 129, 130, 132, 133, 134
geodesics, 113
Gibbs state, 654
Glauber dynamics, 654
Glauber–Sudarshan symbol, 378, 386
Gram–Schmidt, 408
graph Laplacian, 631
Green’s formula, 215
Green’s function, 111, 112, 129, 139, 205, 228, 231, 233, 259, 260, 270, 271, 302, 303, 308, 310, 314, 315
Green’s function with a pole, 274
Green’s function with a pole at infinity, 259
Green’s theorem, 16
Gross–Nelson semigroup, 650, 658, 659
Grossmann–Morlet–Paul theorem, 380
ground state representation, 622
group extension, 107
group representation, 379
Haar basis, 408, 410, 432
Haar function, 384
Haar measure, 101, 118, 378, 380, 383, 389, 549
Haar wavelet, 384, 408, 423
Hadamard three-circle theorem, 411
Hahn decomposition, 65
Hahn–Banach theorem, 75, 519, 537
Halfspace, Poisson kernel, 186
Hankel matrix, 535, 537, 538
Hankel transform, 245
Hardy space, 440, 444
Hardy space of bounded mean oscillation, 520
Hardy’s convexity theorem, 141, 444
Hardy’s inequality, 323, 325, 345, 548, 549
Hardy’s uncertainty principle, 324, 326
Hardy’s variational principle, 325
Hardy–Littlewood maximal function, 41, 53, 59, 103, 153, 154
Hardy–Littlewood maximal inequality, 41, 45, 52, 55, 77, 83, 90, 91, 147, 158, 161, 179, 192, 592
Hardy–Littlewood maximal theorem, 151
Hardy–Littlewood theorem, 83
Hardy–Littlewood–Sobolev inequality, 335, 544
harmonic conjugate, 505
harmonic distribution, 193
harmonic function, 178, 179, 181, 182, 189, 190, 211, 213, 217, 222, 223, 230
harmonic function, 250, 261, 266, 270, 288, 290, 300
harmonic homogeneous function, 223
harmonic homogeneous polynomial, 239
harmonic measure, 182, 265, 267, 272
harmonic polynomial, 223, 224
Harnack’s inequality, 193, 265, 299, 314, 318, 544
Harnack’s principle, 196, 223, 317
Hausdorff dimension, 251, 277, 280, 329
Hausdorff dimension theory, 679
Hausdorff measure, 274
Hausdorff–Young inequality, 170, 335, 342, 544, 565, 583
Heisenberg commutation relation, 322
Heisenberg group, 321, 382, 397
Heisenberg uncertainty principle, 321
Herglotz function, 287, 499
Herglotz representation, 287, 297, 459
Heisenberg uncertainty principle, 321
Herglotz theorem, 467, 498
Hermite basis, 8, 323, 327
Hermite semigroup, 630
Hilbert inequality, 4, 40, 440, 466, 492, 519, 544, 572, 587, 644, 658
Hilbert–Schmidt, 626
HLS inequality, 559, 562, 676, 682
HMO, 519, 520, 524, 535
Hölder continuous, 574, 671, 673, 685
Hölder continuity, 483
Hölder’s inequality, 4, 40, 440, 466, 492
homogeneous harmonic function, 239
homogeneous harmonic polynomial, 236, 252
homogeneous Sobolev estimates, 570
incomplete family, 390
independent random variables, 10
interlace, 281
interpolation, 15, 518, 615
interpolation estimates, 597
Hölder’s inequality, 4, 40, 440, 466, 492
homogeneous harmonic function, 239
homogeneous harmonic polynomial, 236, 252
invariant measure, 65, 68
invariant probability measure, 112
invariant subspace, 516
Hölder’s inequality, 4, 40, 440, 466, 492
Hölder’s inequality, 4, 40, 440, 466, 492
Hölder’s inequality, 4, 40, 440, 466, 492
homogeneous harmonic function, 239
homogeneous harmonic polynomial, 236, 252
Hölder continuity, 483
Hölder’s inequality, 4, 40, 440, 466, 492
homogeneous harmonic function, 239
homogeneous harmonic polynomial, 236, 252
Hölder’s inequality, 4, 40, 440, 466, 492
homogeneous harmonic function, 239
homogeneous harmonic polynomial, 236, 252
invariant measure, 65, 68
invariant probability measure, 112
invariant subspace, 516
Hölder’s inequality, 4, 40, 440, 466, 492
homogeneous harmonic function, 239
homogeneous harmonic polynomial, 236, 252
Hölder’s inequality, 4, 40, 440, 466, 492
homogeneous harmonic function, 239
homogeneous harmonic polynomial, 236, 252
invariant measure, 65, 68
invariant probability measure, 112
invariant subspace, 516
inverse Fourier transform, 7
irreducible rotations, 94
irrational rotations, 94
Jacobi parameters, 281, 283, 292
Jacobi theta function, 391, 404
John–Nirenberg inequality, 473, 490
joint probability distribution, 10
HLS inequality, 559, 562, 676, 682
jointly continuous, 266

K-systems, 97
Kac return time theorem, 3, 90
Kadec 1 theorem, 406
Kakeya conjecture, 684, 685
Kakeya dimension conjecture, 685
Kakeya maximal function conjecture, 685
Kakeya problem, 685
Kakeya set, 685
Kato's inequality, 544
Kellogg–Evans theorem, 260, 265, 274
Kelvin transform, 187, 201, 221, 233, 260
Khinchin recurrence theorem, 90
Khinchin’s constant, 111
Khinchin’s theorem, 110
Kingman ergodic theorem, 133
Knapp scaling, 681
Knapp's counterexample, 678
Kolmogorov 0-1 law, 154, 162
Kolmogorov three-series theorem, 161
Kolmogorov's inequality, 152
Kolmogorov’s random $L^2$ series theorem, 154
Kolmogorov’s theorem, 162, 163, 174
Koopman unitary, 67
Krein–Millman theorem, 172
Kronecker’s lemma, 166
Kronecker–Weyl theorem, 98

Laplace’s method, 568
Laplace–Beltrami operator, 178, 288
large deviations, 654
lattice, 119
Laurent polynomials, 478
Laurent series, 426
Laurent series coefficients, 502
law of large numbers, 10
Lebesgue differentiation theorem, 53
Lebesgue measure, 190
Lebesgue point, 201
Lebesgue spine, 220
Legendre–Fejér theorem, 53
Legendre polynomials, 249
Legendre relation, 392
Levy 0-1 law, 154

Lévy’s constant, 111
Lévy’s theorem, 111
