STUDENT MATHEMATICAL LIBRARY
Volume 12

Problems in Mathematical Analysis II

Continuity and Differentiation

W. J. Kaczor
M. T. Nowak

AMS
American Mathematical Society

http://dx.doi.org/10.1090/stml/012

Selected Titles in This Series

12 **W. J. Kaczor and M. T. Nowak,** Problems in mathematical analysis II: Continuity and differentiation, 2001

11 **Michael Mesterton-Gibbons,** An introduction to game-theoretic modelling, 2000

10 **John Oprea,** The mathematics of soap films: Explorations with Maple®, 2000

9 **David E. Blair,** Inversion theory and conformal mapping, 2000

8 **Edward B. Burger,** Exploring the number jungle: A journey into diophantine analysis, 2000

7 **Judy L. Walker,** Codes and curves, 2000

6 **Gérald Tenenbaum and Michel Mendès France,** The prime numbers and their distribution, 2000

5 **Alexander Mehlmann,** The game's afoot! Game theory in myth and paradox, 2000

4 **W. J. Kaczor and M. T. Nowak,** Problems in mathematical analysis I: Real numbers, sequences and series, 2000

3 **Roger Knobel,** An introduction to the mathematical theory of waves, 2000

2 **Gregory F. Lawler and Lester N. Coyle,** Lectures on contemporary probability, 1999

1 **Charles Radin,** Miles of tiles, 1999

Problems in Mathematical Analysis II

Continuity and Differentiation

STUDENT MATHEMATICAL LIBRARY
Volume 12

Problems in Mathematical Analysis II

Continuity and Differentiation

W. J. Kaczor
M. T. Nowak

American Mathematical Society

Originally published in Polish, as
Zadania z Analizy Matematycznej. Część Druga
Funkcje Jednej Zmiennej—Rachunek Różniczowy

Translated, revised and augmented by the authors.

2000 *Mathematics Subject Classification*. Primary 00A07;
Secondary 26A06, 26A15, 26A24.

Library of Congress Cataloging-in-Publication Data

Kaczor, W. J. (Wieslawa J.), 1949–
[Zadania z analizy matematycznej. English]
Problems in mathematical analysis. I. Real numbers, sequences and series / W. J. Kaczor, M. T. Nowak.
p. cm. — (Student mathematical library, ISSN 1520-9121 ; v. 4)
Includes bibliographical references.
ISBN 0-8218-2050-8 (softcover ; alk. paper)
1. Mathematical analysis I. Nowak, M. T. (Maria T.), 1951– II. Title. III. Series.
QA300K32513 2000
515′.076–dc21 99-087039

---

10 9 8 7 6 5 4 3 2 1 06 05 04 03 02 01

Contents

Preface

This is the second volume of a planned series of books of problems in mathematical analysis. The book deals with real functions of one real variable, except for Section 1.7 where functions in metric spaces are discussed. Like the first volume, *Problems in Mathematical Analysis I, Real Numbers, Sequences and Series*, the book is divided into two parts. The first part is a collection of exercises and problems, and the second contains their solutions. Although often various solutions of a given problem are possible, we present here only one. Moreover, problems are divided into sections according to the methods of their solutions. For example, if a problem is in the section *Convex Functions* it means that in its solution properties of convex functions are used. While each section begins with relatively simple exercises, one can still find quite challenging problems, some of which are actually theorems. Although the book is intended mainly for mathematics students, it covers material that can be incorporated by teachers into their lectures or be used for seminar discussions. For example, following Steven Roman (Amer. Math. Monthly, 87 (1980), pp. 805-809), we present a proof of the well known Faà di Bruno formula for the nth derivative of the composition of two functions. Applications of this formula to real analytic functions given in Chapter 3 are mainly borrowed from the book *A Primer of Real Analytic Functions* by Steven G. Krantz and Harold R. Parks. In fact, we found this book so stimulating that we could not resist borrowing a few theorems from it. We would like also to mention here a generalization of Tauber's theorem due to Hardy and Littlewood. The proof of this result that we give is based on Karamata's paper (Math. Zeitschrift, 2 (1918)).

Many problems have been borrowed freely from problem sections of journals like the American Mathematical Monthly, Mathematics Today (Russian) and Delta (Polish), and from many textbooks and problem books. The complete list of books is given in the bibliography. As in the first volume, it was beyond our scope to trace all original sources, and we offer our sincere apologies if we have overlooked some contributions.

All the notations and definitions used in this volume are standard. However, in an effort to make the book consistent and to avoid ambiguity, we have included a list of notations and definitions. In making references we write, for example, 1.2.33 or I, 1.2.33, which denotes the number of the problem in this volume or in Volume I, respectively.

We owe much to many friends and colleagues with whom we have had many fruitful discussions. Special mention should be made, however, of Tadeusz Kuczumow for suggestions of several problems and solutions, and of Witold Rzymowski for making his manuscript [28] available to us. We are very grateful to Armen Grigoryan, Małgorzata Koter-Mórgowska, Stanisław Prus and Jadwiga Zygmunt for drawing the figures and for their help with incorporating them into the text. We are deeply indebted to Professor Richard J. Libera, University of Delaware, for his unceasing help with the English translation and for his valuable suggestions and corrections which we feel have greatly improved both the form and the content of the two volumes. Finally, we would like to thank the staff at the AMS for their dedicated assistance (via e-mail) in bringing our work to fruition.

