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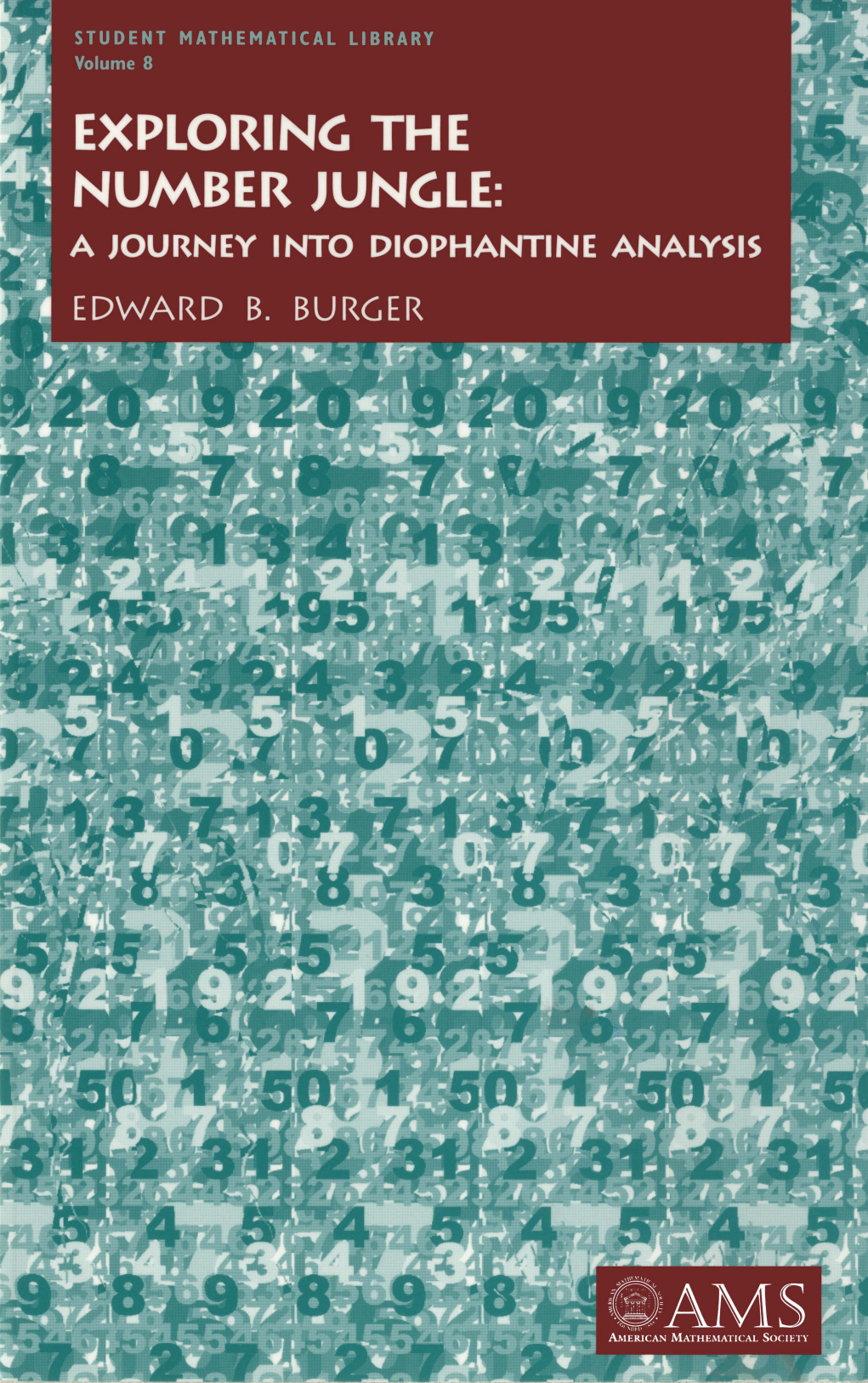
EXPLORING THE NUMBER JUNGLE:

A JOURNEY INTO DIOPHANTINE ANALYSIS

EDWARD B. BURGER



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EXPLORING THE NUMBER JUNGLE:
A JOURNEY INTO DIOPHANTINE ANALYSIS

EDWARD B. BURGER

This highly readable book brings to life the fundamental ideas and theorems from diophantine approximation and equations, geometry of numbers, diophantine geometry, and p-adic analysis. Through his engaging style, the author creates an inviting atmosphere where readers actively build intuition, develop ideas, and prove results as they journey through some beautiful areas of number theory. The minimal background requirements and the author's fresh approach make this book enjoyable and accessible to a wide range of students, mathematicians, and fans of number theory.

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EDWARD B. BURGER is a professor of mathematics at Williams College. He has published widely in number theory, is the author of several virtual, video mathematics textbooks, and co-author of *The Heart of Mathematics: An invitation to effective thinking*. He is also known for his inspiring and enlightening mathematical performances and is rumored to have made appearances as a stand-up comedian. Professor Burger is the recipient of The Mathematical Association of America's national Haimo Award for Distinguished Teaching, and was named an MAA Polya Lecturer.

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