Lie algebras, 122
Lie groups, 122
Lie product formula, 128
Lie–Thirring bounds, 670
Lie–Thirring inequality, 657, 658
Lifschtiz tails, 295
limit set, 126
Lindelöf spaces, 51
Liouville number, 296
Liouville’s theorem, 65
Liouville–Picard theorem, 197
Lipschitz boundary, 274
Littlewood–Paley decomposition, 233, 603, 604, 610, 676, 682
local constant, 636
local norm, 636
localization, 345
log Sobolev inequality, 636
logarithmic Sobolev estimates, 615
logarithmic Sobolev inequality, 636
lognormal distribution, 419
Lorentz quasinorm, 548
Lorentz spaces, 172, 548
low-pass filter, 416
lower envelope theorem, 281
lower order, 606
lower symbol, 377
$L^p$ Fourier multiplier, 598
$L^p$-contractive semigroup, 615
$L^p$-convergence of Fourier series, 297
$L^p$-multiplier, 599
$L^p$-norms, 27
lsc, 258
LT bounds, 653
LT inequality, 653
Lusin’s theorem, 250
Lyapunov behavior, 141
Lyapunov exponent, 133
M. Riesz’s theorem, 472, 171
MacDonald function, 566
magic of maximal functions, 23
magnetic fields, 669
Malgrange–Ehrenpreis theorem, 666
Marcinkiewicz interpolation, 619

745
Marcinkiewicz interpolation theorem, 32, 540, 555, 569, 691, 698
Markov semigroup, 622, 634, 651, 654
Markov’s inequality, 5
Martin boundary, 276
martingale, 148, 149, 152, 157
martingale convergence, 153
martingale convergence theorem, 158
mass gap, 656
maximal ergodic inequality, 73, 83, 88
maximal ergodic theorem, 76
maximal function, 22, 23
maximal Hilbert transform, 512, 539
maximal inequality, 24, 544
maximum principle, 180, 184, 191, 207, 227, 250, 264, 279, 309, 326
mean ergodic theorem, 72
mean oscillation, 519
mean value property, 178, 179
measurable dynamical system, 66, 67, 73, 83, 89, 91, 133, 137
measurable semiflow, 68
measure space, 22
measure-preserving, 120
measure-preserving map, 66
measure-preserving semiflow, 68, 77, 87
Mehler’s formula, 372
Menshov’s theorem, 172
Mercedes frame, 403
Mercer’s theorem, 620, 628
meromorphic function, 316
metrical transitivity, 71
Mexican hat wavelet, 384
Meyer wavelets, 408, 417
microlocal analysis, 352, 368
midpoint convexity, 203
minimal measure, 84
minimal superharmonic majorant, 307
minimum principle, 180
Minkowski’s inequality, 41, 132, 544, 549
mixing, 58, 80, 83
modified Bessel function of the second kind, 566
modular function, 379, 389
modulus of continuity, 483
moment problem, 295
monotone convergence theorem, 5, 27, 410, 466
Montel’s theorem, 156, 289
Montel’s theorem for harmonic functions, 192
Morrera’s theorem, 192
mother wavelet, 383, 407, 419
MRA, 412, 414, 415, 420, 422, 427, 429
431, 433
Muirhead maximal function, 36, 41, 548
multiplication operator, 589
multiplicative ergodic theorem, 144
multiplier, 599
multipole expansion, 242, 243
multiresolution analysis (MRA), 411
mutual energy, 258
MVP, 179, 184, 188, 195
Nash estimate, 572, 619
Nehari’s theorem, 537, 538
Nelson’s best hypercontractive estimate, 642
Neumann boundary conditions, 270
Neumann problem, 202, 270
Neumann series, 275
Nevanlinna function, 499
Nevanlinna space, 13, 440
Nevanlinna theory, 444
Newton’s potential, 249
non-uniqueness of exterior Dirichlet problem, 260
nonatomic measure, 37
noncommutative integration, 658
nonpolar, 288, 289
nonpolar set, 289
nontangential boundary value, 500
nontangential limits, 58, 145
nontangential maximal function, 58, 146
nontangential maximal inequality, 144
norm convergence of Fourier series, 496
normal number, 94
normal number theorem, 94
normalized coherent states, 375
normalized surface measure, 183
nowhere dense, 278
off-diagonal kernel, 589, 597, 603
open cover, 274
operator core, 655
OPRL, 285
optimal hypercontractive estimates, 640
OPUC, 298
ordinary distribution, 209
Subject Index

orgy of interpolation theory, 624
Orlicz spaces, 172
Ornstein–Uhlenbeck semigroup, 630
orthogonal polynomial, 238, 280
orthogonal projection, 341, 343
orthogonality relation, 380, 387
orthonormal basis, 233, 238, 398, 399
orthonormal polynomial, 280
orthonormal set, 411, 423
Orselec’s theorem, 143
outer boundary, 258
outer function, 469
outward pointing normal, 181
overcomplete, 384
overcomplete family, 390
overcomplete lattice, 396
pacman, 231
Paley–Littlewood decomposition, 607
Paley–Wiener coherent states, 376
Paley–Wiener ideas, 421
Paley–Wiener theorem, 502
parabolic Riemann surface, 307
paraboloid, 680
paramatrix, 365
Parseval relation, 397, 399
partial differential equations, 565
partial differential operator, 852
PDO, 852
Peelecs inequality, 579, 615
percolation model, 139
periodic Schrödinger operator, 666
Ferron construction, 307
Perron family, 300, 309, 317
Perron method, 220, 221, 224, 226, 265
Perron modification, 222, 223, 298, 317
Perron solution, 221
Perron theory, 261
Perron trials, 221
Perron’s principle, 300, 317
Perron–Frobenius theorem, 622, 654
Phragmén–Lindelöf method, 826
Picard’s theorem, 170, 213
Pick function, 499
Plancherel formula, 8
Plancherel theorem, 874, 882, 883, 102
plane wave expansion, 249
Plémenj–Privalov theorem, 184, 189
Poincaré recurrence theorem, 86
Poincaré conjecture, 654
Poincaré metric, 115
Poincaré sequence, 99
Poincaré’s criterion, 229
Poincaré’s inequality, 578, 581
pointwise a.e. convergence, 25
pointwise convergence, 465