W. J. Kaczor, M. T. Nowak

Notation and Terminology

This is a supplement to the notation and terminology of *Problems in Mathematical Analysis I, Real Numbers, Sequences and Series.*

If $(\mathbf{X}, d)$ is a metric space, $x \in \mathbf{X}$ and $\mathbf{A}$ is a nonempty subset of $\mathbf{X}$, then

- $\mathbf{A}^c = \mathbf{X} \setminus \mathbf{A}$ is the complement of the set $\mathbf{A}$,
- $\mathbf{B}_{\mathbf{X}}(x, r)$, $\overline{\mathbf{B}}_{\mathbf{X}}(x, r)$ denote the open and the closed ball centered at x and of radius $r > 0$, respectively. If $\mathbf{X}$ is fixed we omit the index and simply write $\mathbf{B}(x, r)$ or $\overline{\mathbf{B}}(x, r)$,
- $\mathbf{A}^\circ$ is the interior of $\mathbf{A}$ in the metric space $(\mathbf{X}, d)$,
- $\overline{\mathbf{A}}$ denotes the closure of $\mathbf{A}$ in the metric space,
- $\partial\mathbf{A} = \overline{\mathbf{A}} \cap \overline{\mathbf{X} \setminus \mathbf{A}}$ is the boundary of $\mathbf{A}$,
- $\operatorname{diam}(\mathbf{A}) = \sup\{d(x, y) : x, y \in \mathbf{A}\}$ denotes the diameter of the set $\mathbf{A}$,
- $\operatorname{dist}(x, \mathbf{A}) = \inf\{d(x, y) : y \in \mathbf{A}\}$ denotes the distance between x and the set $\mathbf{A}$,
- $\mathbf{A}$ is of type $\mathcal{F}_\sigma$ if it is a union of countably many sets which are closed in $(\mathbf{X}, d)$,
- $\mathbf{A}$ is of type $\mathcal{G}_\delta$ if it is an intersection of countably many sets which are open in $(\mathbf{X}, d)$,

- $\mathbf{X}$ is said to be connected if there do not exist two nonempty disjoint open subsets $\mathbf{B}$ and $\mathbf{C}$ of $\mathbf{X}$ such that $\mathbf{X} = \mathbf{B} \cup \mathbf{C}$,
- $$\chi_{\mathbf{A}}(x) = \begin{cases} 1 & \text{if} \quad x \in \mathbf{A}, \\ 0 & \text{if} \quad x \in (\mathbf{X} \setminus \mathbf{A}) \end{cases}$$
  is the characteristic function of $\mathbf{A}$,
- if $\mathbf{A} \subset \mathbf{X}$ and if f is a function defined on $\mathbf{X}$, then $f_{|\mathbf{A}}$ denotes the restriction of f to $\mathbf{A}$.

If f and g are real functions of a real variable, then

- $f(a^+)$ and $f(a^-)$ denote the right-hand and the left-hand limit of f at a, respectively,
- if the quotient $f(x)/g(x)$ tends to zero (or remains bounded) as $x \to x_0$, then we write $f(x) = o(g(x))$ (or $f(x) = O(g(x))$),
- $C(\mathbf{A})$ - the set of all continuous functions on $\mathbf{A}$,
- $C(a,b)$ - the set of all continuous functions on an open interval (a,b),
- $f^{(n)}$ - the nth derivative of f,
- $C^n(a,b)$ - the set of all functions n times continuously differentiable on (a,b),
- $f'_+(a)$, $f'_-(a)$ - the right- and the left-hand derivative of f at a, respectively,
- $C^1([a,b])$ denotes the set of all functions continuously differentiable on $[a,b]$, where at the endpoints the derivative is right- or left-hand, respectively. The set $C^n([a,b])$ of all functions n times continuously differentiable on $[a,b]$ is defined inductively,
- $C^\infty(a,b)$, $C^\infty([a,b])$ - the set of functions infinitely differentiable on (a,b) and $[a,b]$, respectively.

Bibliography - Books

References

[1] J. Banaś, S. Wędrychowicz, *Zbiór zadań z analizy matematycznej*, Wydawnictwa Naukowo-Techniczne, Warszawa, 1994.

[2] V. I. Bernik, O.V. Melnikov, I. K. Žuk,, *Sbornik olimpiadnych zadač po matematike*, Narodnaja Asveta, Minsk, 1980.

[3] P. Biler, A. Witkowski, *Problems in Mathematical Analysis*, Marcel Dekker, Inc, New York and Basel, 1990.

[4] T. J. Bromwich, *An Introduction to the Theory of Infinite Series*, Macmillan and Co., Limited, London, 1949.

[5] R. P. Boas, *A Primer of Real Analytic Functions*, Birkhäuser Verlag, Basel Boston Berlin, 1992.

