Inversion Theory and Conformal Mapping

David E. Blair
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About the cover: The picture on the cover is M. C. Escher’s “Hand with Reflecting Sphere”, a self-portrait of the artist. Reflection (inversion) in a sphere is a conformal map; thus, while distances are distorted, angles are preserved and the image is recognizable. It is for this reason that we commonly use spherical mirrors, e.g., the right-hand rear view mirror on an automobile.
To Marie and Matthew
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It is rarely taught in an undergraduate, or even graduate, curriculum that the only conformal maps in Euclidean space of dimension greater than 2 are those generated by similarities and inversions (reflections) in spheres. This contrasts with the abundance of conformal maps in the plane, a fact which is taught in most complex analysis courses. The principal aim of this text is to give a treatment of this paucity of conformal maps in higher dimensions. The result was proved in 1850 in dimension 3 by J. Liouville [22]. In Chapter 5 of the present text we give a proof in general dimension due to R. Nevanlinna [26] and in Chapter 6 give a differential geometric proof in dimension 3 which is often regarded as the classical proof, though it is not Liouville’s proof. For completeness, in Chapter 4 we develop enough complex analysis to prove the abundance of conformal maps in the plane.

In addition this book develops inversion theory as a subject along with the auxiliary theme of “circle preserving maps”.

The text as presented here is at the advanced undergraduate level and is suitable for a “capstone course”, topics course, senior seminar, independent study, etc. The author has successfully used this material for capstone courses at Michigan State University. One particular feature is the inclusion of the paper on circle preserving transformations by C. Carathéodory [6]. This paper divides itself up nicely into small sections, and students were asked to present the paper to the
class. This turned out to be an enjoyable and profitable experience for the students. When there were more than enough students in the class for this exercise, some of the students presented Section 2.8.

The author expresses his appreciation to Dr. Edward Dunne and the production staff of the American Mathematical Society for their kind assistance in producing this book.
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It is rarely taught in an undergraduate or even graduate curriculum that the only conformal maps in Euclidean space of dimension greater than two are those generated by similarities and inversions in spheres. This is in stark contrast to the wealth of conformal maps in the plane.

The principal aim of this text is to give a treatment of this paucity of conformal maps in higher dimensions. The exposition includes both an analytic proof in general dimension and a differential-geometric proof in dimension three. For completeness, enough complex analysis is developed to prove the abundance of conformal maps in the plane. In addition, the book develops inversion theory as a subject, along with the auxiliary theme of circle-preserving maps. A particular feature is the inclusion of a paper by Carathéodory with the remarkable result that any circle-preserving transformation is necessarily a Möbius transformation—not even the continuity of the transformation is assumed.

The text is at the advanced undergraduate level and is suitable for a capstone course, topics course, senior seminar or independent study. Students and readers with university courses in differential geometry or complex analysis bring with them background to build on, but such courses are not essential prerequisites.