pointwise ergodic theorem, 173
pointwise limits, 145, 155, 156
Poisson formula, 141, 145, 165
Poisson integral, 225
Poisson kernel, 52, 183, 186, 188
Poisson kernel of the ball, 187
Poisson representation, 141, 189, 401
Poisson representation theorem, 337
Poisson–Jensen formula, 123, 130, 142
polar decomposition, 140
polar set, 254, 256, 260, 261, 263, 264
polar singularities, 202
Polish space, 4
polynomial, 279
potential, 206, 252, 256, 279, 280
potential theory, 111, 252, 273, 276
predictable, 148
predictive, 165
principle of descent, 272, 284
probabilistic potential theory, 276
probability distribution, 10, 320
probability measure space, 355
product of distributions, 346
product of pseudodifferential operators, 364
product of two distributions, 346
prolate spheroidal function, 338
propagation of singularities, 350, 371
pseudodifferential operator, 320, 350
pseudolocal, 350
ΨDO, 350, 604
punctured ball, 220
punctured disk, 231
q.e., 254
quantum mechanics, 320
quasi-everywhere, 254
Rademacher functions, 409
radial maximal function, 444
Radon–Nikodym derivative, 288
random matrix product, 107
random series, 147
rcm, 27, 29
real interpolation method, 656
rearrangement, 29
recurrence theorem, 85
recursion relations, 281
refinable function, 412
refinement equation, 412
reflection principle, 199
reflection principle for harmonic functions, 199
reflectionless Jacobi matrices, 293
reflexive relation, 3
region, 178
regular, 281, 285, 286, 289
regular directed point, 347
regular hypersurfaces, 672
regular point, 221, 229, 245
regularity, 292
Rellich embedding theorem, 578
Rellich’s inequality, 560
Rellich–Kondrachov embedding theorem, 570, 582
removable singularities theorem, 193
reproducing kernel Hilbert space, 373, 375, 385
restricted dyadic filtration, 148
restriction conjecture, 683, 685
restriction to submanifolds, 671
return time theorem, 85, 90
Ricker wavelet, 384
Rickman’s lemma, 216
Riemann integral, 375
Riemann map, 269
Riemann mapping theorem, 268, 172
Riemann surface, 298, 300, 310, 311
316, 318
Riemann–Hilbert problem, 487
Riemann–Lebesgue lemma, 398
Riesz basis, 395, 396, 398, 401
Riesz decomposition, 217
Riesz decomposition theorem, 212, 213
Riesz factorization, 452, 454, 455, 458
462, 470, 507
Riesz factorization theorem, 457
Riesz maximal equality, 18
Riesz potentials, 276
Riesz projection, 489
Riesz transform, 514
Riesz–Markov theorem, 182
Riesz–Thorin interpolation, 623
Riesz–Thorin theorem, 15, 492, 556, 677
right continuous monotone, 27
right limits, 293
right regular representation, 120
Robin constant, 253, 275
Robin potential, 274
Robin’s problem, 274
root asymptotics, 281
roots, 281
Rosen’s lemma, 644, 653
rotations, 68
Ruelle–Oseledec theorem, 141, 145
scale covariance, 590
scaling filter, 410, 420, 422
scaling function, 411, 429
Schrödinger operators, 644
Schrödinger–Robertson uncertainty relations, 334
Schur function, 464
Schur product, 420
Schur’s lemma, 379, 381
Schur–Lebesco–Weyl inequality, 544
Schwartz kernel theorem, 357
Schwartz space, 5
Schwarz alternation method, 275
Schwarz inequality, 322
Schwarz kernel, 145
second kind polynomials, 294
Segal–Bargmann transform, 81, 327, 337, 374, 387, 402
self-adjoint operator, 381, 610
separating hyperplane theorem, 246
Shannon entropy, 334
Shannon’s inequality, 334, 344
Shilov boundary, 277
σ-finite measure space, 3
signal analysis, 357
simply connected, 304, 310
single-layer potentials, 276
singular inner function, 169
singular integral operator, 588, 599
singular point, 221, 229, 266, 276, 345
singular Riesz potential, 590, 599
singular support, 345
singular values, 140
skew shift, 106, 108, 123
Subject Index

SMP, 202, 203, 223, 224, 230, 231, 232, 233, 234, 235
Sobolev embedding theorem, 570, 573
Sobolev estimates, 570, 571, 573, 608
Sobolev inequality, 523, 544, 583, 588
Sobolev norm, 568
Sobolev spaces, 544, 568, 582, 583, 681
Sobolev spaces for fractional exponent, 566
space of Bessel potentials, 566
space-time bounds, 683
space-time estimates, 682
spectral representation, 620
spectral theorem, 102, 616
spherical Bessel function, 249
spherical coordinates, 198
spherical harmonic, 197, 232, 233, 234, 235, 236
spherical harmonic expansion, 241
spherical harmonic expansion of plane waves, 248
spherical maximal function, 49, 51
square integrable representation, 579
stability of hydrogen, 523
stability of matter, 699
Stahl–Totik theorem, 297
stationary phase, 130
stationary phase ideas, 681
stationary phase method, 674
statistical mechanics, 654
Stein interpolation, 619
Stein–Weiss inequality, 590
Stieltjes measure, 28
Stieltjes transform, 62
Stokes’ theorem, 177, 419
Stone–von Neumann uniqueness theorem, 336, 337, 338
stopping time, 144, 165
Strichartz estimates, 679, 680
strong law of large numbers, 10, 92, 93
strong maximal theorem, 48
strongly mixing, 88
strongly overcomplete family, 390
structure constants, 418
sub-Dirichlet bound, 631
sub-Dirichlet inequality, 630, 632, 638
sub-Markovian, 623
sub-Markovian semigroup, 622, 632
subadditive ergodic theorem, 134
subadditive sequence, 134
subcritical Gabor lattice, 390
subharmonic, 240
subharmonic function, 202, 203, 208
subharmonic function, discontinuous, 240
submartingale, 148, 149, 152, 157, 165