[6] L. Carleson, T. W. Gamelin, *Complex Dynamics*, Springer-Verlag, New York Berlin Heidelberg, 1993.

[7] B. P. Demidovič, *Sbornik zadač i upražnenij po matematičeskomu analizu*, Nauka, Moskva, 1969.

[8] J. Dieudonné, *Foundations of Modern Analysis*, Academic Press, New York San Francisco London, 1969.

[9] A. J. Dorogovcev, *Matematičeskij analiz. Spravočnoe posobe*, Vyščaja Škola, Kiev, 1985.

[10] A. J. Dorogovcev, *Matematičeskij analiz. Sbornik zadač*, Vyščaja Škola, Kiev, 1987.

[11] G. M. Fichtenholz, *Differential-und Integralrechnung, I, II, III*, V.E.B. Deutscher Verlag Wiss., Berlin, 1966-1968.

[12] B. R. Gelbaum, J. M. H. Olmsted, *Theorems and Counterexamples in Mathematics*, Springer-Verlag, New York Berlin Heidelberg, 1990.

[13] E. Hille, *Analysis Vol.I*, Blaisdell Publishing Company, New York Toronto London, 1964.

[14] W. J. Kaczor, M. T. Nowak, *Problems in Mathematical Analysis I. Real Numbers, Sequences and Series*, American Mathematical Society, Providence, RI, 2000.

[15] G. Klambauer, *Mathematical Analysis*, Marcel Dekker, Inc., New York, 1975.

[16] G. Klambauer, *Problems and Propositions in Analysis*, Marcel Dekker, Inc., New York and Basel, 1979.

[17] K. Knopp, *Theorie und Anwendung der Unendlichen Reihen*, Springer-Verlag, Berlin and Heidelberg, 1947.

[18] S. G. Krantz, H. R. Parks, *A Primer of Real Analytic Functions*, Birkhäuser Verlag, 1992.

[19] L. D. Kudriavtsev, A. D. Kutasov, V. I. Chejlov, M. I. Shabunin, *Problemas de Análisis Matemático. Límite, Continuidad, Derivabilidad (Spanish)*, Mir, Moskva, 1989.

[20] K. Kuratowski, *Introduction to Calculus*, Pergamon Press, Oxford-Edinburgh-New York; Polish Scientific Publishers, Warsaw, 1969.

[21] S. Łojasiewicz, *An Introduction to the Theory of Real Functions*, A Wiley-Interscience Publication, John Wiley & Sons, Ltd., Chichester, 1988.

[22] D. S. Mitrinović, *Elementary Inequalities*, P. Noordhoff Ltd., Groningen, 1964.

[23] G. Pólya, G. Szegö, *Problems and theorems in analysis I*, Spriger-Verlag, Berlin Heidelberg New York, 1978.

[24] R. Remmert, *Theory of Complex Functions*, Springer-Verlag, New York Berlin Heidelberg, 1991.

[25] J.I. Rivkind, *Zadači po matematičeskomu analizu*, Vyšejšaja Škola, Mińsk, 1973.

[26] W. I. Rozhov, G. D. Kurdevanidze, N. G. Panfilov, *Sbornik zadač matematičeskich olimpiad*, Izdat. Univ. Druzhby Narodov, Moskva, 1987.

[27] W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill Book Company, New York, 1964.

[28] W. Rzymowski, *Convex Functions*, preprint.

[29] W. A. Sadovničij, A. S. Podkolzin, *Zadači studenčeskich olimpiad po matematike*, Nauka, Moskva, 1978.

[30] R. Sikorski, *Funkcje rzeczywiste*, PWN, Warszawa, 1958.

[31] H. Silverman, *Complex variables*, Houghton Mifflin Company, Boston, 1975.

[32] E.C. Titchmarsh, *The Theory of Functions*, Oxford University Press, London, 1944.

[33] G. A. Tonojan, W. N. Sergeev, *Studenčeskije matematičeskije olimpiady*, Izdatelstwo Erevanskogo Universiteta, Erevan, 1985.

Index

Problems in Mathematical Analysis II
Continuity and Differentiation

W. J. Kaczor and M. T. Nowak

We learn by doing. We learn mathematics by doing problems. And we learn more mathematics by doing more problems. This is the sequel to *Problems in Mathematical Analysis I* (Volume 4 in the Student Mathematical Library series). If you want to hone your understanding of continuous and differentiable functions, this book contains hundreds of problems to help you do so. The emphasis here is on real functions of a single variable.

The book is mainly geared toward students studying the basic principles of analysis. However, given its selection of problems, organization, and level, it would be an ideal choice for tutorial or problem-solving seminars, particularly those geared toward the Putnam exam. It is also suitable for self-study. The presentation of the material is designed to help student comprehension, to encourage them to ask their own questions, and to start research. The collection of problems will also help teachers who wish to incorporate problems into their lectures. The problems are grouped into sections according to the methods of solution. Solutions for the problems are provided.

ISBN 978-0-8218-2051-3

9 780821 820513

STML/12