submartingale convergence theorem, 156
submean property, 202, 212, 263, 307
sunrise lemma, 47, 51, 52, 77
supercontractive semigroup, 613, 649
supercontractivity, 646, 653
supercritical Gabor lattice, 390
superharmonic function, 202, 203, 266
superharmonic majorant, 307
supermartingale, 148, 149
surface measure, 181
suspension, 88
symbol, 533, 604
symmetric envelope, 54
symmetric rearrangement, 40
symmetric relation, 8
symmetry, 311
T1 theorem, 603
Tauberian theorem, 686
Taylor series, 233
tempered distribution, 343, 673
tensor power trick, 556
theta function, 391
thin set, 276
Thouless formula, 283, 289, 291, 295
Tomas–Stein theorem, 174, 175, 194, 198
topological dynamical system, 107
topological group, 101
toral automorphisms, 100
trace class, 626
transitive relation, 3
Triebel–Lizorkin space, 583
trigonometric polynomial, 435
<table>
<thead>
<tr>
<th>Subject Index</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>ultracontractive</strong></td>
</tr>
<tr>
<td><strong>ultracontractive semigroup</strong></td>
</tr>
<tr>
<td><strong>ultracontractivity</strong></td>
</tr>
<tr>
<td><strong>unbounded component</strong></td>
</tr>
<tr>
<td><strong>unbounded operator</strong></td>
</tr>
<tr>
<td><strong>uncertainty principle</strong></td>
</tr>
<tr>
<td><strong>uniform boundedness principle</strong></td>
</tr>
<tr>
<td><strong>uniform lattice</strong></td>
</tr>
<tr>
<td><strong>uniform measure</strong></td>
</tr>
<tr>
<td><strong>uniformization theorem</strong></td>
</tr>
<tr>
<td><strong>unimodular</strong></td>
</tr>
<tr>
<td><strong>unimodular group</strong></td>
</tr>
<tr>
<td><strong>unique ergodicity</strong></td>
</tr>
<tr>
<td><strong>uniquely ergodic</strong></td>
</tr>
<tr>
<td><strong>uniquely ergodic measure</strong></td>
</tr>
<tr>
<td><strong>uniqueness for Dirichlet problem</strong></td>
</tr>
<tr>
<td><strong>unit ball</strong></td>
</tr>
<tr>
<td><strong>unitary operator</strong></td>
</tr>
<tr>
<td><strong>upcrossing inequality</strong></td>
</tr>
<tr>
<td><strong>upcrossing methods</strong></td>
</tr>
<tr>
<td><strong>upcrossings</strong></td>
</tr>
<tr>
<td><strong>upper envelope theorem</strong></td>
</tr>
<tr>
<td><strong>upper half-plane</strong></td>
</tr>
<tr>
<td><strong>upper semicontinuous</strong></td>
</tr>
<tr>
<td><strong>upper symbol</strong></td>
</tr>
<tr>
<td>usc</td>
</tr>
<tr>
<td><strong>van der Corput’s difference theorem</strong></td>
</tr>
<tr>
<td><strong>vanishing mean oscillations</strong></td>
</tr>
<tr>
<td><strong>variation</strong></td>
</tr>
<tr>
<td><strong>variational principle for Green’s function</strong></td>
</tr>
<tr>
<td><strong>Varopoulos–Fabes–Stroock theorem</strong></td>
</tr>
<tr>
<td><strong>Verblunsky coefficients</strong></td>
</tr>
<tr>
<td><strong>Vitali’s convergence theorem</strong></td>
</tr>
<tr>
<td><strong>Vitali’s convergence theorem for harmonic functions</strong></td>
</tr>
<tr>
<td><strong>Vitali’s covering lemma</strong></td>
</tr>
<tr>
<td><strong>Vitali’s covering theorem</strong></td>
</tr>
<tr>
<td><strong>Vitali’s theorem</strong></td>
</tr>
<tr>
<td><strong>VMO</strong></td>
</tr>
<tr>
<td><strong>von Neumann ergodic theorem</strong></td>
</tr>
<tr>
<td><strong>von Neumann lattice</strong></td>
</tr>
<tr>
<td><strong>von Neumann trick</strong></td>
</tr>
</tbody>
</table>

**wavefront set** | 347, 350, 371 |
| **wavelet theory** | 357, 433 |
| **wavelets** | 383, 418 |
| **weak barrier** | 224 |
| **weak Hausdorff–Young inequality** | 564 |
| **weak L¹ bound** | 659 |
| **weak L¹ inequality** | 658, 660 |
| **weak mixing** | 85 |
| **weak Stein–Weiss estimate** | 581 |
| **weak Young inequality** | 646, 561, 580 |
| **weak-L¹ bounds** | 594 |
| **weak-¹ topology** | 500 |
| **weakly harmonic function** | 191 |
| **weakly mixing** | 85, 92 |
| **wedding-cake representation** | 28, 36, 40 |
| **Weierstrass approximation theorem** | 230 |
| **Weierstrass P-function** | 392 |
| **Weyl’s criterion** | 98 |
| **Weyl’s equidistribution** | 94 |
| **Weyl’s equidistribution theorem** | 95, 98 |
| **Weyl’s law** | 101 |
| **Weyl’s theorem** | 112 |
| **Widom’s theorem** | 280, 291, 292 |
| **Wigner distribution** | 369, 370 |
| **Wigner–Ville distribution** | 370 |
| **Wirtinger calculus** | 312 |
| **Wirtinger’s theorem** | 122 |
| **Zak transform** | 8, 9, 387, 398, 400, 402 |
| **Zaremba’s criterion** | 230 |
| **zero capacity** | 111, 263 |
| **zero counting measure** | 280, 281 |
| **zeros** | 280 |
| **zonal harmonic** | 238 |
Author Index

Abels, H., 367, 603, 691
Adams, R. A., 583, 652, 691
Agnon, S., 683, 691
Ahlfors, L. V., 298, 691
Aikawa, H., 177, 691
Aizenman, M., 513, 669, 691
Amrein, W. O., 337, 691
Ané, C., 690, 691
Armitage, D., 177, 692
Arnol’d, V. I., 177, 692
Aronszajn, N., 276, 681, 692
Artin, E., 125, 692
Aslaksen, E. W., 387, 692
Aubin, T., 582, 692
Aubry, S., 296, 692
Avez, A., 79, 99, 692
Avila, A., 145, 292, 692
Avron, J., 291, 296, 692
Axler, S., 177, 692
Babenko, K. I., 125, 692
Bacry, H., 401, 402, 692
Báez-Duarte, L., 161, 692
Baggett, L., 403, 692
Bakry, D., 653, 692
Balian, R., 402, 692
Banach, S., 24, 166, 191, 692, 693
Bañuelos, R., 162, 692
Bargmann, V., 385, 387, 693
Bari, N. K., 401, 606, 693
Barut, A. O., 386, 693
Battle, G., 402, 405, 433, 434, 693
Bauer, H., 177, 276, 693
Beals, R., 614, 693
Beardon, A. F., 127, 693
Beckner, W., 336, 652, 693
Ben-Aroya, A., 538, 693
Benedetto, J. J., 335, 693
Benedicks, M., 337, 693
Benford, F., 99, 693
Bennett, C., 556, 658, 693
Berezanskiï, Ju. M., 292, 694
Berezin, F. A., 386, 402, 694
Bergh, J., 556, 683, 694
Bernstein, S., 291, 694
Berthier, A. M., 337, 694
Besicovitch, A. S., 50, 684, 694
Besov, O. V., 583, 694
Beurling, A., 177, 276, 470, 517, 694
Bialynicki-Birula, I., 335, 694
Biane, P., 189, 694
Bienvenu, L., 160, 694
Billingsley, P., 179, 125, 694
Birkhoff, G., 406, 695
Birkhoff, G. D., 65, 70, 82, 125, 406, 694
Bishop, E., 64, 695
Blachman, N. M., 652, 695
Blanchard, Ph., 669, 695
Blatter, C., 383, 695
Bliedtner, J., 177, 695
Bloch, F., 386, 387, 695
Blumenthal, R. M., 177, 695
Böcher, M., 197, 695
Bochi, J., 177, 692
Bochner, S., 574, 603, 695
Author Index

Boggess, A., 433, 695
Bohl, P., 48, 695
Bohr, H. A., 49, 695
Bokobza, J., 367, 733
Boltzman, L., 79, 80, 695
Bonami, A., 337, 652, 695
Boole, G., 513, 696
Boon, M., 402, 696
Borel, E., 97, 696
Bosma, W., 125, 696
Bouligand, G., 231, 696
Bourdon, P., 177, 692
Bourgain, J., 49, 84, 85, 682, 683, 696
Bowen, R., 126, 696
Brascamp, H. J., 563, 696
Bratteli, O., 433, 696
Brelot, M., 177, 231, 273, 274, 276, 696
Breiz, H., 336, 697
Brown, J. R., 65, 697
Burkholder, D. L., 25, 162, 697
Busemann, H., 48, 697
Butera, P., 101, 693
Calderón, A.-P., 36, 83, 276, 387, 542
Calkin, J. W., 581, 697
Callahan, J. J., 17, 697
Calvin, C., 387, 702
Carbery, A., 684, 698
Carey, A. L., 387, 698
Carlen, E. A., 652, 653, 698
Carleson, L., 172, 698
Carmona, R., 294, 698
Cartan, H., 274, 276, 698
Casasnovas, G., 160
Chacon, R. V., 86, 698
Champanoix, D. G., 97, 698
Chang, Y.-C., 684, 701
Chemin, J.-Y., 685, 698
Cho, Y., 172, 698
Choquet, G., 274, 698
Chousionis, V., 603, 698
Christ, M., 172, 603, 654, 698, 699
Christensen, O., 401, 699
Christoffel, E. B., 291, 699
Chung, K. L., 155, 699
Cima, J. A., 489, 699
Clarke, F. H., 652, 691
Conlon, J. G., 669, 699
Constantinescu, C., 177, 699
Copeland, A. H., 97, 699
Cordes, H. O., 367, 614, 699
Cordoba, A., 48, 685, 699
Corne, A., 172, 699
Cotlar, M., 93, 542, 613, 699
Courant, R., 17, 699
Craig, W., 291, 295, 297, 699
Crépel, P., 160, 699
Croft, H. T., 63, 700
Cwikel, M., 63, 699
Cycon, H. L., 294, 700
Dahlberg, B. E. J., 274, 700
Dajani, K., 123, 700
Damanik, D., 293, 700
Damelin, S. B., 401, 700
Darboux, G., 291, 700
Daubechies, I., 401, 403, 433, 434, 700
David, G., 602, 700
Davies, E. B., 336, 682, 650, 655, 700
Davis, B., 162, 514, 693, 701
Davis, K. M., 684, 701
de Bruijn, N. G., 99, 701
de Guzmán, M., 20, 701
de la Vallée Poussin, C., 64, 701
de Leeu, K., 172, 701
de Wolf, R., 654, 693
Del Pino, M., 582, 701
del Rio, R., 514, 701
Dellacherie, C., 161, 701
Demange, B., 337, 695
Demengel, F., 583, 701
Demengel, G., 583, 701
Denisov, S. A., 293, 701
Denjoy, A., 99, 701
Denker, J., 373, 701
Deny, J., 172, 274, 276, 698, 695, 701
Derriennic, Y., 155, 702
Deuschel, J-D., 652, 654, 702
Devore, R. A., 534, 694
Diaconis, P., 653, 702
DiBenedetto, E., 501, 702
Dirichlet, P. G. L., 273
Doebelin, W., 124, 702
Dolbeault, J., 582, 701
Donoho, D., 339, 702
Doob, J. L., 84, 160, 163, 165, 177, 276, 702
Duffin, R. J., 401, 403, 702
Duistermaat, J. J., 85, 367, 702
Dunford, N., 80, 702
Duren, P., 337, 401, 403, 702
Durrett, R., 162, 163, 702
Dvir, Z., 685, 702
Dym, H., 537, 702
Egorov, Y. V., 367, 368, 703
Ehrenfest, P., 80, 703
Ehrenfest, T., 80, 703
Einsiedler, M., 79, 123, 126, 703
Ekholm, T., 669, 703
Emerson, R. W., 1, 703
Émery, M., 652, 692, 703
Erdős, P., 97, 291, 292, 699, 703
Ermenko, A. E., 218, 703
Eskin, G. I., 368, 703
Essén, M., 177, 488, 691, 703
Evans, G. C., 273, 274, 703
Faber, G., 291, 703
Fabes, E. B., 653, 704
Faris, W. G., 654, 704
Farkas, H. M., 316, 704
Fatou, P., 653, 704
Etkin, P. G., 651, 704
Fefferman, R., 48, 699
Feichtinger, H. G., 390, 704
Fejér, L., 334, 704
Feller, W., 48, 699
Figalli, A., 654, 704
Fifirtin, A., 294, 723
Findley, E., 292, 703
Flandrin, P., 337, 704
Folland, G. B., 333, 338, 534, 342, 704
Ford, L. R., 516, 705
Fournier, J. F., 583, 691
Frank, R. L., 564, 669, 670, 703, 705
Fréchet, M., 31, 705
Freund, G., 292, 705
Friedrichs, K. O., 367, 581, 705
Fristedt, B., 162, 705
Froese, R. G., 294, 700
Frostman, O., 273, 274, 276, 705
Fukushima, M., 177, 276, 705
Füredi, Z., 50, 705
Furstenberg, H., 84, 128, 125, 146, 706, 705, 706
Gabor, D., 334, 386, 401, 706
Gagliardo, E., 682, 681, 706
Gamelin, T. W., 310, 706
Garban, C., 650, 706
Gardiner, S. J., 177, 692
Garling, D. J. H., 177, 1866, 547, 691, 650, 706
Garnett, J. B., 273, 338, 690, 706
Garsia, A. M., 49, 508, 690, 91, 161, 706
Gauss, C. F., 121, 197, 706
Gelfand, I. M., 102, 706
Genčay, C., 333, 706
Getoor, R. K., 177, 692
Gilbarg, D., 177, 276, 706
Gillmore, R., 356, 706
Ginibre, J., 683, 706
Giradello, L., 386, 401, 693
Glasner, E., 99, 706
Glauber, R. J., 386, 386, 707
Glimm, J., 631, 651, 656, 707
Godement, R., 386, 707
Gohberg, I., 603, 707
Goldberg, M., 653, 707
Gordon, A., 296, 707
Grafakos, L., 534, 535, 603, 652, 684, 707
Gray, L., 162, 705
Green, B., 683, 707
Green, G., 121, 127, 707
Greenleaf, A., 652, 707
Gröchenig, K., 390, 707
Gross, L., 650, 652, 654, 656, 704, 707
Grossmann, A., 386, 880, 587, 401, 102, 692, 700, 708
Guenicher, J., 99, 708
Guionnet, A., 322, 650, 654, 708
Gundy, R. F., 162, 697
Haar, A., 433, 708
Hadamard, J., 437, 652, 708
Halmos, P. R., 177, 708
Hammersley, J. M., 310, 708
Han, Q., 177, 708
Hansen, W., 177, 695
Hardy, G. H., 56, 10, 52, 98, 213, 335, 337, 144, 488, 684, 685, 557, 559, 656, 709
<table>
<thead>
<tr>
<th>Author</th>
<th>Page References</th>
</tr>
</thead>
<tbody>
<tr>
<td>Harnack, A.</td>
<td>198, 709</td>
</tr>
<tr>
<td>Haroske, D.</td>
<td>583, 709</td>
</tr>
<tr>
<td>Hartman, P.</td>
<td>83, 536, 709</td>
</tr>
<tr>
<td>Hartogs, F.</td>
<td>213, 709</td>
</tr>
<tr>
<td>Hasselblatt, B.</td>
<td>83, 713</td>
</tr>
<tr>
<td>Havin, V.</td>
<td>333, 709</td>
</tr>
<tr>
<td>Hayman, W. K.</td>
<td>177, 253, 709</td>
</tr>
<tr>
<td>Hedlund, G. A.</td>
<td>125, 709</td>
</tr>
<tr>
<td>Heil, C.</td>
<td>401, 403, 710</td>
</tr>
<tr>
<td>Heisenberg, W.</td>
<td>333, 710</td>
</tr>
<tr>
<td>Helms, L.</td>
<td>177, 710</td>
</tr>
<tr>
<td>Hensley, D.</td>
<td>123, 125, 710</td>
</tr>
<tr>
<td>Herbert, D.</td>
<td>291, 710</td>
</tr>
<tr>
<td>Herbst, I. W.</td>
<td>664, 710</td>
</tr>
<tr>
<td>Herglotz, G.</td>
<td>468, 513, 710</td>
</tr>
<tr>
<td>Hernández, E.</td>
<td>433, 710</td>
</tr>
<tr>
<td>Herz, C.</td>
<td>684, 710</td>
</tr>
<tr>
<td>Hilbert, D.</td>
<td>273, 316, 387, 710</td>
</tr>
<tr>
<td>Hirschman, I. L.</td>
<td>335, 710</td>
</tr>
<tr>
<td>Hoegh-Krohn, R.</td>
<td>652, 749</td>
</tr>
<tr>
<td>Hollenbeck, B.</td>
<td>489, 710</td>
</tr>
<tr>
<td>Holmes, P.</td>
<td>399, 703</td>
</tr>
<tr>
<td>Hopf, E.</td>
<td>81, 89, 89, 91, 125, 710</td>
</tr>
<tr>
<td>Hörmander, L.</td>
<td>218, 350, 366, 387, 320</td>
</tr>
<tr>
<td>Horváth, J.</td>
<td>614, 711</td>
</tr>
<tr>
<td>Howe, R.</td>
<td>330, 365, 614, 711</td>
</tr>
<tr>
<td>Hruščev, S. V.</td>
<td>514, 711</td>
</tr>
<tr>
<td>Hubbard, B. B.</td>
<td>341, 711</td>
</tr>
<tr>
<td>Hundertmark, D.</td>
<td>369, 670, 711</td>
</tr>
<tr>
<td>Hunt, G. A.</td>
<td>177, 240, 711</td>
</tr>
<tr>
<td>Hunt, R. A.</td>
<td>179, 550, 711</td>
</tr>
<tr>
<td>Husimi, K.</td>
<td>586, 712</td>
</tr>
<tr>
<td>Hwang, I. L.</td>
<td>614, 712</td>
</tr>
<tr>
<td>Indrei, E.</td>
<td>654, 712</td>
</tr>
<tr>
<td>Ionescu Tulcea, A.</td>
<td>161, 683, 712</td>
</tr>
<tr>
<td>Ionescu Tulcea, C.</td>
<td>161, 712</td>
</tr>
<tr>
<td>Iosevich, A.</td>
<td>682, 712</td>
</tr>
<tr>
<td>Iosifescu, M.</td>
<td>128, 712</td>
</tr>
<tr>
<td>Ishii, K.</td>
<td>291, 712</td>
</tr>
<tr>
<td>Issac, R.</td>
<td>161, 712</td>
</tr>
<tr>
<td>Izu, S.</td>
<td>337, 339, 712</td>
</tr>
<tr>
<td>Jaffé, A.</td>
<td>651, 654, 707</td>
</tr>
<tr>
<td>Jager, H.</td>
<td>125, 696</td>
</tr>
<tr>
<td>Janson, S.</td>
<td>653, 712</td>
</tr>
<tr>
<td>Janssen, A. J. E. M.</td>
<td>402, 403, 700, 712</td>
</tr>
<tr>
<td>Jentzsch, R.</td>
<td>654, 712</td>
</tr>
<tr>
<td>Jerison, M.</td>
<td>161, 712</td>
</tr>
<tr>
<td>Jessen, B.</td>
<td>48, 712</td>
</tr>
<tr>
<td>Jitomirskaya, S.</td>
<td>294, 296, 514, 701, 712</td>
</tr>
<tr>
<td>John, F.</td>
<td>177, 534, 691, 712</td>
</tr>
<tr>
<td>Jones, R.</td>
<td>291, 710</td>
</tr>
<tr>
<td>Jones, R. L.</td>
<td>83, 713</td>
</tr>
<tr>
<td>Jørgensen, P.</td>
<td>433, 696</td>
</tr>
<tr>
<td>Jorick, B.</td>
<td>333, 709</td>
</tr>
<tr>
<td>Journé, J.-L.</td>
<td>602, 700</td>
</tr>
<tr>
<td>Kac, M.</td>
<td>85, 713</td>
</tr>
<tr>
<td>Kadec, M. I.</td>
<td>106, 713</td>
</tr>
<tr>
<td>Kahane, J.-P.</td>
<td>337, 713</td>
</tr>
<tr>
<td>Kahn, J.</td>
<td>652, 654, 713</td>
</tr>
<tr>
<td>Kaiser, G.</td>
<td>339, 713</td>
</tr>
<tr>
<td>Kakeya, S.</td>
<td>684, 713</td>
</tr>
<tr>
<td>Kakutani, S.</td>
<td>83, 713</td>
</tr>
<tr>
<td>Kalai, G.</td>
<td>652, 654, 713</td>
</tr>
<tr>
<td>Kalikow, S.</td>
<td>79, 84, 97, 713</td>
</tr>
<tr>
<td>Kamae, T.</td>
<td>145, 713</td>
</tr>
<tr>
<td>Karamata, J.</td>
<td>689, 713</td>
</tr>
<tr>
<td>Karatzas, I.</td>
<td>101, 713</td>
</tr>
<tr>
<td>Kato, T.</td>
<td>337, 614, 713</td>
</tr>
<tr>
<td>Katok, A.</td>
<td>83, 713</td>
</tr>
<tr>
<td>Katok, S.</td>
<td>127, 713</td>
</tr>
<tr>
<td>Katz, N. H.</td>
<td>685, 696, 713</td>
</tr>
<tr>
<td>Katznelson, Y.</td>
<td>145, 639, 713</td>
</tr>
<tr>
<td>Kaufman, R.</td>
<td>82, 713</td>
</tr>
<tr>
<td>Kawohl, B.</td>
<td>86, 713</td>
</tr>
<tr>
<td>Keane, M.</td>
<td>86, 128, 124, 135, 714</td>
</tr>
<tr>
<td>Keel, M.</td>
<td>653, 713</td>
</tr>
<tr>
<td>Keller, W.</td>
<td>333, 714</td>
</tr>
<tr>
<td>Kellogg, O. D.</td>
<td>177, 273, 274, 714</td>
</tr>
<tr>
<td>Kelvin, Lord.</td>
<td>196, 273</td>
</tr>
<tr>
<td>Kemp, T.</td>
<td>653, 714</td>
</tr>
<tr>
<td>Kennard, E. H.</td>
<td>835, 714</td>
</tr>
<tr>
<td>Kennedy, P. B.</td>
<td>177, 253, 709</td>
</tr>
<tr>
<td>Kesavan, S.</td>
<td>30, 714</td>
</tr>
<tr>
<td>Kesten, H.</td>
<td>145, 706</td>
</tr>
<tr>
<td>Khinchin, A.</td>
<td>83, 90, 123, 124, 714</td>
</tr>
<tr>
<td>Killip, R.</td>
<td>293, 700</td>
</tr>
<tr>
<td>King, J. L.</td>
<td>99, 714</td>
</tr>
<tr>
<td>Kingman, J. F. C.</td>
<td>145, 714</td>
</tr>
<tr>
<td>Kirsch, W.</td>
<td>293, 700</td>
</tr>
<tr>
<td>Kiselev, A.</td>
<td>172, 698, 699, 714</td>
</tr>
<tr>
<td>Klauder, J. R.</td>
<td>385, 387, 401, 692, 693, 714</td>
</tr>
<tr>
<td>Knapp, A. W.</td>
<td>613, 714</td>
</tr>
<tr>
<td>Knopp, K.</td>
<td>125, 714</td>
</tr>
<tr>
<td>Koebe, P.</td>
<td>197, 316, 715</td>
</tr>
<tr>
<td>Koh, E.</td>
<td>172, 698</td>
</tr>
<tr>
<td>Kohn, J. J.</td>
<td>367, 715</td>
</tr>
<tr>
<td>Kolmogorov, A.</td>
<td>35, 65, 79, 162, 167</td>
</tr>
<tr>
<td>Kondrachov, V. I.</td>
<td>582, 715</td>
</tr>
</tbody>
</table>
Koopman, B. O., 79, 82, 124, 695, 710
Koosis, P., 439, 554, 710
Kotani, S., 296, 715
Kra, I., 310, 703
Kraikamp, C., 123, 700, 712
Krasnosel’ski, M., 36, 715
Krein, S. G., 556, 715
Krengel, U., 79, 145, 715
Kronecker, L., 98, 715
Krupnik, N., 603, 707
Kufner, A., 336, 557, 715, 722
Kuijpers, L., 123, 715
Kumanogo-go, H., 367, 716
Kuttler, K., 50, 716
Kuzmin, R., 124, 716
Łaba, L., 685, 716
Lacey, M., 172, 716
Lacroix, J., 294, 698
Lagrange, J.-L., 273
Lakey, J., 337, 339, 712
Landau, E., 557, 716
Landau, H. J., 337, 338, 716
Landkof, N. S., 177, 276, 716
Laplace, S., 124, 249, 273, 716
Laptev, A., 340, 697, 703, 711, 716
Last, Y., 292, 694, 697, 716
Lax, P. D., 367, 705
Lebesgue, H., 59, 231, 273, 716, 717
Lee, S., 172, 692, 698, 717
Legendre, A. M., 249, 273, 717
Lemarié, P. G., 331, 717
Lenard, A., 344, 717
Leoni, G., 583, 717
Levin, D., 669, 717
Levin, E., 292, 717
Lévy, P., 124, 192, 717
Lewis, J. L., 618, 703, 717
Li, P., 669, 717
Lieber, H. E., 36, 275, 356, 503, 554, 653, 669, 691, 696, 698, 703, 711, 717
Liggett, T. M., 145, 161, 718
Lin, F., 177, 708
Lindley, D., 250, 18
Linial, N., 652, 654, 713
Lions, J.-L., 650, 718
Littlewood, J. E., 30, 50, 52, 98, 213
458, 454, 488, 557, 592, 564, 603
409, 718
Lizorkin, P. I., 583, 718
Loeb, P. A., 65, 703
Löfström, J., 556, 583, 694
Loomis, L. H., 513, 715
López, F., 556, 718
Lorentz, G. G., 36, 37, 556, 718
Loss, M., 36, 275, 554, 717
Loupitas, G., 308, 703
Low, F. E., 409, 718
Lu, G., 682, 712
Lubinsky, D. S., 292, 717, 718
Luttinger, J. M., 503, 696
Lyubarski, Y. I., 101, 718
MacRobert, T. M., 177, 718
Maggi, F., 654, 704
Makarov, N. G., 274, 718
Maligranda, L., 336, 557, 715
Mallat, S., 439, 434, 719
Mansuy, R., 100, 719
Marcinkiewicz, J., 15, 603, 712, 719
Marcon, D., 654, 712
Marcus, M., 336, 697
Marshall, D. E., 274, 703
Martin, R. S., 274, 719
Maslova, V. P., 368, 719
Matheson, A. L., 489, 699
Maz’ya, V., 683, 719
McCUTCHEON, R., 79, 97, 713
McKean, H. P., 337, 702
Melas, A. D., 49, 719
Menshov, D., 172, 719
Meyer, P.-A., 161, 177, 273, 701, 719
Meyer, Y., 433, 434, 614, 699, 719, 720
Michlin, S. G., 603, 720
Mizuta, Y., 177, 720
Mockenhaupt, G., 15, 683, 684, 720
Montanaro, A., 654, 720
Montgomery, H. L., 128, 129, 720
Montiel, S., 177, 720
Morawetz, C. S., 682, 720
Morgan, G. W., 537, 720
Morgan, J., 654, 720
Morlet, J., 386, 387, 720
Morrey, Ch. B., Jr., 581, 720
Morse, A. P., 50, 720
Morse, M., 83
Mosser, J., 653, 720
Moyal, J. E., 370, 386, 720
Mueller, P., 407, 433, 733
Muirhead, R. F., 336, 720
Muscalu, C., 682, 721
Muskhelishvili, N. I., 603, 721
Mycielski, J., 335, 694
Nadkarni, M. G., 79, 721
Najmi, A-H., 433, 721
Narcowich, F., 433, 695
Nash, J., 582, 653, 721
Nason, G. P., 436, 721
Naumann, J., 551, 721
Nazarov, F. L., 337, 721
Nehari, Z., 538, 721
Nelson, E., 197, 651, 652, 721
Netuka, I., 197, 721
Neumann, C., 273, 721
Nevanlinna, F., 444, 721
Nevanlinna, R., 197, 444, 457, 513, 721
Neveu, J., 161, 722
Newcomb, S., 100, 722
Niederreiter, H., 123, 715
Nievergelt, Y., 433, 722
Nigrini, M. J., 100, 722
Nikodym, O., 531, 722
Nirenberg, L., 352, 367, 534, 582, 712

db
Noether, E., 543
O'Neil, R., 557, 722
Oguntuase, J. A., 557, 722
Olkiewicz, R., 553, 722
Opic, B., 336, 657, 722
Orlicz, W., 562, 722
Ornstein, D. S., 65, 83, 97, 698, 722
Ortega-Cerdà, J., 562, 722
Oscledec, V. I., 145, 722
Otto, F., 654, 722
Palais, R. S., 367, 722
Paley, R. E. A. C., 406, 164, 603, 718
Parry, W., 79, 123, 723
Pastur, L. A., 294, 723
Paul, T., 386, 708
Peetre, J., 556, 715, 723
Peller, V. V., 536, 723
Peller, V. V., 536, 723
Percival, D. B., 536, 723
Perelman, G., 654, 723
Perelomov, A. M., 386, 101, 723
Perron, O., 212, 251, 273, 723
Persson, L. E., 536, 551, 723, 722
Petersen, K., 79, 83, 115, 714, 723
Philipp, W., 123, 723
Philipp, J., 587, 723
Phillips, R., 53, 723
Phong, D. H., 386, 703
Picard, É., 497, 723
Pichorides, S. K., 538, 723
Pick, G., 513, 723
Pinsky, M. A., 433, 723
Plamenevskii, B. A., 367, 723
Plenljl, J., 459, 724
Plessner, A., 403, 723
Poincaré, H., 80, 85, 212, 231, 273, 275
Port, S. C., 177, 723
Post, K. A., 99, 701
Pratelli, A., 653, 724
Privatov, L. I., 380, 724
Prössdorf, S., 403, 720
Quéfflec, M., 97, 723
Rademacher, H., 400, 725
Ragunathan, M. S., 143, 725
Rakhmanov, E. A., 292, 725
Ramey, W., 177, 692
Ransford, T., 177, 723
Rao, M. R., 161, 723
Rauzy, G., 123, 725
Reed, M., 654, 653, 725
Regev, O., 654, 693
Relli, F., 382, 725
Remling, C., 293, 725
Renzende, J., 169, 725
Rickman, S., 218, 725
Riemann, G. F. B., 197, 723
Riesz, F., 406, 313, 513, 522, 523, 723
Riesz, M., 400, 513, 522, 523, 723
Robertson, H. P., 82, 334, 720
Robin, G., 274, 726
Roeck, A. M., 123, 720
Rogers, C. A., 564, 726
Romberg, J., 389, 698
Ros, A., 17, 720
Rosen, J., 553, 726
Rosenblatt, J. M., 84, 718
Ross, W. T., 589, 699
Author Index

Rota, G-C., 161, 406, 695, 726
Rothaus, O. S., 692, 726
Royer, G., 650, 726
Rozenbljum, G. V., 669, 726
Ruch, D.-K., 433, 726
Rudin, W., 439, 172, 701, 726
Ruelle, D., 145, 726
Rumin, M., 670, 727
Runst, T., 583, 727
Rutickii, Ya., 36, 715
Ryll-Nardzewski, C., 124, 727
Sadosky, C., 603, 614, 727
Sagher, Y., 534, 700
Saint-Raymond, X., 367, 727
Saks, S., 64, 727
Saloff-Coste, L., 653, 702
Sarason, D., 534, 727
Sarason, I., 385, 652, 683, 727
Saso, M., 367, 727
Schaeffer, A. C., 401, 103, 702
Schlag, W., 682, 683, 727
Schrödinger, E., 334, 727
Schulze, B.-W., 670, 727
Schur, I., 488, 727
Schwartz, J. T., 684, 720
Schwartz, H. A., 250, 727
Seeger, A., 694, 720
Seelye, R., 687, 727
Segal, I. E., 385, 652, 683, 727
Seip, K., 686, 722, 728
Seiringer, R., 669, 705
Seltuk, F., 488, 726
Schweitzer, J. T., 687, 727
Schwarz, H. A., 250, 727
Seeger, A., 694, 720
Seelye, R., 687, 727
Segal, I. E., 385, 652, 683, 727
Seip, K., 686, 722, 728
Seiringer, R., 669, 705
Seltuk, F., 488, 726
Semenov, E. M., 356, 714
Series, C., 126, 696, 728
Shafer, G., 104, 724
Shakarchi, R., 387, 682, 730
Shannon, C., 343, 728
Sharples, R., 651, 656, 658, 694
Shelley, P. B., 319, 728
Shen, A., 690, 724
Shreve, S., 101, 713
Shubin, M. A., 367, 368, 371, 728
Shvartsman, P., 434, 700
Sickel, W., 533, 727
Sierpinsinis, W., 97, 73, 728
Silva, C. E., 79, 728
Silverstein, M. L., 162, 697
Simon, B., 36, 127, 146, 197, 250
201, 207, 383, 540, 850, 187, 172
514, 583, 640, 841, 669, 676, 698
689, 692, 699, 701, 704, 708, 711
712, 716, 717, 725, 728, 729

Sinai, Ya. G., 161, 726
Sitaram, A., 333, 338, 342, 704
Slepian, D., 333, 729
Smirnov, V. I., 170, 726
Smith, C., 250, 729
Smith, K. T., 276, 681, 692
Smith, P. A., 60, 695
Sobolev, S. L., 562, 582, 729, 730
Soddi, M. L., 218, 703
Sogge, C. D., 664, 684, 720, 730
Solomyak, M., 340, 669, 716, 717
Song, R., 102, 701
Spanne, S., 189, 730
Spitzer, F., 163, 730
Stahl, H., 291, 293, 730
Stam, A. J., 652, 730
Stark, P., 339, 702
Steele, M. J., 145, 730
Stein, J., 650, 706
Stein, E. M., 25, 48, 49, 251, 368, 487
189, 513, 514, 534, 563, 564, 601
603, 613, 681, 682, 704, 705, 714
730, 731
Stein, P., 188, 192, 731
Stone, C. J., 177, 731
Stone, M. H., 51, 83, 380, 731
Stauss, W. A., 482, 720
Strichartz, R. S., 682, 683, 731
Strohmer, T., 390, 704
Strömberg, J.-O., 367, 731
Strock, D. W., 650, 652, 653, 702, 704
731
Strobe, B., 669, 695
Sudarshan, E. C. G., 385, 386, 731
Sullivan, J. M., 50, 731
Szegő, G., 201, 731
Szűsz, P., 124, 726
Tait, P. G., 190, 250, 732
Talenti, G., 682, 731
Tao, T., 41, 339, 555, 557, 682, 685
696, 698, 713, 714, 731
Tartar, L., 583, 731
Taylor, M., 367, 731
Thiele, C., 172, 716
Thirring, W., 669, 714, 718
Thomas, L. E., 669, 711
Thompson, S. P., 250, 732
Thompson, W., 196, 250, 723, 732
Thouless, D. J., 291, 732
Tian, G., 651, 720
Tolsa, X., 603, 698
Tomas, P. A., 682, 732
Tonelli, L., 581, 732
Totik, V., 391, 293, 730, 732
Trèves, F., 367, 732
Triebel, H., 357, 583, 709, 732
Trudinger, N. S., 177, 276, 700
Tsuij, M., 177, 274, 732
Turán, P., 291, 292, 708
Ullman, J. L., 291, 732
Unterberger, A., 367, 733
Vaillancourt, R., 613, 697
Van Assche, W., 292, 733
van den Berg, J. C., 448, 724
van der Corput, J. G., 123, 733
Van Fleet, P.-J., 433, 726
Vargas, A., 682, 717
Vasilescu, F., 274, 733
Veblen O., 82
Velo, G., 683, 708
Verbitsky, I., 489, 710
Veselý, J., 197, 724
Vidakovic, B., 407, 433, 733
Villani, C., 654, 722
Vigne, J., 560, 728
Vinogradov, S. A., 514, 711
Vitali, G., 419, 513, 733
von Neumann, J., 80, 82, 129, 336, 410
Walden, A. T., 438, 728
Wallstén, R., 401, 728
Walnut, D. F., 333, 133, 410, 733
Walsh, J. L., 83, 291, 733
Walters, P., 700
Ward, T., 700, 716, 733
Warzel, S., 513, 691
Watson, C., 290, 734
Weidl, T., 660, 670, 703, 711, 716, 733
Weil, A., 492, 734
Weiss, B., 83, 81, 99, 145, 706, 713, 722
Weiss, G., 291, 583, 733, 734
Weissler, F. B., 652, 734
Welsh, D. J. A., 177, 708
Wermer, J., 747
Weyl, H., 485, 722
Whitcher, B., 583, 700
Whitney, H., 83
Widder, D., 83
Widom, H., 290, 633, 710, 734
Wiedijk, F., 125, 734
Wiener, N., 469, 83, 84, 231, 273, 274, 278
Wierdl, M., 84, 713
Wigner, E. P., 83
Wilk, I., 533, 724
Willard, W., Jr., 401, 700
Wirsing, E., 125, 733
Wise, M. N., 250, 729
Wolff, T., 685, 734
Wynken, M. F., 291, 732, 734
Xu, C.-J., 685, 698
Yau, S. T., 669, 717
Yosida, K., 83, 734
Young, R. M., 401, 406, 734
Yukich, J. E., 652, 703
Zaanen, A. C., 30, 735
Zak, J., 401, 402, 689, 696, 735
Zaremba, S., 291, 723, 735
Zegarlinski, B., 622, 650, 653, 654, 708
Zhang, Q. S., 654, 735
Zhou, Z.-F., 653, 755
Zichenko, M., 513, 724
Ziegler, L., 291, 732
Zucker, I. J., 402, 686
Zund, J. D., 81, 735
Zygmund, A., 30, 45, 172, 463, 464, 481
Zygmund, E., 556, 601, 604, 697, 712, 723, 735
Zenan, A. C., 36, 735
Index of Capsule Biographies

Birkhoff, G. D., 82
Calderón, A., 601
Cotlar, M., 613
Hardy, G. H., 46
Hörmander, L., 368
John, F., 534
Kelvin, Lord, 250
Littlewood, J. E., 47
Marcinkiewicz, J., 556
Riesz, M., 489
Zygmund, A., 601
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